Commutativity in non-Abelian Groups

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Abstract.

Let $P_2(G)$ be defined as the probability that any two elements selected at random from the group G, commute with one another. If G is an Abelian group, $P_2(G) = 1$, so our interest lies in the properties of the commutativity of non-Abelian groups.

Particular results include that the maximum commutativity of a non-Abelian group is $5/8$, and this degree of commutativity only occurs when the order of the center of the group is equal to one fourth the order of the group. Explicit examples will be provided of arbitrarily large non-Abelian groups that exhibit this maximum commutativity, along with a proof that there are no $5/8$ commutative groups of order 4 mod 8.

Further, we prove that no group exhibits commutativity 0, there exist examples of groups whose commutativity is arbitrarily close to 0. Then, we show that for every positive integer n there exists a group G such that $P_2(G) = 1/n$. Finally we prove that the commutativity of a factor group G/N of a group G is always greater than or equal to the commutativity of G.

0 Introduction:

The way we define Abelian groups (see Definitions 0.2 and 0.3) provides us with a simple way of understanding what might be termed the commutativity of such groups. In particular, as each pair of elements in an Abelian group necessarily commutes, we can say that the group has complete commutativity, or 100% commutativity, or, on a decimal scale of zero to one, commutativity 1. As is known by those with even a little background in the study of group theory, however, Abelian groups account for only a small proportion of all groups.

Given that these other, so-called non-Abelian groups, represent the vast majority of all groups, we would like to develop a method for classifying the degree, or the percent, to which a given non-Abelian group is commutative. As we will see directly from the following definitions, even non-Abelian groups cannot have commutativity 0 on the zero to one scale, since at least one group element, namely the identity (see Definition 0.2), will always commute with every other element. Also, as a non-Abelian group is, by definition, not Abelian, it must contain some elements that do not commute, so we cannot have a non-Abelian group with commutativity 1.

Since, at a glance, we already can rule out 0 and 1 as possible values for the commutativity of a given non-Abelian group, our initial investigations will consist primarily of showing what values, between 0 and 1, are possible degrees of non-Abelian commutativity. In addition, we will spend much of our time describing the ways in which many the characteristics of commutativity are exhibited by finite non-Abelian groups, such as the dihedral groups. Finally, we will conclude by demonstrating additional properties of the commutativity of non-Abelian groups, such as the relationship between the commutativity of a group, and the commutativity of one of its factor groups.

To begin our investigation of the commutativity of non-Abelian groups we will first provide a few definitions. Unless otherwise specified, all notation and terminology used will be as in [1].

Definition 0.1: *Binary Operation*.
Let G be a set. A *binary operation* on G is a function that assigns each ordered pair of elements of G an element of G.

From the definition of a binary operation, we next develop the notion of a group.

Definition 0.2: *Group*
Let G be a nonempty set together with a binary operation (usually called multiplication) that assigns to each ordered pair (a, b) of elements of G an element of G denoted by ab. We say that G is a *group* under this operation if the following three properties are satisfied.

> **1.** *Associativity.* The operation is associative; that is, $(ab)c = a(bc)$ for all a, b, c in G.
>
> **2.** *Identity.* There is an element e (called the *identity*) in G such that $ae = ea = a$ for all a in G.
>
> **3.** *Inverses.* For each element a in G, there is an element b in G (called the *inverse* of a) such that $ab = ba = e$.

Now, to characterize the difference between Abelian and non-Abelian groups, we provide the following definition.

Definition 0.3: ***Abelian Group***
If a group has the property that $ab = ba$ for every pair of elements a and b, we say that the group is *Abelian*. A group is *non-Abelian* if there is some pair of elements a and b for which $ab \neq ba$.

Finally, the next two definitions require particular attention, as they will be directly referred to numerous times throughout this article.

Definition 0.4: ***Center of a Group***
The *center*, $Z(G)$, of a group G is the subset of elements in G that commute with every element of G. In symbols, $Z(G) = \{a \in G \mid ax = xa \text{ for all } x \text{ in G}\}$.

Definition 0.5: ***Centralizer of a in G***
Let a be a fixed element of a group G. The *centralizer of a in G*, $C(a)$, is the set of all elements in G that commute with a. In symbols, $C(a) = \{g \in G \mid ga = ag\}$.

In addition to the five definitions above, we will later refer to both the Class Equation and Lagrange's Theorem, both of which will be stated and explained prior to their usage.

1 Commutativity

Using the definition of the center of a group and of the centralizer of a group element, we are prepared to begin developing a measure for the commutativity of a given group. First, though, we define a certain family of groups.

Dihedral groups

The following definition describes the family of groups that will play the most important role in providing examples of the properties of non-Abelian commutativity throughout the remainder of this article.

Definition 1.1: ***Dihedral group*** [1]
The dihedral group D_n is the symmetry group of an n-sided regular polygon for $n > 1$. The group order of D_n is $2n$.

[1] See [4] for definition. A thorough explanation of the properties and construction of the dihedral groups can be found in [1].

By *order* we mean the number of elements of a group (finite or infinite). We will use $|G|$ to denote the order of a group G. Now, to further explain this definition, we will consider as an example the dihedral group, D_4, of order eight.[2]

From the definition, we know that the dihedral group D_4 is the symmetry group of a 4-sided regular polygon, that is, a square. To elaborate, this means that the group D_4 consists of the eight flips and rotations which comprise the natural transformations that can be performed on a square. The four rotations in the plane are by degrees of 0, 90, 180, and 270. These are denoted R_0, R_{90}, R_{180}, and R_{270} respectively. The four remaining elements are two flips, H and V, across the horizontal and vertical axes, and two flips, D and D', across the two diagonal axes.

These eight transformation elements are shown in Figure 1.

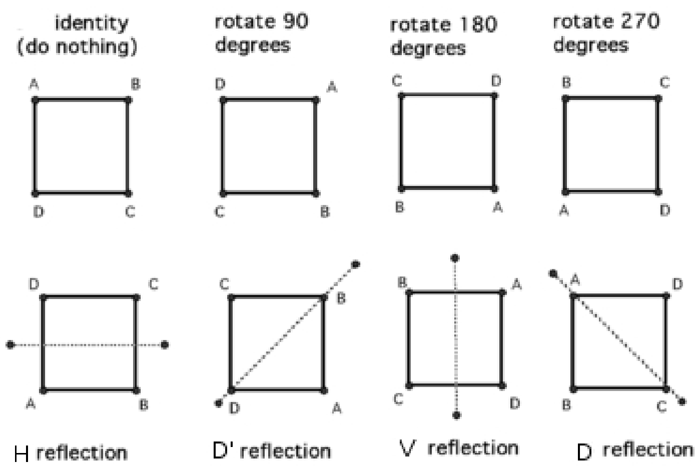

Figure 1: $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$

Now that we have gained a knowledge of the elements that comprise the dihedral group D_4 we begin to consider products of these elements in different orders, so as to determine how many pairs of elements commute out of the total number of possible commutative pairs.

[2]The reader should note that the properties of this particular group D_4 will be invaluable in later sections of this article, as they exhibit many of the properties of commutativity of non-Abelian groups that will be discussed and proved.

The easiest way to summarize the result of taking the product of each pair of elements in the group is to construct a Cayley table for D_4 [see Table 1]. A Cayley table for group elements can be thought of in the same way as a multiplication table for integers. That is, to determine the value of the product ij, one may refer to the value in the intersection of the ith row and the jth column.

D_4	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_0	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_{90}	R_{90}	R_{180}	R_{270}	R_0	D'	D	H	V
R_{180}	R_{180}	R_{270}	R_0	R_{90}	V	H	D'	D
R_{270}	R_{270}	R_0	R_{90}	R_{180}	D	D'	V	H
H	H	D	V	D'	R_0	R_{180}	R_{90}	R_{270}
V	V	D'	H	D	R_{180}	R_0	R_{270}	R_{90}
D	D	V	D'	H	R_{270}	R_{90}	R_0	R_{180}
D'	D'	H	D	V	R_{90}	R_{270}	R_{180}	R_0

Table 1: Cayley table for the group D_4

As an example, notice that $HR_{90} = D$. Further, to demonstrate that the group D_4 is non-Abelian, we may note that $R_{90}H = D'$. It will be left to the reader to verify that the elements D and D' do indeed represent the result of the product of the elements H and R_{90} in the orders shown above.

A measure of commutativity

Now that we have one family of non-Abelian groups (the dihedral groups) to examine, we can begin to develop a measure for the commutativity of groups in general.

There are at least two ways to approach the problem of specifying a measure of commutativity for non-Abelian groups. One uses the simple definition of the centralizer of each element of the group, while the other uses the notions of *conjugates* and *conjugacy classes*.

As it turns out, both of these methods lead to the same measure of commutativity. We will discuss both methods, initially focusing on the perhaps more intuitive approach involving centralizers, and later detailing the somewhat more elegant conjugacy class idea.

Henceforth, we will refer to the commutativity of a group G as $P_2(G)$ (notice that this notation applies even when the group G is Abelian, but that $P_2(G)$ trivially equals 1 if this is the case). The rationale behind this notation is that the commutativity of a group as we will be measuring it is the probability that any two group elements selected at random commute with one another, hence: P_2.

Before defining the P_2 measure of commutativity in terms of the centralizers of the group elements, we note that the center of a group by itself does not provide a particularly accurate measure for commutativity. In order to demonstrate this fact, we introduce some of the other small dihedral groups, specifically D_3, D_5, and D_6.[3]

As previously defined, the small dihedral groups D_3, D_5, and D_6 consist of elements corresponding to the rotation- and flip- symmetries of regular polygons with 3, 5, and 6 sides respectively. In particular, R_m represents a rotation by m degrees, and the other variables V, H, D, F and their primes represent flips across the axes that run either from one vertex to the opposite vertex or from the midpoint of one side to the midpoint of the opposite side (in the case of the even-sided polygons such as the hexagon) or from one vertex to the midpoint of the side opposite it (in the case of the odd-sided polygons such as the triangle and the pentagon).

Thus, including D_4 as defined earlier, we have:

$$\begin{aligned}
&D_3 = \{R_0, R_{120}, R_{240}, V, D, D'\},\\
&D_4 = \{R_0, R_{90}, R_{180}, R_{270}, V, H, D, D'\},\\
&D_5 = \{R_0, R_{72}, R_{144}, R_{216}, R_{288}, V, D, D', F, F'\},\\
&D_6 = \{R_0, R_{60}, R_{120}, R_{180}, R_{240}, R_{300}, V, V', V'', D, D', D''\},
\end{aligned}$$

where, in each case, R_0 is the identity element of the group. Now, examining the Cayley table of each group,[4] we can conclude that the centers of the four dihedral groups are:

$$\begin{aligned}
Z(D_3) &= \{R_0\},\\
Z(D_4) &= \{R_0, R_{180}\},\\
Z(D_5) &= \{R_0\},\\
Z(D_6) &= \{R_0, R_{180}\}.
\end{aligned}$$

[3]In general, we will not consider the first dihedral group D_2, as it only consists of 4 elements and is thereby Abelian. In fact, whereas [4] defines the dihedral groups as beginning with D_2, [1] only considers the dihedral groups D_n for $n \geq 3$.

[4]The reader may consult Table 1 for an example of the Cayley table of D_4. The Cayley table of each of the other three groups can be easily constructed.

By the regular structural nature of the polygons that comprise the D_n groups, we can conclude that when n is even the two elements in the center of D_n are R_0 and R_{180}, whereas when n is odd, the only element in $Z(D_n)$ is R_0. To see this, we make note of the following facts.

Clearly R_0, a rotation by zero degrees, is the identity of every dihedral group and is thus in the center of D_n for all n. Then, in all dihedral groups, every rotation element commutes with every other rotation element, whereas every flip element does not commute with most rotation elements. Since in the case of even-sided polygon dihedral groups, every flip element commutes with the identity, R_0, and the element R_{180}, but not with any other rotation element, we conclude the $Z(D_n) = \{R_0, R_{180}\}$ when n is even. On the other hand, every flip element in an odd-sided polygon dihedral group does not commute with any rotation element besides the identity, so $Z(D_n) = \{R_0\}$ when n is odd.

Since the center will continue to either be of order 1 or order 2, no matter how large the group is, the sizes of the centers may rapidly become an inaccurate measure of the degree to which a dihedral group is non-Abelian.

Given that the centers of the dihedral groups do not provide us with an accurate method for measuring group commutativity,[5] we proceed to compute the centralizers of every element of each of the four dihedral groups detailed above. We may again refer to the Cayley table of each of these groups in order to determine the centralizers of their elements according to Definition 0.5.

The elements of the centralizer of each element of D_3 are listed below.

$$\begin{aligned}
C(R_0) &= \{R_0, R_{120}, R_{240}, V, D, D'\},\\
C(R_{120}) &= \{R_0, R_{120}, R_{240}\},\\
C(R_{240}) &= \{R_0, R_{120}, R_{240}\},\\
C(V) &= \{R_0, V\},\\
C(D) &= \{R_0, D\},\\
C(D') &= \{R_0, D'\}.
\end{aligned}$$

[5] It should be noted, however, that the center of a group will be an important consideration when determining the maximum commutativity of non-Abelian groups.

The elements of the centralizer of each element of D_4 are listed below.

$$\begin{aligned}
C(R_0) &= \{R_0, R_{90}, R_{180}, R_{270}, V, H, D, D'\},\\
C(R_{90}) &= \{R_0, R_{90}, R_{180}, R_{270}\},\\
C(R_{180}) &= \{R_0, R_{90}, R_{180}, R_{270}, V, H, D, D'\},\\
C(R_{270}) &= \{R_0, R_{90}, R_{180}, R_{270}\},\\
C(V) &= \{R_0, R_{180}, V, H, \},\\
C(H) &= \{R_0, R_{180}, V, H, \},\\
C(D) &= \{R_0, R_{180}, D, D'\},\\
C(D') &= \{R_0, R_{180}, D, D'\}.
\end{aligned}$$

The elements of the centralizer of each element of D_5 are listed below.

$$\begin{aligned}
C(R_0) &= \{R_0, R_{72}, R_{144}, R_{216}, R_{288}, V, D, D', F, F'\},\\
C(R_{72}) &= \{R_0, R_{72}, R_{144}, R_{216}, R_{288}\},\\
C(R_{144}) &= \{R_0, R_{72}, R_{144}, R_{216}, R_{288}\},\\
C(R_{216}) &= \{R_0, R_{72}, R_{144}, R_{216}, R_{288}\},\\
C(R_{288}) &= \{R_0, R_{72}, R_{144}, R_{216}, R_{288}\},\\
C(V) &= \{R_0, V\},\\
C(D) &= \{R_0, D\},\\
C(D') &= \{R_0, D'\},\\
C(F) &= \{R_0, F\},\\
C(F') &= \{R_0, F'\}.
\end{aligned}$$

The elements of the centralizer of each element of D_6 are listed below.

$$\begin{aligned}
C(R_0) &= \{R_0, R_{60}, R_{120}, R_{180}, R_{240}, R_{300}, V, V', V'', D, D', D''\},\\
C(R_{60}) &= \{R_0, R_{60}, R_{120}, R_{180}, R_{240}, R_{300}\},\\
C(R_{120}) &= \{R_0, R_{60}, R_{120}, R_{180}, R_{240}\},\\
C(R_{180}) &= \{R_0, R_{60}, R_{120}, R_{180}, R_{240}, R_{300}, V, V', V'', D, D', D''\},\\
C(R_{240}) &= \{R_0, R_{60}, R_{120}, R_{180}, R_{240}, R_{300}\},\\
C(R_{300}) &= \{R_0, R_{60}, R_{120}, R_{180}, R_{240}, R_{300}\},\\
C(V) &= \{R_0, R_{180}, V, D\},\\
C(V') &= \{R_0, R_{180}, V', D'\},\\
C(V'') &= \{R_0, R_{180}, V'', D''\},\\
C(D) &= \{R_0, R_{180}, V, D\},\\
C(D') &= \{R_0, R_{180}, V', D'\},\\
C(D'') &= \{R_0, R_{180}, V'', D''\}.
\end{aligned}$$

As the order of the centralizer of a certain group element gives us the number of group elements that commute with that certain element, we can rapidly determine the total number of commutative pairs of elements (including the product of an element with itself, which is clearly commutative) simply by summing the orders of the centralizers of every element in a group. By this standard, we can then also say that the total possible number of commutative pairs of elements in a given group (that is, the total number of pairs) is simply the order of the group squared.

Thus, the previously mentioned $P_2(G)$ measure of commutativity can be written as the fraction $c/|G|^2$, where c represents the sum of the the centralizers of all of the group elements.

To better understand this definition of $P_2(G)$, we will compute the $P_2(D_n)$ for $n = 3, 4, 5, 6$. As we have already computed the centralizers for every element in each of these four dihedral groups, it remains to sum the order of each centralizer in a given group, and divide this value by the order of the group squared.

In this way, we find that in D_3, the total number of pairs that commute is $c = 6 + 3 + 3 + 2 + 2 + 2 = 18$ (the sum of the order of each of the six centralizers), while the order of the group squared is $|D_3|^2 = 6^2 = 36$. Thus, the degree to which D_3 is commutative, is:

$$18/36 = 0.5$$

Likewise, for D_4, the number of commutative pairs is $c = 8 + 4 + 8 + 4 + 4 + 4 + 4 + 4 = 40$, the total number of pairs is $|D_4|^2 = 8^2 = 64$, and the degree to which it is commutative is:

$$40/64 = 0.625$$

For D_5, the number of commutative pairs is $c = 10+5+5+5+5+2+2+2+2+2 = 40$, the total number of pairs is $|D_5|^2 = 10^2 = 100$, and the degree to which it is commutative is:

$$40/100 = 0.4$$

Finally, for D_6, the number of commutative pairs is $c = 12 + 6 + 6 + 12 + 6 + 6 + 4 + 4 + 4 + 4 + 4 + 4 = 72$, the total number of pairs is $|D_6|^2 = 12^2 = 144$, and the degree to which it is commutative is:

$$72/144 = 0.5$$

From these examples, we can begin to suppose that even-polygon dihedral groups are more commutative than similarly-sized odd-polygon dihedral groups, and that smaller dihedral groups are more commutative than larger ones. We will shortly see that both of these ideas are correct, and can be easily proven.

Deriving P_2 from conjugacy classes

As promised, we will now discuss a second method by which we may arrive at the commutativity measure P_2. Since this method uses the idea of conjugacy classes, let us begin with a definition.

Definition 1.2: *Conjugacy Class of* a.
Let a and b be elements of a group G. We say that a and b are *conjugate* in G (and call b a *conjugate* of a) if $xax^{-1} = b$ for some x in G. The *conjugacy class of* a is the set $\mathrm{cl}(a) = \{xax^{-1} \mid x \in G\}$.

The reader may notice that conjugacy is an *equivalence relation* on a group. That is, conjugacy satisfies the reflexive, symmetric, and transitive properties, as shown below.

Let G be a group and let $a \in G$. Notice that $a = eae^{-1}$, where $e \in G$ is the identity of the group. Thus, conjugacy satisfies the reflexive property. Next, let $b \in G$ and that $a = xbx^{-1}$ for some $x \in G$. Then $x^{-1}ax = x^{-1}xbx^{-1}x = b$, such that conjugacy satisfies the symmetric property. Finally, let $c \in G$, and suppose that in addition to $a = xbx^{-1}$, there exists $y \in G$ such that $b = ycy^{-1}$. Then $a = x(ycy^{-1})x^{-1} = (xy)c(xy)^{-1}$, which shows that conjugacy satisfies the transitive property.

This straightforward verification that conjugacy satisfies the three defining properties of an equivalence relation allows us to henceforth partition any group into disjoint conjugacy classes where necessary.

It is worth noting that proofs of several of the following theorems rely one one of the most important results in finite group theory, Lagrange's Theorem, which is included here for reference.

Theorem 1.1. Lagrange's Theorem: $|H|$ divides $|G|$.
If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$. Moreover, the number of distinct left cosets of H in G is $|G|/|H|$.

The reader is encouraged once again to consult [1] for the definition of a *coset*. We call the number of distinct left cosets of a subgroup H, in a group, G, the *index* of the subgroup, and denote this by $|G : H|$. Hence, as a consequence of Lagrange's Theorem, we have the following corollary.

Corollary 1.2. $|G : H| = |G|/|H|$**.**
If G is a finite group and H is a subgroup of G, then $|G : H| = |G|/|H|$.

Now, as stated in [1], we have a theorem relating the size of the conjugacy class of an element a in a group G with the size of the centralizer of a in G.

Theorem 1.3. The Number of Conjugates of a
Let G be a finite group and let a be an element of G. Then,

$$|\mathrm{cl}(a)| = |G : C(a)| = |G|/|C(a)|.$$

Proof. Consider the function T that sends the coset $xC(a)$ to the conjugate xax^{-1} of a. A routine calculation shows that T is well defined, is one-to-one, and maps the set of left cosets onto the conjugacy class of a. Thus, the number of conjugates of a is the index of the centralizer of a.

□

We previously stated that the $P_2(G)$ is defined to be the probability that two elements chosen randomly with replacement from the finite group G commute. That is, the total number of commutative elements divided by the total possible number of element pairs, or the sum of the order of the centralizer of each element of G divided by the order of G squared.

As a consequence of Theorem 1.3, if a and b are in the same conjugacy class, then $|C(a)| = |C(b)|$. Thus, if $\mathrm{cl}(b) = \{b_1, b_2, \ldots, b_k\}$,

$$\begin{aligned} |C(b_1)| + |C(b_2)| + \cdots + |C(b_k)| &= k|C(b)| \\ &= |G : C(b)|\ |C(b)| \\ &= |G|. \end{aligned}$$

Now, the sum of the centralizers of all the elements of G can be separated into sums of the centralizers of all the elements from each conjugacy class of G. That is,

$$\sum_{a \in G} |C(a)| = \sum_{x \in A_1} |C(x)| + \sum_{y \in A_2} |C(y)| + \cdots + \sum_{z \in A_m} |C(z)|.$$

where $A_1, A_2, \ldots, A_m$ are the conjugacy classes of G.

Thus, if we choose one element from each conjugacy class, say $a_1, a_2, \ldots, a_m$, we have that

$$\sum_{a \in G} |C(a)| = \sum_{i=1}^{m} |G : C(a_i)|\ |C(a_i)| = \sum_{i=1}^{m} |G| = m|G|.$$

In general, if m is the number of conjugacy classes in G, and $|G| = n$, then

$$P_2(G) = \frac{m \cdot n}{n^2} = \frac{m}{n}.$$

Maximizing commutativity

Now that we have seen examples of different values of commutativity for non-Abelian groups, we would like to determine what the maximum value of non-Abelian commutativity is, given that we know it must be strictly less than 1.

In order to aid our investigation, we make note of the following three theorems, as stated in [1].

Theorem 1.4. $Z(G)$ **is a group.**
The center of a group G is a subgroup of G.

Theorem 1.5. $C(a)$ **is a group.**
For each a in a group G the centralizer of a is a subgroup of G.

In order to determine the maximum non-Abelian commutativity, we begin by determining the maximum order of the center of a non-Abelian group. A brief analysis, given the theorems above, leads to the following theorem.

Theorem 1.6. Maximum size of the center of non-Abelian groups.
If G is a finite non-Abelian group, then the maximum possible order of the center of G is $|G|/4$.

Proof. Let $b \in Z(G)$. Since G is non-Abelian, $Z(G) \neq G$. Thus, there exists $a \in G$ such that $a \notin Z(G)$. Note that this implies that $C(a) \neq G$. By definition, every element in G commutes with b, so $ab = ba$. It follows that $b \in C(a)$. Since $a \in C(a)$, we find that $Z(G) \subset C(a)$. In fact, since a group that is a subset of a subgroup under the same operation is itself a subgroup of the subgroup, we find that $Z(G)$ is a proper subgroup of $C(a)$. By Lagrange's Theorem, it follows that $|Z(G)| \leq |C(a)|/2$.

Now, since we assumed $C(a) \neq G$, $C(a) \subset G$, so by Lagrange's Theorem and the fact that the centralizer of any group element must be a subgroup of the group, we find that $|C(a)| \leq |G|/2$. Hence, we conclude that $|Z(G)| \leq |C(a)|/2 \leq |G|/4$. This completes the proof.

□

Now we proceed to determine the maximum value of $P_2(G)$ where G is non-Abelian. Recall that the value of the denominator of P_2 is always taken to be $|G|^2$. Thus, it is the value of the numerator, c, that must be maximized in order to maximize the value of the fraction $c/|G|^2$.

We can represent c as the sum of the sizes of each centralizer of the group. That is,

$$\sum_{a \in G} |C(a)|.$$

We will suppose that the center of G takes on the maximum order of $|G|/4$, so the number of centralizers that are equal to G is equal to $|G|/4$. We claim that the centralizers of the remaining three-fourths of the group elements each have order equal to $|G|/2$. To see this, notice that by assumption $(G : Z(G)) = 4$. Suppose that $a \in G$ is not an element of the center of G. By Lagrange's Theorem, $(G : Z(G)) = (G : C(a))(C(a) : Z(G))$. Since $a \notin Z(G)$ and $a \in C(a)$, we find that $C(a) \neq G$ and $C(a) \neq Z(G)$, so $(C(a) : Z(G)) = 2$. Thus, $(G : C(a)) = 2$, which implies that $|C(a)| = |G|/2$.

It follows that the maximum possible number of commutative pairs in a non-Abelian group is

$$\begin{aligned} c &= \left(\frac{|G|}{4}\right)(|G|) + \left(\frac{3|G|}{4}\right)\left(\frac{1}{2}|G|\right), \\ &= \frac{|G|^2}{4} + \frac{3|G|^2}{8}, \\ &= \frac{5|G|^2}{8}. \end{aligned}$$

Hence, the maximum size of the measure $P_2(G)$ is

$$\begin{aligned} P_2(G) &= \frac{5|G|^2/8}{|G|^2}, \\ &= \frac{5}{8}. \end{aligned}$$

We have thus proven the following theorem.

Theorem 1.7. Maximum commutativity of non-Abelian groups
If G is a finite non-Abelian group, then the maximum possible value of the commutativity measure $P_2(G)$ is $5/8$. Furthermore, this maximum value of P_2 occurs if and only if $|Z(G)| = |G|/4$.

As an example of a group that exhibits this maximum commutativity, consider the dihedral group D_4. Recall that $Z(D_4) = \{R_0, R_{180}\}$, so

$$|Z(D_4)| = 2 = 8/4 = |D_4|/4.$$

Thus, by Theorem 2.5, we expect that $P_2(D_4) = 5/8$, and indeed we found previously that this is the case, as $P_2(D_4) = c/|D_4|^2 = 40/8^2 = 5/8$.

Commutative properties of D_4

We showed earlier that among the four dihedral groups, D_4 is the only one that exhibits $5/8$ commutativity. It turns out that we can easily prove that D_4 is the smallest (although not the unique smallest) group with $5/8$ commutativity. In other words, there are no $5/8$ commutativity groups of order less than 8.

To see this, we begin by listing the groups with order less than or equal to 8 (the reader may consult [2] for a more detailed discussion of small groups).

Order	Group(s)
1	C_1 (trivial group)
2	C_2
3	C_3
4	$C_4,\ C_2 \times C_2$
5	C_5
6	C_6, D_3
7	C_7
8	$C_8,\ C_4 \times C_2,\ C_2 \times C_2 \times C_2,\ D_4,\ Q_8$

Table 2: Groups of small order

The notation in Table 2 is easily explained. The groups C_n which appears for each order $1, 2, \ldots, 8$ are the cyclic groups of order 1 through 8 respectively. Here we refer the reader to [1] and [3] for more detail on cyclic groups, and we will suffice it to say that a property of cyclic groups is that they are Abelian.

This simple observation rules out the majority of the groups with order less than or equal to 8 as possible 5/8 commutativity groups.

Now, the Fundamental Theorem of Finite Abelian groups states that every finite Abelian group is a direct product of cyclic groups of prime-power order. This allows us to conclude that the direct product groups,[6] $C_2 \times C_2$, $C_4 \times C_2$, and $C_2 \times C_2 \times C_2$, are also Abelian.

The only remaining groups to consider are D_3, D_4, and Q_8. We know that D_3 has commutativity $1/2 < 5/8$, so we have shown that there is no 5/8 commutative group of order less than 5/8. Now we must determine if Q_8 is 5/8 commutative, since if it is not, D_4 must be the unique smallest non-Abelian group with maximum commutativity.

By Q_8, we mean the group of quaternions.[7] That is, $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$. By examining Table 3 we find that $Z(Q_8) = \{1, -1\}$. Since $|Z(Q_8)| = 2 = |Q_8|/4$, we conclude that the group of quaternions is, in fact, 5/8 commutative.

Q_8	-1	$-i$	$-j$	$-k$	1	i	j	k
-1	1	i	j	k	-1	$-i$	$-j$	$-k$
$-i$	i	-1	k	$-j$	$-i$	1	$-k$	j
$-j$	j	$-k$	-1	i	$-j$	k	1	$-i$
$-k$	k	j	$-i$	-1	$-k$	$-j$	i	1
1	-1	$-i$	$-j$	$-k$	1	i	j	k
i	$-i$	1	$-k$	j	i	-1	k	$-j$
j	$-j$	k	1	$-i$	j	$-k$	-1	i
k	$-k$	$-j$	i	1	k	j	$-i$	-1

Table 3: Cayley table for the group of quaternions

Thus D_4 has the non-unique property (because it is shared with Q_8) of being the smallest non-Abelian group to exhibit maximum commutativity of 5/8.

5/8 commutative groups of order $8n$, $n \in \mathbb{Z}^+$

We would like to be able to find examples of groups other than just D_4 that exhibit the maximum commutativity measure of 5/8. Among well-known finite non-Abelian groups, however, there appear to be no other obvious examples.

[6]See the following subsection and section 2 of this article, or consult [1], for more detail on direct products.

[7]See [5]. The reader may verify that D_4 is not isomorphic to Q_8.

After all, for P_2 to be $5/8$, the size of the center of the group must be $1/4$ the size of the group. This can easily be seen not to occur in dihedral groups other than D_4, and which certainly never occurs for any symmetric or alternating groups.[8]

Instead, we may perform group composition by way of the external direct product in order to obtain a non-Abelian group with $5/8$ commutativity.

Definition 1.3: ***External Direct Product***
Let $G_1, G_2, \ldots, G_n$ be a finite collection of groups. The *external direct product* of $G_1, G_2, \ldots, G_n$, written as $G_1 \oplus G_2 \oplus \cdots \oplus G_n$, is the set of all n-tuples for which the ith component is an element of G_i and the operation is component-wise.

Suppose that $G = \{a_1, a_2, \ldots, a_n\}$ is a finite Abelian group, where $|G| = n$. We will show that $D_4 \oplus G$ is a non-Abelian group with $P_2 = 5/8$. An arbitrary element from $D_4 \oplus G$ will be of the form (d_i, a_x), where $d_i \in D_4$ and $a_x \in G$. Suppose that $(d_i, a_x), (d_j, a_y) \in D_4 \oplus G$. Then

$$(d_i, a_x) \cdot (d_j, a_y) = (d_i d_j, a_x a_y).$$

Now, when $d_i d_j = d_j d_i$, which happens $5/8$ of the time for elements of D_4, we have

$$(d_i, a_x) \cdot (d_j, a_y) = (d_j d_i, a_y a_x) = (d_j, a_y) \cdot (d_i, a_x),$$

since G is Abelian so $a_x a_y$ always equals $a_y a_x$. The other $3/8$ of the time, when $d_i d_j \neq d_j d_i$, we have

$$(d_i, a_x) \cdot (d_j, a_y) \neq (d_j, a_y) \cdot (d_i, a_x).$$

Thus, we find that $D_4 \oplus G$ is a group of order $8n$ which exhibits the $5/8$ measure of commutativity. Since the order, n, of G was arbitrary, we can thus find arbitrarily large groups of order $8n$ that exhibit $5/8$ commutativity.

No $5/8$ commutative groups of order $4 \bmod 8$

It should be clear that there are no $5/8$ commutative groups of order $k \bmod 8$ for $k = 1, 2, 3, 5, 6, 7$, since for the order of the center to equal one-fourth the order of the group, the order of the group must be a multiple of four.

[8] We do not discuss symmetric or alternating groups in this article, but information can be found on them in [1].

It is tempting, however, to suppose that we can find a 5/8 commutative group of order 4 mod 8, just as we found one (infinitely many, in fact) of order 0 mod 8. What follows is a justification for why this cannot be the case.

As the proof of the following theorem relies on the Class Equation, it will be stated here as in [1].

Theorem 1.8. The Class Equation
For any finite group G, $|G| = \sum |G : C(a)|$, where the sum runs over one element a from each conjugacy class of G.

Given the Class Equation, we are now prepared to state and prove the following theorem.

Theorem 1.9. Groups of order $4 \bmod 8$
There are no $5/8$ commutative groups of order $4 \bmod 8$.

Proof. Suppose G has order $4 \bmod 8$ and G is 5/8 commutative. By the Class Equation, $|G| = \sum |G : C(a)|$, where the sum runs over one element a from each conjugacy class of G. Since 5/8 commutativity requires $|Z(G)| = |G|/4$, $|G| = |Z(G)| + \sum |G : (C(a))|$ where now the sum runs over one element a from each conjugacy class outside of the center.

Since the centralizer of each element of G is a subgroup of G, it's order must divide $|G|$. The centralizer of an element $a \in G$ that satisfies $a \notin Z(G)$ can be at most $|G|/2$. To see that $|C(a)| = |G|/2$, notice that a commutes with itself as well as with each of the $|G|/4$ elements of center of $Z(G)$. That is, $|C(a)| = |G|/4 + 1 > |G|/4$. By assumption, $|G|$ is even, so 3 does not divide $|G|$. Therefore, $|C(a)| = |G|/2$. Since a was an arbitrary element of G that was not in the center of G, it follows that the centralizer of each element outside of the center must have order $|G|/2$, implying that $|G : C(a)| = |G|/(|G|/2) = 2$.

Thus, $\sum |G : C(a)|$ is even, but since $|G| \equiv 4 \pmod 8$, $|Z(G)| = |G|/4$ is odd. Since the sum of an even number, $\sum |G : C(a)|$, and an odd number, $|Z(G)|$, can never equal an even number, $|G|$, we have arrived at a contradiction.

$\square$

2 External direct products:

We saw, when developing a 5/8 commutative group of order $8n$, that by taking the external direct product of a 5/8 commutative group and an Abelian group (with commutativity 1), that the commutativity of the resulting group was $5/8 \cdot 1 = 5/8$. We will now generalize this idea to determine the commutativity of the external direct product of any number of groups of any commutativity.

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite group of order n and let $B = \{b_1, b_2, \ldots, b_m\}$ be a finite group of order m. Let x represent the commutativity of A, where $0 < x \leq 1$ and let y represent the commutativity of B, where $0 < y \leq 1$.

An arbitrary element from $A \oplus B$ will be of the form (a_i, b_s), where $a_i \in A$ and $b_s \in B$. Suppose that $(a_i, b_s), (a_j, b_t) \in A \oplus B$. Then

$$(a_i, b_s) \cdot (a_j, b_t) = (a_i a_j, b_s b_t).$$

Now, when $a_i a_j = a_j a_i$, which happens x of the time for elements of A, we have

$$(a_i, b_s) \cdot (a_j, b_t) = (a_j a_i, b_s b_t),$$

and when $b_s b_t = b_t b_s$, which happens y of the time for elements of B, we have

$$(a_i, b_s) \cdot (a_j, b_t) = (a_i a_j, b_t b_s).$$

This means that $x \cdot y$ of the time, for elements of $A \oplus B$, we have

$$(a_i, b_s) \cdot (a_j, b_t) = (a_j a_i, b_t b_s) = (a_j, b_t) \cdot (a_i, b_s).$$

That is, $x \cdot y$ is the probability that any two elements of $A \oplus B$ commute, so we can conclude that the commutativity of $A \oplus B$ is the product of the commutativity of A and of B.

This result can be trivially generalized to apply to external direct products of any finite number of finite groups, allowing us to explicitly find groups of very large order with very small commutativity.

A lower bound for P_2

An obvious conclusion to draw from the result above is that 0 is a lower bound for $P_2(G)$ where G is a non-Abelian group.

Clearly, an element will always commute with itself, and the identity will always commute with every element, so $P_2(G)$ can never equal 0, but as we will see, it is possible to find groups whose commutativity is arbitrarily close to 0.

An easy way to find groups of such arbitrarily small commutativity is to consider the group formed from the external direct product of a small group with itself a certain number of times.

Suppose, for example, we wish to find a group with $P_2(G) < .00001$. We may simply consider the group formed by taking the external direct product of D_3 with itself several times. Specifically, if we consider

$$G = \underbrace{D_3 \oplus D_3 \oplus \cdots \oplus D_3}_{17 \text{ times}}$$

we will have a group with commutativity $P_2(G) = 1/2^{17} \approx .000008 < .00001$ as desired.

Also, note that if we consider the external direct product of D_5 with itself, we will be able to obtain a similarly small commutativity with fewer direct products, since $P_2(D_5) = 40/100 = 2/5$. That is,

$$G = \underbrace{D_5 \oplus D_5 \oplus \cdots \oplus D_5}_{8 \text{ times}}$$

yields a group with commutativity $P_2(G) = (2/5)^8 \approx .000005 < .00001$. Hence, 0 is a lower bound for $P_2(G)$, and is in fact the greatest lower bound, as we can find examples of groups, G, such that $P_2(G)$ is arbitrarily close to 0.

3 The reduced fraction P_2

With such a simple way to determine the commutativity of groups we can create through the method of external direct products, we now would like to determine what numbers can appear in the denominator of the reduced fractional expression for $P_2(G)$.

We have seen that $P_2(D_3) = 1/2$, so clearly we can generate any power of 2 in the denominator by taking the external direct product of D_3 with itself the correct number of times. Similarly, $P_2(D_6) = 2/5$, so we can generate any power of 5 in the denominator by a similar process. Further, combining these two groups in various direct products allows us to create groups with any product of a 2-power and a 5-power in the denominator of the measure of commutativity.

The next fact to determine is whether or not it is possible to find a group, G, such that the denominator of $P_2(G)$ is a 3 in reduced fractional form. If this is indeed possible, then in turn it is possible that there exist groups such the the denominator of the expression for commutativity is any prime, whereas if it is not possible to obtain a 3 in the denominator, then clearly not every prime can appear in the denominator of the expression.

Since dihedral groups often provide nice examples of commutativity properties, we will begin this search for a 3 in the denominator of the commutativity measure of a group by extrapolating the commutativity values from the first four dihedral group so that they apply to arbitrarily large dihedral groups.

First, consider the dihedral groups, D_n, where n is even. That is, where $n = 2k$ for $k \in \mathbb{Z}^+$. Recall that we previously determined the sizes of the centralizers of every element in each of the first four dihedral groups. For D_4 and D_6, we have that

> D_4 contains:
>
> 2 centralizers of size 8 (all of D_4)
>
> 2 centralizers of size 4 (just rotations)
>
> 4 centralizers of size 4 (a combination of $Z(D_4)$ and two flips)

> D_6 contains:
>
> 2 centralizers of size 12 (all of D_6)
>
> 4 centralizers of size 6 (just rotations)
>
> 6 centralizers of size 4 (a combination of $Z(D_6)$ and two flips)

We previously determined that by the regular nature of the polygons whose flips and rotations comprise the dihedral groups, $Z(D_4) = Z(D_6) = Z(D_{2k}) = \{R_0, R_{180}\}$ for $k \in \mathbb{Z}^+$. Thus, for any even-sided polygon dihedral group D_{2k}, we will have two centralizers of size $4k$.

The remaining $2k - 2$ rotation elements in the group each commute only with all of the other rotation elements, since none of the flip elements commute with rotation elements other than R_0 and R_{180}. Thus, there will be $2k-2$ centralizers of size $2k$. Finally, of the remaining $2k$ elements, all of which correspond to flip symmetries, each commutes only with itself, the two elements of the center, and the directly perpendicular flip element.

In summary, we have that

> D_{2k} contains:
>
> 2 centralizers of size $4k$ (all of D_{2k})
>
> $2k - 2$ centralizers of size $2k$ (just rotations)
>
> $2k$ centralizers of size 4 (a combination of $Z(D_{2k})$ and two flips)

Similarly, we will now determine a list of the sizes of the centralizers of a group D_n where $n = 2k+1$ such that n is odd. First, listing the sizes of the centralizers of D_3 and D_5, we have that

D_3 contains:

1 centralizer of size 6 (all of D_3)

2 centralizers of size 3 (just rotations)

3 centralizers of size 2 ($Z(D_3)$ and one flip)

D_5 contains:

1 centralizer of size 10 (all of D_5)

4 centralizers of size 5 (just rotations)

5 centralizers of size 2 ($Z(D_5)$ and one flip)

As with the even-sided polygon dihedral groups, the odd-sided polygon dihedral groups are comprised of the flips and symmetries of regular polygons. Thus, all the odd-sided polygon groups follow a structured pattern in terms of the distribution of the orders of the centralizers of the elements. Specifically, since $Z(D_3) = Z(D_5) = Z(D_{2k+1}) = \{R_0\}$ for all $k \in \mathbb{Z}^+$, the group $Z(D_{2k+1})$ contains just one centralizer of size $4k + 2$.

Further, each rotation element outside of the center commutes only with every other rotation element, so there are $2k$ centralizers of size $2k+1$. The remaining $2k + 1$ elements are all flip elements, each of which commutes only with itself and the identity.

Thus, we have that

D_{2k+1} contains:

1 centralizer of size $4k + 2$ (all of D_{2k+1})

$2k$ centralizers of size $2k + 1$ (just rotations)

$2k + 1$ centralizers of size 2 ($Z(D_{2k+1})$ and one flip)

It follows that in order to determine the commutativity of any dihedral group, we may simply add up the appropriate numbers of elements (depending on the order of the group and whether it is even or odd) and divide this number by the the square of the total number of elements in the group.

Summarized below are the P_2-measures of commutativity for the first 9 odd and first 9 even dihedral groups,

G	$P_2(G)$ (n odd)	G	$P_2(G)$ (n $even$)
D_3	$18/6^2 = 1/2$	D_4	$40/8^2 = 5/8$
D_5	$40/10^2 = 2/5$	D_6	$72/12^2 = 1/2$
D_7	$70/14^2 = 5/14$	D_8	$112/16^2 = 7/16$
D_9	$108/18^2 = 1/3$	D_{10}	$160/20^2 = 2/5$
D_{11}	$154/22^2 = 7/22$	D_{12}	$216/24^2 = 3/8$
D_{13}	$208/26^2 = 4/13$	D_{14}	$280/28^2 = 5/14$
D_{15}	$270/30^2 = 3/10$	D_{16}	$352/32^2 = 11/32$
D_{17}	$340/34^2 = 5/17$	D_{18}	$432/36^2 = 1/3$
D_{19}	$418/38^2 = 11/30$	D_{20}	$520/40^2 = 13/40$

Table 4: The first 18 dihedral groups

A formula for $P_2(D_{2k})$ and for $P_2(D_{2k+1})$

Now that we know the sizes of all the centralizers in a given dihedral group, and we know that the size of the denominator, c, of the $P_2 = d/c$ measure of commutativity, is simply the order of the group squared, we can derive explicit formulas for the measure of commutativity. This will allow us to easily determine if a given prime can appear in the denominator of the reduced expression for the P_2-measure of commutativity.

First, consider the dihedral groups D_{2k}. We know the sizes of the centralizers of these groups, so summing these values gives us the numerator of the unreduced P_2 fraction.

The denominator is clearly $(4k)^2$, so we have

$$\begin{aligned}
P_2(D_{2k}) &= \frac{(2)(4k) + (2k-2)(2k) + (2k)(4)}{(4k)^2} \\
&= \frac{8k + 4k^2 - 4k + 8k}{16k^2} \\
&= \frac{4k^2 + 12k}{16k^2} \\
&= \frac{k+3}{4k}.
\end{aligned}$$

Similarly, if we consider the dihedral groups D_{2k+1}, since the denominator is clearly $(4k+2)^2$, we have that

$$\begin{aligned}
P_2(D_{2k+1}) &= \frac{(4k+2)+(2k)(2k+1)+(2k+1)(2)}{(4k+2)^2} \\
&= \frac{4k+2+4k^2+2k+4k+2}{4(2k+1)^2} \\
&= \frac{4k^2+10k+4}{4(2k+1)^2} \\
&= \frac{2(k+2)(2k+1)}{4(2k+1)^2} = \frac{k+2}{2(2k+1)} = \frac{k+2}{4k+2}.
\end{aligned}$$

From the above formulas we obtain the following theorem.

Theorem 3.1.
For each positive integer k, $P_2\left(D_{2k+1}\right) = P_2\left(D_{2(2k+1)}\right)$.

Proof. Let $k \in \mathbb{Z}^+$. We know that

$$P_2(D_{2(2k+1)}) = \frac{(2k+1)+3}{4(2k+1)},$$

and

$$P_2(D_{2k+1}) = \frac{k+2}{4k+2}.$$

Thus, we find that

$$\begin{aligned}
P_2(D_{2(2k+1)}) - P_2(D_{2k+1}) &= \frac{(2k+1)+3}{4(2k+1)} - \frac{k+2}{4k+2} \\
&= \frac{2k+4}{4(2k+1)} - \frac{2(k+2)}{2(4k+2)} \\
&= \frac{2k+4-2k-4}{4(4k+2)} = 0.
\end{aligned}$$

Hence, $P_2(D_{2k+1}) = P_2(D_{2(2k+1)})$.

$\square$

Thus it is convenient and appropriate to simply consider the even-polygon dihedral groups, D_{2k}, as their formula for commutativity is simpler than that of the odd-polygon groups.

It may be observed that the preceding theorem is also a consequence of the following.

Theorem 3.2.
For each positive integer k, $D_{2(2k+1)}$ is isomorphic to $\langle R_{180}\rangle \oplus D_{2k+1}$, where $\langle R_{180}\rangle = \{R_0, R_{180}\} = Z\left(D_{2(2k+1)}\right)$.

Proof. Let $k \in \mathbb{Z}^+$. We will first show that the mapping $\phi : \langle R_{180}\rangle \oplus D_{2k+1} \to D_{2(2k+1)}$ defined by $\phi((x,y)) = x \cdot y$ is a homomorphism.

Let (x_1, y_1) and (x_2, y_2) be elements of $\langle R_{180}\rangle \oplus D_{2k+1}$. Notice that x_1 and x_2 are $\langle R_{180}\rangle$, the center of $D_{2(2k+1)}$, so

$$\begin{aligned}
\phi\left((x_1, y_1) \cdot (x_2, y_2)\right) &= \phi\left((x_1x_2, y_1y_2)\right) \\
&= x_1x_2 \cdot y_1y_2 \\
&= x_1y_1 \cdot x_2y_2 \\
&= \phi\left((x_1, y_1)\right) \cdot \phi\left((x_2, y_2)\right).
\end{aligned}$$

Hence, ϕ is a homomorphism. By a result proven in [1], if we can show that the kernel of the homomorphism, Kerϕ (the subset of elements of the group that are mapped to the identity element of the codomain), is the identity element $\langle R_{180}\rangle \oplus D_{2k+1}$, then ϕ must be one-to-one.

Let $(x, y) \in \text{Ker}\phi$. Then $\phi\left((x,y)\right) = x \cdot y = R_0$. Since $x \in Z(D_{2(2k+1)})$,

$$xxy = x \;\Rightarrow\; y = x.$$

It follows that either $y = R_0$ or $y = R_{180}$. As the group D_{2k+1} is an odd-polygon dihedral group, however, it does not contain the rotation element R_{180}. Thus, $x = y = R_0$, so that Kerϕ is indeed comprised only of the identity element of $\langle R_{180}\rangle \oplus D_{2k+1}$.

Hence, ϕ is a one-to-one homomorphism, and since $D_{2(2k+1)}$ and $\langle R_{180}\rangle \oplus D_{2k+1}$ both have the same finite order for any positive integer k, it follows that ϕ is also onto. Hence, $D_{2(2k+1)}$ is isomorphic to $\langle R_{180}\rangle \oplus D_{2k+1}$.

□

Figure 2 shows how the related symmetries of D_{2k+1} and $D_{2(2k+1)}$ can be helpful in visualizing the result of the preceding theorem.

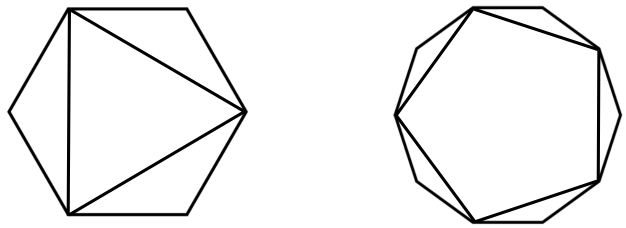

Figure 2: Related symmetry of even- and odd-sided polygons.

Valid values for the fraction P_2

Now that we can express $P_2(D_n)$, where $n = 2k$ or $n = 2k + 1$, in terms of k, we can directly test whether a given fraction can appear as the commutativity of a dihedral group. Before beginning to search for fractions that do not appear in Table 4, let us demonstrate the process with some known examples.

Suppose we wish to determine which dihedral groups have commutativity 1/3. First, testing the odd-polygon dihedral groups, we will solve the following equation for k,

$$\begin{aligned} \frac{1}{3} = \frac{k+2}{4k+2} &\Rightarrow 4k + 2 = 3k + 6, \\ &\Rightarrow k = 4. \end{aligned}$$

As we know from Table 4, the group $D_{2(4)+1} = D_9$ does indeed have commutativity 1/3. Now, testing the even-polygon dihedral groups, we solve the following equation for k,

$$\begin{aligned} \frac{1}{3} = \frac{k+3}{4k} &\Rightarrow 4k = 3k + 9, \\ &\Rightarrow k = 9. \end{aligned}$$

As before, the table confirms that $D_{2(9)} = D_{18}$ does indeed have commutativity 1/3.

Now, recall that among the first 18 dihedral groups we were unable to find an example of a group with a 7 in the denominator of the reduced expression for P_2. We will now use the known equations for the commutativity of dihedral groups to test for the existence of a group with $P_2 = 1/7$. First, examining the odd-polygon dihedral groups, we solve,

$$\begin{aligned}\frac{1}{7} = \frac{k+2}{4k+2} &\Rightarrow 4k+2 = 7k+14,\\ &\Rightarrow 3k = -12,\\ &\Rightarrow k = -4.\end{aligned}$$

Since obviously we only concern ourselves with positive integer values of k, this shows that there is no odd-polygon dihedral group with commutativity $1/7$. Similarly, solving for k in the case of the even-polygon dihedral groups, we obtain $k = -7$, so there can be no dihedral group with commutativity $1/7$.

We are interested in the denominator containing 7, however, with the only restriction on the numerator begin that it is relatively prime to 7 so that the fraction cannot be further reduced. Thus, solving the equation

$$\begin{aligned}\frac{2}{7} = \frac{k+2}{4k+2} &\Rightarrow 8k+4 = 7k+14,\\ &\Rightarrow k = 10,\end{aligned}$$

tells us that the group $D_{2(10)+1} = D_{21}$ has commutativity $2/7$. Similarly, in the case of even-polygon dihedral groups, we find that

$$\begin{aligned}\frac{2}{7} = \frac{k+3}{4k} &\Rightarrow 8k = 7k+21,\\ &\Rightarrow k = 21,\end{aligned}$$

which shows that $D_{2(21)} = D_{42}$ also has commutativity $2/7$.

We have now found denominators of size $2, 3, 5, 7, 13, 17$, so to complete the list of the first seven primes it would be nice to find at least one dihedral group with a P_2 measure containing 11 in the denominator.

As in the case of finding a 7 in the denominator of P_2, the equations show that it is not possible to find a dihedral group with commutativity $1/11$.

The same is true for 2/11, but for 3/11 we find that

$$\frac{3}{11} = \frac{k+2}{4k+2} \Rightarrow 12k + 6 = 11k + 22,$$
$$\Rightarrow k = 18,$$

and

$$\frac{3}{11} = \frac{k+3}{4k} \Rightarrow 12k = 11k + 33,$$
$$\Rightarrow k = 33,$$

which shows that $D_{2(18)+1} = D_{37}$ and $D_{2(33)} = D_{66}$ both have commutativity 3/11.

Having found examples of dihedral groups exhibiting the first seven primes as the denominator of their reduced fractional measure of commutativity it is tempting to suppose that, in fact, any prime can appear as the denominator of the P_2 measure of some dihedral group.

Every prime as the denominator of P_2

Note that every odd prime is either of the form $4n+1$ or $4n+3$ for some integer n (otherwise it would clearly be an even number). We now state, with proof, the fact that the previous pages of hypotheses were leading up to.

Theorem 3.2. Every prime as the denominator of P_2
Every prime can appear as the denominator of the reduced fractional expression for $P_2(G)$ where G is some dihedral group.

Proof. Note that D_6 exhibits commutativity 1/2, so the result is seen to be true in the case of the only even prime. Now, suppose that we wish to find a dihedral group that exhibits a prime of the form $4n + 3$ in the denominator of P_2. Recall that our simplified expression for the commutativity of even-polygon dihedral group, D_{2k}, is

$$\frac{k+3}{4k}.$$

Then, we set the above expression equal to $(n+1)/(4n+3)$, where, since $n+1 < 4n+3$ and $4n+3$ is prime, $\gcd(n+1, 4n+3) = 1$, so the expression for P_2 is indeed fully reduced and has the desired prime value as its denominator.

Thus, we have that

$$\frac{n+1}{4n+3} = \frac{k+3}{4k}.$$

Then, cross-multiplying and simplifying, we find that

$$\begin{aligned}
4k(n+1) &= k(4n+3) + 3(4n+3),\\
4k(n+1) - 4k(n) - 3k &= 12n+9,\\
4k - 3k &= 12n+9,\\
k &= 12n+9.
\end{aligned}$$

Hence, the dihedral group $D_{2(12n+9)} = D_{24n+18}$ exhibits commutativity $(n+1)/4n+3)$. This means that we can find examples of dihedral groups with a measure of commutativity that exhibits any prime of the form $4n+3$ as its denominator.

We now consider primes of the form $4n+1$. Given the same expression for the commutativity of an even-polygon dihedral group, D_{2k}, we set it equal to $(n+1)/(4n+1)$ (where, as before, $n+1$ and $4n+1$ are clearly relatively prime when $4n+1$ is prime, so the fraction is fully reduced and its denominator is as desired). Thus,

$$\frac{n+1}{4n+1} = \frac{k+3}{4k}.$$

As before, we simplify to obtain

$$\begin{aligned}
4k(n+1) &= k(4n+1) + 3(4n+1),\\
4k(n+1) - 4k(n) - k &= 12n+3,\\
4k - k &= 12n+3,\\
3k &= 12n+3,\\
k &= 4n+1.
\end{aligned}$$

Hence, the dihedral group $D_{2(4n+1)} = D_{8n+2}$ exhibits commutativity $(n+1)/(4n+1)$. Again, this means that we can find examples of dihedral groups with a measure of commutativity that exhibits any prime of the form $4n+1$ as its denominator. We conclude that every prime can appear as the denominator of $P_2(G)$ for some dihedral group G.

$\square$

Commutativity of the form $1/n$

Having shown that for every prime p, there exists a dihedral group such that the p is equal to the denominator of $P_2(G)$, it is easy to show that we can construct groups with commutativity $1/p$.

Theorem 3.3. Commutativity of the form $1/p$.
For every prime, p, there exists a group G such that $P_2(G) = 1/p$.

Proof. This will be a proof by induction. Since D_3 exhibits commutativity $1/2$ and D_9 exhibits commutativity $1/3$, the result holds for the base cases. Now, suppose that for every prime q, where $2 \leq q \leq p-1$, there exists a group, G, such that $P_2(G) = 1/q$.

Then, for the prime p, by the previous theorem we know there exists a group H such that $P_2(H) = (n+1)/p$, where $p = 4n+1$ or $p = 4n+3$. Since $n+1 < p$, by the Fundamental Theorem of Arithmetic $n+1$ can be uniquely expressed as the product of prime-powers where each prime in the decomposition must be less than p.

Hence, if we let $n+1 = q_1^{m_1} q_2^{m_2} \cdots q_k^{m_k}$, where clearly $q_i < p$ for each $1 \leq i \leq k$, then

$$\frac{1}{p} = \frac{n+1}{p} \cdot \frac{1}{q_1^{m_1}} \frac{1}{q_2^{m_2}} \cdots \frac{1}{q_k^{m_k}}.$$

By the induction hypothesis, there exist groups with commutativity $1/q_i$ for $1 \leq i \leq k$. Then, by the method of external direct products, we can create groups with commutativity $1/q_1^{m_1}, 1/q_2^{m_2}, \ldots, 1/q_k^{m_k}$ (by taking the external direct product of a group with commutativity $1/q_i$ with itself m_i times for each $1 \leq i \leq k$). Hence, by the Principle of Strong Mathematical Induction, for every prime, p, we can find or construct a group, G, such that $P_2(G) = 1/p$.

□

If we can construct $1/p$ commutative groups for any prime p, the Fundamental Theorem of Arithmetic (FTA) gives us the following corollary.

Corollary 3.4.
For every integer, n, there exists a group G such that $P_2(G) = 1/n$.

The validity of this corollary should be immediately obvious. By the FTA, each integer has a unique prime-power decomposition. Since we can find groups with commutativity $1/p$ for every prime, p, and since the external direct product of a $1/p$-commutative group with itself m-times results in a $1/p^m$-commutative group, we can construct a group with commutativity $1/n$ for each $n \in \mathbb{Z}^+$.

The following table provides examples of the corollary for the first several integers.

Group	Commutativity
A, where A is Abelian	$1 = 1/1$
D_3	$1/2$
D_9	$1/3$
$D_3 \oplus D_3$	$(1/2)^2 = 1/4$
$D_3 \oplus D_5$	$(1/2)(2/5) = 1/5$
$D_3 \oplus D_9$	$(1/2)(1/3) = 1/6$
$D_3 \oplus D_{21}$	$(1/2)(2/7) = 1/7$
$D_3 \oplus D_3 \oplus D_3$	$(1/2)^3 = 1/8$
$D_9 \oplus D_9$	$(1/3)^2 = 1/9$

Table 5: Groups with commutativity $1/n$

Every positive integer as the numerator of P_2

Although hardly as relevant as the fact that every integer can appear as the denominator of $P_2(G)$ for some group, it is possible to show that every integer can also appear as the numerator of $P_2(G)$. Of course, this does not mean that every numerator/denominator-pair is a possible commutativity fraction for non-Abelian groups, since the maximum commutativity value of $5/8$ ensures that non-Abelian commutativity cannot equal fractions greater than $5/8$.

Theorem 3.5. Every integer as the numerator of P_2.
For every positive integer, n, there exists a single dihedral group G such that n is the numerator of the reduced expression for $P_2(G)$.

Proof. We have seen examples of groups that exhibit 1 as the numerator of their reduced expression for commutativity. Let $m = n + 1$ for $n \geq 1$ be a positive integer. Then, set $(n+1)/(4n+3)$ equal to the measure of commutativity for the even-polygon dihedral group D_{2k}, and solve for k. That is,

$$\begin{aligned}
\frac{n+1}{4n+3} &= \frac{k+3}{4k}, \\
4k(n+1) &= 4kn + 12n + 3k + 9, \\
4k(n+1) - 4k(n) - 3k &= 12n + 9, \\
4k - 3k &= 12n + 9, \\
k &= 12n + 9.
\end{aligned}$$

Thus, the dihedral group $D_{2(12n+9)} = D_{24n+18}$ exhibits commutativity with the integer m in the numerator.

To ensure that the P_2 fraction is, in fact, in reduced form, we use the Euclidean Algorithm to verify that the greatest common divisor of $4n+3$ and $n+1$ is one.

$$\begin{aligned} 4n+3 &= 3(n+1)+n, \\ n+1 &= 1(n)+1. \end{aligned}$$

We see that the numerator and denominator of this fraction are indeed relatively prime. Hence, every positive integer can appear as the numerator of the reduced fractional expression for the commutativity of some dihedral group. This completes the proof.

□

4 Commutativity of factor groups:

The last property of the commutativity of non-Abelian groups that we will consider relates the commutativity of a group with the commutativity of any one of its factor groups.

Definition 4.1: ***Normal subgroup***.
A subgroup H of a group G is called a *normal* subgroup of G if $aH = Ha$ for all a in G. We denote this $N \lhd G$.

Definition 4.2: ***Factor group***.
Let G be a group and H a normal subgroup of G. The set $G/H = \{aH \mid a \in G\}$ is a group under the operation $(aH)(bH) = abH$. This group is often referred to as the factor group of G by H.

The reader may recognize the factor group of G by H as the set of cosets of H in G. Recall that cosets were previously defined in the context of Theorem 1.1.

The property of commutativity that we will prove, given these definitions, is that $P_2(G) \leq P_2(G/N)$, where G is a non-Abelian group and $N \lhd G$. This entails showing that

$$|N|^2 d_{G/N} \geq d_G,$$

where $d_{G/N}$ and d_G are the actual number of pairs of commuting elements in G/N and G respectively. We wish to show that there are at most $|N|^2$ pairs of commutative elements in G for each pair of commutative cosets in G/N.

Proving the Inequality

Let $a, b \in G$, where G is any group, and let N be a normal subgroup of G. Then $aN = \{a_i \ : \ a_i = an_i, \ n_i \in N, \ 1 \leq i \leq |N|\}$ and $bN = \{b_i \ : \ b_i = bn_i, \ n_i \in N, \ 1 \leq i \leq |N|\}$ are two elements of G/N. Since $aNbN = abN$ and $bNaN = baN$, aN and bN commute if a and b commute.

Now, since a_i, $1 \leq i \leq |N|$, is one of the $|N|$ elements of aN and b_j, $1 \leq j \leq |N|$, is one of the $|N|$ elements of bN, and both a_i and b_j are elements of G for $1 \leq i, j \leq |N|$, the total number of pairs of elements of G that may commute for each pair of commutative cosets is $|N|^2$.

That is, $|N|^2$ is the maximum possible number of commutative pairs of elements of G for each pair of commutative cosets in G/N. Thus, $|N|^2 d_{G/N} \geq d_G$.

Examples of the Inequality

We have shown that the commutativity of any factor group of a non-Abelian group can be no greater than the commutativity of the group itself. Before demonstrating that the equality of these two quantities is, in fact, possible, we here provide two general examples of the types of groups that are strictly less commutative than at least one of their factor groups.

First, let G be a non-Abelian group with maximum commutativity $5/8$. Let H denote the center of G. Since $P_2(G) = 5/8$, the order of the center must be $|G|/4$, and since the center of G is normal in the G, we conclude that G/H is a factor group of G of order $|G|/(|G|/4) = 4$. Since all groups of order 4 are Abelian, the group G/H is Abelian. Thus,

$$P_2(G) = 5/8 < 1 = P_2(G/H).$$

As a second example, let D be any dihedral group and let k denote the commutativity of D. Recall that since all dihedral groups are non-Abelian, $k < 1$. Now let R denote the subgroup of D consisting of all the rotation symmetries of D. Since R contains precisely half the elements of D, $|R| = |D|/2$. Thus, the subgroup R has index 2 in the group D, so by a result stated in [1] we conclude that R is normal in D. Since all groups of order 2 are Abelian, the factor group D/R is Abelian. Hence,

$$P_2(D) = k < 1 = P_2(D/R).$$

Example of the Equality

In order to prove that the equality $P_2(G) = P_2(G/N)$ is possible, for some non-Abelian group G and some $N \lhd G$, it is sufficient to provide a single example where this is the case. First, though, we describe the specific circumstances that are necessary in order for this equality to occur.

We previously saw that if two elements, a and b, of a group G commute, then the cosets aN and bN of G/N must also commute. If, in addition, for every pair of commutative cosets aN and bN of the factor group of G by N the elements $a, b \in G$ commute, then we will have established the equality $|N|^2 d_{G/N} = d_G$. That is, if

$$ab = ba \iff aNbN = bNaN,$$

then for each pair of commutative cosets in G/N we will have exactly $|N|^2$ pairs of commutative elements in G. This, then, leads to the desired equality $P_2(G) = P_2(G/N)$.

We will now demonstrate an example of this biconditional implication giving rise to a group and a factor group that share the same commutativity.

Let $A = D_3$, and let B be an Abelian group of order 3. Let $G = A \oplus B$, and let $N = \{(e_A, x) \mid x \in B\} = \{e_A\} \oplus B$. We know that $P_2(G) = 1/2 = P_2(A)$, and N is a normal subgroup of G that is isomorphic to A. Thus,

$$P_2(G) = P_2(A) = P_2(G/N).$$

Notice that this example was constructed so that little emphasis was placed on the actual values of P_2 for each of the groups in consideration. The only matter of importance was that the measure of commutativity for each was equal. Thus, simply dropping the condition that A be a particular dihedral group, we can construct the following generalization of the example.

Let A be a non-Abelian group and let B be an Abelian group. Then let $G = A \oplus B$ and let $N = \{e_A\} \oplus B$. Since $P_2(G) = P_2(A) \cdot P_2(B) = P_2(A)$, and $N \approx A$ is normal in G, $P_2(G) = P_2(G/N)$.

5 Conclusion

The results alluded to, explained, and proven in this article have covered a broad range of topics in the realm of the commutativity of groups. Beyond defining one method for classifying group commutativity, we proceeded to show that whereas the commutativity of an Abelian group is always 1, the commutativity of a non-Abelian group can assume a value ranging between arbitrarily close to 0 and a maximum of 5/8. Furthermore, infinitely many groups of order $8n$, $n \in \mathbb{Z}^+$, can be generated with commutativity 5/8, whereas no groups of order $4 \mod 8$ are 5/8 commutativity.

In large part, the latter half of the article details the possible reduced fractions that can appear as the P_2-measure of commutativity of non-Abelian groups. However, despite the result that for each integer n there exists a group G such that $P_2(G) = 1/n$, a larger question was left unanswered. That is, what specific range of values between 0 and 5/8 can appear as the reduced fractional expression for the commutativity of a non-Abelian group?

The final section of the article describes a relationship between the commutativity of a group and any one of its factor groups. The results of this particular investigation were was not explained in great detail, and the exploration of the commutativity of groups and their factor groups is recommended to the interested reader as a possible area of further study.

As a final note, we assure the reader that, although the vast majority of the groups used in the examples and theorems in this article are dihedral groups, the study of the commutativity of non-Abelian groups is in no way limited to symmetry groups of regular polygons.

References

[1] Gallian, Joseph A. *Contemporary Abstract Algebra: Fifth Edition.* Boston, MA: Houghton Mifflin Co., 2002.

[2] Pedersen, John. *Groups of Small Order.* University of South Florida. <http://shell.cas.usf.edu/~wclark/algctlg/small_groups.html>.

[3] Weisstein, Eric W. "Cyclic Group." From MathWorld – A Wolfram Web Resource. <http://mathworld.wolfram.com/CyclicGroup.html>.

[4] Weisstein, Eric W. "Dihedral Group." From MathWorld – A Wolfram Web Resource. <http://mathworld.wolfram.com/DihedralGroup.html>.

[5] Weisstein, Eric W. "Quaternion Group." From MathWorld – A Wolfram Web Resource. <http://mathworld.wolfram.com/QuaternionGroup.html>.