Classifying Some Infinite Abelian Groups
and Answering Kaplansky's Test Questions

by

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A senior project submitted in partial fulfillment of the requirements
for graduation with Honors in Mathematics.

Whitman College
2019

Certificate of Approval

This is to certify that the accompanying senior project by Alexander Francis Shaw has been accepted in partial fulfillment of the requirements for graduation with Honors in Mathematics.

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May 8th, 2019

Abstract

In his influential title *Infinite Abelian Groups*, Irving Kaplansky posed two general questions designed to test classifications of abelian groups. This work answers the questions for a subclass of abelian p-groups that are entirely characterized by their socles (the subgroups with 0 and all elements of order p). The socle is generalized as a valuated vector space and much of this work is dedicated to classifying this generalization in terms of Ulm invariants. For these groups, the questions can thus be translated in two steps: first into the terms of socles and then into the terms of Ulm invariants. The first step is made by Fuchs and Irwin in [1]. This work makes the second step, building up the results and classifications while assuming only working knowledge of introductory algebra. In culmination, this work answers Kaplansky's test questions and gives an example to which the results apply.

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1. Introduction

Irving Kaplansky states the goal of studying abelian groups as: "we seek to classify all abelian groups, or to give a complete set of invariants for abelian groups, or to give necessary and sufficient conditions for two abelian groups to be isomorphic" [2].

In the finite case, this is already done. We know that every finite abelian group may be decomposed uniquely as a sum of cyclic groups of prime-power order. However, this does not extend to a general abelian group, for which there are no such structure theorems.

If we restrict the class of abelian groups (which are called "groups" from now on) we are working to understand, there are prospects for structure theorems. These are steps toward the goal Kaplansky described, as long as the structure theorem is effective as a tool of understanding the group it classifies.

When is a theorem effective? Consider the finite case: the decomposition allows questions about the finite groups to be asked and answered with lucidity. A structure theorem does not have to be so powerful, though. It could be unwieldy and full of so many invariants and conditions that to apply it would bog down understanding.

Kaplansky proposes two questions that are designed to test a structure theorem's effectiveness, for groups G and H:

1. If G is isomorphic to a direct summand of H, and H is isomorphic to a direct summand of G, are G and H necessarily isomorphic?

2. If $G \oplus G$ and $H \oplus H$ are isomorphic, are G and H isomorphic?

This paper answers these questions for a specific class of abelian groups, which we will build an intuition for as we progress. For the reader familiar with this field, the groups are separable $p^{\omega+1}$-projective groups with complete cofree subsocles, i.e. G is such a separable $p^{\omega+1}$-projective group if there exists a complete $P \subseteq G[p]$ such that $G[p]/P$ is free. If the previous sentence was unintelligible, have no fear. This paper assumes only working knowledge at an introductory algebra level, and each new term in the sentence will find definition in the proceeding pages.

Before giving a full overview, to aid the reader new to the subject, we briefly discuss in the following section some fundamental notions of abelian group theory. We use notation from *Contemporary Abstract Algebra* by Joseph Gallian [3] and reference Kaplansky from [2]. All groups are be expressed with additive notation.

1.1. Torsion Groups and p-Groups

A *torsion* group is a group such that every element has finite order. The extreme opposite to torsion groups are *torsion-free* groups, groups in which every element but 0 has infinite order. The theory of abelian groups is vast and well developed, but generally follows three paths: The study of torsion groups, the study of torsion-free groups and the study of groups that are the combination of torsion and torsion-free groups.

This work focuses on torsion groups. A *p-group*, or *primary* group, is a group such that every element has order a power of an arbitrary, but fixed, prime p. Finite groups are torsion groups and as p-groups play a crucial role in their classification (the Fundamental Theorem of Finite Abelian Groups [3]), they are fundamental to the class of groups we seek to understand.

This is seen in the following theorem that is a generalization of the existence part of the finite classification theorem:

Theorem 1 ([2] Theorem 1)**.** *Any torsion group is a direct sum of primary groups.*

The distinction between the direct sum and direct product is discussed in Section 4, but at this point, it is enough to recognize that primary groups are fundamental in the study of torsion groups. Torsion marks our first general restriction in terms of the groups we are trying to classify, for this is implicit in any p-group and is assumed for all groups henceforth.

Let A be a p-group. There are some fundamental constructions that should be discussed. If $k < \omega$, or equivalently $k \in \mathbb{N}$ if $0 \in \mathbb{N}$, then consider the following:

$$p^k A := \{p^k a : a \in A\} \text{ and } A[p^k] := \{a \in A : p^k a = 0\}$$

Recall that $p^k a$ is just a added to itself p^k times, so $p^k a \neq 0$ iff the order of a is greater than p^k. Inversely, we have $A[p^k]$ containing all elements of A with order less than or equal to p^k.

For this section, let $|a|$ for $a \in A$ denote the *height* of a, which is the largest k such that p^k divides a in A. Define $A(k) := \{a \in A : |a| \geq k\}$, all elements of A with height greater than or equal to k. Note that $A(k) = p^k A$. The notation is necessary because if A is a subgroup of some p-group G, then $A(k)$ means the elements of A with height in G at least k. In this case, we would have that $A(k) = A \cap p^k G$.

The subgroup $A[p]$ is called the *socle* of A; it contains 0 and all elements of order p. This entity is crucial to our classification, as is the following sequence of subgroups:

$$A[p] \supseteq A[p](1) \supseteq A[p](2) \supseteq A[p](3) \supseteq \ldots$$

This sequence is discussed later on. Note that $A[p]$ is a vector space over $\mathbb{Z}_p$. To clarify, we can take $n \in \mathbb{Z}_p$ and note that if $a \in A[p]$, then we have $na \in A[p]$. Also, we know $A[p]$ is inherently an abelian group under addition. The perspective that $A[p]$ is a vector space is fundamental to this work. It allows us to take concepts from linear algebra and apply them to the socle of a p-group. With this in mind, we now give an overview:

1.2. Overview

A p-group A is called $p^{\omega+1}$-*projective* if it has a subgroup $P \subseteq A[p]$ such that A/P is a direct sum of cyclic groups.

At this point, the only necessary concept to understand about these groups is that they are entirely characterized by their socles. This result was shown by Fuchs and Irwin in [1] and is discussed in Section 8. Essentially, this result allows the translation of Kaplansky's test questions about these groups into questions about their socles.

However, the questions cannot be answered through this result alone. First, know that a group is *separable* if it has no elements of infinite height besides 0. This work gives a classification regarding a subclass of socles of separable $p^{\omega+1}$-projective groups, which translates the questions again into questions mostly about cardinal numbers. The classification is given in Section 7 and the questions are answered in Section 8.

The restriction of separability allows us to capitalize on the perspective that the socle is a vector space and develop significant results using methods related to those of topology.

We give our classification in terms of a generalization of the socle, the *valuated vector space*. Thus, Section 2 to Section 6 builds results around this generalization. After answering the questions, we give a relevant example of a group to which our results apply in Section 9, alongside motivating the further study of these groups. We conclude in Section 10.

We thus begin with the valuated vector space.

2. Value and Valuated Vector Spaces

Let F be a field and V be a vector space over F. Define ∞ by the property that, if α is an ordinal, then $\alpha < \infty$.

A function $|\cdot|$ from a vector space V to the finite ordinal numbers (and ∞) is a *valuation* if,

1. $|0| = \infty$.
2. If $a \in F$ with $a \neq 0$ and $v \in V$, then $|av| = |v|$.
3. If $v, w \in V$, then $|v + w| \geq \min(|v|, |w|)$.

A vector space together with a valuation is called a *valuated vector space*, and for $v \in V$, we call $|v|$ the *value* of v. The use of the same notation for height reflects that the height induces a valuation on the socle of a p-group, i.e. for a p-group A, the value of $a \in A[p]$ as treated in $A[p]$ is given by the height of a in A. One can intuitively think of a value as a magnitude, taking elements close to zero to small numbers and elements far from zero to larger numbers.

The socle is a valuated vector space over $\mathbb{Z}_p$, so we implicitly assume that $F = \mathbb{Z}_p$.

Lemma 2. *For every ordinal α, the following is a subspace of V:*

$$V(\alpha) := \{v \in V : |v| \geq \alpha\}$$

Proof. Certainly 0 is in $V(\alpha)$, for by definition it has value larger than α. So, let $a \in F$ with $a \neq 0$ and $v \in V(\alpha)$. By the second property of a valuation, $|av| = |v| \geq \alpha$, so $av \in V(\alpha)$.

Let $w \in V(\alpha)$. The third property implies $|v + w| \geq \min\{|v|, |w|\} \geq \alpha$. So, we have $v + w \in V(\alpha)$. □

This lemma confirms that the sequence of subgroups of Section 1.1 is valid. Next, a useful lemma regarding arithmetic and valuation.

Lemma 3. *If $v, w \in V$ and $|v| \neq |w|$, then $|v + w| = \min\{|v|, |w|\}$.*

Proof. Without loss of generality, let $|v| > |w|$. We know by the third property of a valuation that $|v + w| \geq |w|$. Also, see that:

$$|w| = |(v + w) + (-v)| \geq \min\{|v + w|, |-v|\} = \min\{|v + w|, |v|\} = |v + w|,$$

where the second property of a valuation was used on $|-v|$, and the last equality follows as $|v| > |w|$, so it could not be that $|w| \geq |v|$. Therefore, we have $|v+w| = |w|$.

□

Using this above technique with the second property of a valuation, we know that if $|v| \neq |w|$, then $|v-w| = |v+w| = |w-v|$. This is used later on.

Lemma 4. *If α is an ordinal and $|w| < \alpha$, then every element in the coset $w+V(\alpha)$ has the same value.*

Proof. Let $w+v \in w+V(\alpha)$. Since $|v| > |w|$, Lemma 3 implies $|w+v| = |w|$, a value not dependent on v. □

This lemma hints that one can define a natural value for the factor space $V/V(\alpha)$, and therefore that $V/V(\alpha)$ will be a valuated vector space. The following section proves this.

3. Metrics and Separable Valuated Vector Spaces

To the ends of establishing $V/V(\alpha)$ as a valuated vector space, we now start to incorporate a natural metric.

Recall that a *metric* on a set X is a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$,

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$.

Intuitively, a metric measures distance between elements in X, where the last condition represents the familiar triangle inequality.

From now on, it is implicit (except in formal statements) that our valuated vector spaces are *separable*, meaning that their only element of infinite value is 0. In other words, such a vector space V has $V(\omega) = 0$. This allows us to define a metric using the valuation.

We briefly mention that what follows explicitly applies only to separable p-groups, ones with no nonzero elements of infinite height. However, much of this work is very relevant in the non-separable case, as discussed in [1]. We return to this later.

Continuing, this metric is:

$$d(v,w) := \begin{cases} 0 & \text{if } v = w \\ 1/|v-w| & \text{if } v \neq w. \end{cases}$$

Over a separable V, the values d takes are always finite. Recalling the intuitive understanding of a valuation, this metric is quite natural.

Lemma 5. *The function d is a metric on V.*

Proof. Let $u, v, w \in V$. First, we check that $d(v,w) = d(w,v)$. If $v = w$, then they are both 0. When $v \neq w$, let $c = v - w \neq 0$. Since $-1 \neq 0 \in F$, we have that $|v-w| = |c| = |-c| = |w-v|$.

Second, the triangle inequality must be checked. See that,

$$\frac{1}{|u-w|} = \frac{1}{|(u-v)+(v-w)|} \leq \frac{1}{\min(|u-v|, |v-w|)} \leq \frac{1}{|u-v|} + \frac{1}{|v-w|}.$$

If some of u, v or w are equal and zeros are substituted into the equation above appropriately, it is seen that the last inequality always holds, implying the triangle inequality in all cases. □

Using this metric, we can talk about limits of sequences using the usual ϵ-δ definition. Often it is more convenient to rephrase the convergence criteria in terms of value.

A *closed subspace* is a subspace that is closed under limits of sequences. The culmination of this section proves that these subspaces produce valuated factor spaces. Notably, we have the following result.

Lemma 6. *The subspace $V(\alpha)$ is a closed subspace of V.*

Proof. Let $\{v_k\} \subseteq V(\alpha)$ and let $\lim_{k\to\infty} v_k = w$. We must show that $|w| \geq \alpha$. Since v_k converges to w, we can find an N such that

$$d(v_N, w) < \frac{1}{\alpha},$$

Or,

$$|w - v_N| > \alpha.$$

Next, we note that $|w| = |w - v_N + v_N| \geq \min(|w - v_N|, |v_N|) \geq \alpha$. So, we find that $w \in V(\alpha)$. $\square$

We need to establish a value using these closed subspaces. To that end, we use the following lemmas:

Lemma 7. *Let $S \subseteq V$ be a subspace of V. Then S is closed if and only if for all $w \in V$, $w + S$ has an element of maximum value.*

Proof. First, the forward direction. If $w \in S$, then we may choose $w - w = 0 \in S$, which by definition has value greater than any other element of $V + S$. If $w \notin S$, assume for contradiction that $w + S$ does not have an element of maximum value. This implies that for all $n < \omega$, there exists a $s_n \in S$ such that $|w - s_n| > n$. Inverting this inequality, we see that $d(w, s_n) < 1/n$. Taking the limit as $n \to \infty$ implies that $\{s_n\}$ converges to w, so by the closure of S, $w \in S$, which is a contradiction. Therefore, there always exists an element of maximum value in $w + S$.

Now, the reverse direction. Let $\{v_k\}$ be a sequence in S that converges to w. We want to show that $w \in S$ under the assumption that $w - m$ is the element of maximum value in $w + S$, for $-m \in S$. We know that $-\{v_k\} \subseteq S$, so that $|w - v_k| \leq |w - m|$. Inverting, we have $d(w, v_k) \geq d(w, m)$. Since we know that v_k converges to w, we can take the limit of this inequality as $k \to \infty$ and see that $d(w, m) = 0$, implying that $w = m$ and $w \in S$. $\square$

We say that the element $w \in V$ is *proper* with respect to S if it is an element of maximum value in V/S.

Lemma 8. *If $S \subseteq V$ is a closed subspace, then $w \in V$ is proper with respect to S if and only if for every $s \in S$, we have $|w + s| = \min(|w|, |s|)$.*

Proof. The forward direction has two cases. Let w be proper with respect to S. For the first case, if $|w| \neq |s|$, then Lemma 3 already implies the result. For the second case, if $|w| = |s|$, then we examine the known relation $|w + s| \geq \min(|w|, |s|) = |w|$. Since w is proper with respect to S, we know that $|w|$ is an upper bound on the value of every element in $w + S$, and $w + s \in w + S$. Therefore, $|w| \geq |w + s|$ and so $|w + s| = |w|$.

For the reverse direction, assume for contradiction that there exists an $s \in S$ such that $|w + s| > |w|$, that is, w is not proper with respect to S. Then, by assumption $|w + s| = \min(|w|, |s|) > |w|$. Whichever this minimum evaluates to, we have a contradiction, and therefore w is proper with respect to S. $\square$

We now implement these results to make a valuated vector space from a closed subspace. Let S be a closed subspace of V and let $w + S \in V/S$ where w is proper with respect to S. Define a value on V/S by

$$|w + S| := |w|.$$

Theorem 9. *With the above value, for a closed $S \subseteq V$, the set of cosets V/S is a separable valuated vector space.*

Proof. For $a \in F$ and $v, w \in V$, we know that V/S is a vector space under operations $(w + S) + (v + S) = (w + v) + S$ and $a(v + S) = av + S$, due to the vector space properties of V.

Since S is closed, each element of V/S has a well-defined value by Lemma 7. No nonzero elements are of infinite value, for if they were, then by definition of the value on V/S, the corresponding nonzero element in V would have infinite value. Therefore V/S is separable.

Finally, we must determine whether the above definition of $|w+S|$ is a valuation. Certainly, since S is the zero element in V/S, i.e. $|S| = |0 + S| = |0| = \infty$. Next, if $0 \neq \alpha \in F$, then

$$|\alpha(v + S)| = |\alpha v + S| = |\alpha v| = |v| = |v + S|.$$

And finally, let $v, w \in V$. In terms of $v + S$ and $w + S$, we may assume that each v and w is proper with respect to S. Therefore, we have

$$|(v + S) + (w + S)| = |(v + w) + S| \geq |v + w| \geq \min(|v|, |w|) = \min(|v + S|, |w + S|).$$

The first inequality follows from Lemma 8. All of the requirements are met, so V/S is a separable valuated vector space. □

These results serve as a basis as we build up language around valuated vector spaces. A main takeaway from this section is the following corollary:

Corollary 10. *For a separable valuated vector space V, we have that $V(\alpha)/V(\alpha+1)$ is a separable valuated vector space with valuation*

$$|v + V(\alpha + 1)| = |v| = \alpha.$$

Proof. We know $V(\alpha + 1)$ is a closed subspace of $V(\alpha)$, so this result is apparent when combining Lemma 6 and Theorem 9. $\square$

Such factor spaces will turn out to be summands of a separable valuated vector space V, and will lend themselves to a very useful decomposition theorem. Recall that a subgroup $G \subseteq H$ is a *summand* of H if there exists a B such that $H = G \oplus B = G \times B$. The distinction between the direct sum—"$\oplus$"—and the direct product—"$\times$"—only manifests itself in the infinite case, as discussed in the following section.

4. Direct Product and Direct Sum

We now build up the notion of direct products and direct sums from an index set I, finite or infinite. This is done so we may give results regarding the structure of p-groups and separable valuated vector spaces, and then use a powerful decomposition to understand this structure better. In Section 4.1 we discuss p-groups, and then transition in Section 4.2 to separable valuated vector spaces, proving this decomposition.

4.1. p-Groups

For each $i \in I$, let there be a corresponding p-group A_i. Then $\boldsymbol{A} := \prod_{i \in I} A_i$, which is the collection of *I-tuples*, is called the *direct product* of $\{A_i\}$. This structure is a group under component-wise addition.

Theorem 11. *The direct product $\boldsymbol{A} = \prod_{i \in I} A_i$ is a torsion group if and only if there is a finite subset $J \subseteq I$ and a $k < \omega$ such that $p^k A_i = 0$ for all $i \in I \setminus J$.*

Proof. In the forward direction, we shall prove the contrapostive. Assume that for all $k < \omega$, there exists an infinite $J_k \subseteq I$ such that

$$J_k = \{i \in I \mid p^k A_i \neq 0\}.$$

Let $j_k \in J_k$, and say that $x_k \in A_{j_k}$ with order greater than k. Do the same for x_{k+1} and note that we may always select $j_{k+1} \neq j_k$ since J_{k+1} is infinite. Define $x := (x_k)_{k<\omega}$ and see that $x \in \mathbf{A}$ with unbounded order, implying that $\mathbf{A}$ is not a torsion group. We therefore have the forward direction.

In the reverse direction, we let $(x_i)_{i \in I} \in \mathbf{A}$. Since each A_j is a p-group for $j \in J$, we may take the maximum order of $\{x_j\}$ since J is finite. Suppose the maximum order is p^h. Then p^k or p^h is an upper bound on the order of $(x_i)_{i\in I}$, and either way we have that $\mathbf{A}$ is a torsion group. $\square$

This theorem serves to give intuition as to the form of direct products that are considered.

Let $v := (a_i)_{i\in I} \in \mathbf{A}$. We denote the *support* of v by:

$$\operatorname{supp}(v) := \{i \in I \ : \ a_i \neq 0\}.$$

Certainly, we know that if $v, w \in \mathbf{A}$, then

$$\operatorname{supp}(v + w) \subseteq \operatorname{supp}(v) \cup \operatorname{supp}(w).$$

Therefore we can define $\bigoplus_{i\in I} A_i \subseteq \prod_{i\in I} A_i$ to be the *direct sum*, the elements of $\mathbf{A}$ with finite support.

Note that $\bigoplus_{i\in I} A_i$ is a torsion subgroup of $\mathbf{A}$, for each element in it has only finitely many nonzero coordinates. The maximum of the order of an element is defined on this finite set, which would in turn be the order of the element in the direct sum.

Familiarly, this is actually the *external* direct sum. If H is a group and each A_i is a subgroup of H, then there is a natural homomorphism from $\bigoplus_{i\in I} A_i \to H$,

$$\phi((a_i)_{i\in I}) = \sum_{i\in I} a_i.$$

This is defined as the *sum map*. See that $\sum_{i\in I} a_i = \sum_{i\in \operatorname{supp}(I)} a_i$, so that this sum is finite. If this map is bijective, then we say H is the *internal* direct sum of $\{A_i\}$. If ϕ is injective, then the A_i are called *linearly independent*.

Lemma 12. *The set of groups $\{A_i\}$ is linearly independent if and only if whenever $a_{i_1} \in A_{i_1}, \ldots, a_{i_m} \in A_{i_m}$ and $a_{i_1} + \ldots + a_{i_m} = 0$, then $a_{i_1} = \ldots = a_{i_m} = 0$.*

Proof. If $\{A_i\}$ is linearly independent, then we know that $\phi((a_i)_{i\in I}) = 0$ implies that $(a_i)_{i\in I} = 0$ in $\bigoplus_{i\in I} A_i$, i.e. it is the zero vector.

In the reverse direction, we know that the kernel of the sum map applied to $\bigoplus_{i\in I} A_i$ is zero. This implies that the sum map is injective, and the set of groups are linearly independent.

$\square$

We see in this lemma how related these constructs are to vector spaces.

4.2. Separable Valuated Vector Spaces

We now translate these concepts to use for valuated vector spaces, which will be very useful language to have. By defining

$$|(v_i)_{i\in I}| := \min(\{|v_i| \ : \ i \in I\})$$

We get that both $\prod_{i\in I} V_i$ and $\bigoplus_{i\in I} V_i$ are valuated vector spaces, when the V_i are such as well.

Let W_1 and W_2 be valuated vector spaces, then a linear transformation $\phi : W_1 \to W_2$ is *valuated* if $|\phi(x)| \geq |x|$ for all $x \in W_1$.

A linear transformation $\phi : W_1 \to W_2$ is said to be an *isometry* if it is bijective and $|\phi(x)| = |x|$ for all $x \in W_1$. From here on, isometry between valuated vector spaces is denoted by " $\doteq$ ", while isomorphism between groups is denoted by " $\cong$ ".

Note that if $\phi : W_1 \to W_2$ is bijective and both ϕ and ϕ^{-1} are valuated, then for any $x \in W_1$, we have $|\phi(x)| \geq |x|$ and $|x| = |\phi^{-1}(\phi(x))| \geq |\phi(x)|$, implying that $|\phi(x)| = |x|$ and so ϕ is an isometry.

Example 13. *For separable groups G and H, with $\phi : G \to H$ a group homomorphism, we have that ϕ restricted to the socle, $\phi' : G[p] \to H[p]$, is a valuated linear transformation.*

Proof. We know $\phi(G[p]) \subseteq H[p]$. If $x \in G[p]$ with $|x| = k$, then there exists a $g \in G$ such that $x = p^k g$. Thus $\phi(x) = \phi(p^k g) = p^k \phi(g)$, implying that $|\phi(x)| \geq |x|$. If ϕ is an isomorphism, we have valuated homomorphisms each way, and we get an isometry between the socles. $\square$

The sum map can be defined for a valuated vector space W, with subspaces V_i for each $i \in I$. This map is valuated in that it is certainly a linear transform and because if $x = (v_i)_{i\in I}$, then $|\phi(x)| = |\sum_{i\in I} v_i| \geq min(\{|v_i| : i \in I\}) = |(v_i)_{i\in I}| = |x|$. The inequality is due to the generalization of the basic properties of a value.

The valuated vector space W is the *valuated direct sum* of $\{V_i\}$ when the sum map is an isometry. Another way to phrase this is that $\bigoplus_{i\in I} V_i$ is a *valuated decomposition* of W.

A valuated vector space W is *free* if there is a basis, say $\{b_i\}_{i\in I}$, such that if $V_i := \langle b_i \rangle$, then W is the valuated direct sum of $\{V_i\}$. These valuated vector spaces are crucial to this work, so we emphasize their form:

$$W \doteq \bigoplus_{i \in I} V_i.$$

We say W is *homogeneous* if there is an $n < \omega$ such that $|w| = n$ for all nonzero $w \in W$.

Lemma 14. *A homogeneous valuated vector space is free.*

Proof. Let $\{b_i\}$, indexed by $i \in I$, be a basis of a homogeneous valuated vector space W such that every nonzero element has value n; in terms of finding a basis, the value of the elements are irrelevant, and we are simply using a basis of a general vector space which exists through the axiom of choice. Consider the sum map $\phi : \bigoplus_{i \in I} \langle b_i \rangle \to W$. By the definition of a basis, a linearly independent spanning set, this map is bijective. Since W is homogeneous, if $x \in \bigoplus_{i \in I} \langle b_i \rangle$, then $|\phi(x)| = n$. If $x = (x_i)_{i \in I}$, then $|x| = \min(\{|x_i| \ : \ i \in I\}) = n$. Therefore, the sum map is an isometry and W is free. $\square$

From Section 3, we know that the valuated vector space $W(\alpha+1)$ is closed in $W(\alpha)$, making $W(\alpha)/W(\alpha+1)$ a valuated vector space. Certainly $W(\alpha)/W(\alpha+1)$ is also homogeneous, so by Lemma 14, it is free. We can combine this fact with the following theorem for the decomposition of valuated vector spaces previously alluded to.

Theorem 15. *Let U be a closed subspace of the separable valuated vector space W with W/U free. Let $\{b_i + U\}$ be the corresponding basis of W/U, and choose b_i so that it is proper with respect to U. Then, by the sum map,*

$$W \doteq U \oplus \left(\bigoplus_{i \in I} \langle b_i \rangle \right).$$

Proof. The sum map ϕ is onto: If $w \in U$, then it is certainly mapped to by the sum map. If $w \notin U$, then consider $w + U$. Since $\{b_i + U\}$ spans W/U, we know there are n_i with $i \in I$ such that,

$$\begin{aligned} w + U &= \sum_{i \in I} (n_i b_i + U) \\ &= \sum_{i \in I} (n_i b_i) + U. \end{aligned}$$

This implies that there exist a u and a u' in U such that,

$$w + u = \sum_{i \in I} (n_i b_i) + u'.$$

Or,

$$w = (u' - u) + \sum_{i \in I} (n_i b_i).$$

Remember that these sums are finite as W/U is free, and from now on, it is implicit for sums that I represents the correct finite subset. Since $u' - u \in U$, we know that this is within the domain of the sum map, and therefore w is in its range.

Now, let us consider the injectivity of ϕ. Suppose that $a \in U \oplus \left(\bigoplus_{i \in I} \langle b_i \rangle\right)$, and that $\phi(a) = 0$. Let u be the element in U and $n_i b_i$ be the elements in $\langle b_i \rangle$. Consider that

$$u + \sum_{i \in I} n_i b_i = 0$$
$$u = - \sum_{i \in I} n_i b_i.$$

We see that $\sum_{i \in I} n_i b_i \in U$, and therefore that $\sum_{i \in I} n_i (b_i + U) = U$. Since $\{b_i + U\}$ is linearly independent, by Lemma 12 we have that $n_i = 0$. This implies that $u = 0$, and thus $a = 0$. By Lemma 12 again, we find that ϕ is injective.

Next, let $x = u \oplus (n_i b_i)_{i \in I} \in U \oplus \left(\bigoplus_{i \in I} \langle b_i \rangle\right)$. Let $w := \phi(x)$. We must check that ϕ preserves values. We know it is valuated, i.e. $|x| = \min(|u|, |n_i b_i|) \leq |w|$. Consider $|w| = |u + \sum_{i \in I} n_i b_i|$ in cases:

If $|u| < \min(|n_i b_i|)$, then we know that $|w| = |u| = |x|$ by Lemma 3.

If $|u| \geq \min(|n_i b_i|; i \in I)$. We start by recognizing that $|w| \leq |w + U|$ by the definition of $|w+U|$ and then apply what we know about $\{b_i + U\}$,

$$\begin{aligned}
|w| &\leq |w+U| \\
&= \left|\left(\sum_{i\in I} n_i b_i\right) + u + U\right| \\
&= \left|\left(\sum_{i\in I} n_i b_i\right) + U\right| \\
&= \left|\left(\sum_{i\in I} n_i b_i + U\right)\right| \\
&= \min(|n_i b_i + U| : i \in I) \\
&= \min(|n_i b_i| : i \in I) \\
&= |x|
\end{aligned}$$

Therefore we have both directions of the inequality and $|w| = |x|$ in general. We conclude that the sum map is an isometry and that we have a valuated decomposition.

□

We also have this very useful corollary:

Corollary 16. *For all separable valuated vector spaces W, we have*

$$W(n) = W(n+1) \oplus \left(\bigoplus_{i\in I_n} \langle b_i^n \rangle\right) = W(n+1) \oplus W(n)/W(n+1)$$

Proof. We can find $\{b_i^n\}$ with $i \in I_n$ that span $W(n)/W(n+1)$ since it is free. By Theorem 15, this is a valuated decomposition. □

Iterating this decomposition, we can expand an arbitrary W like so:

$$\begin{aligned}
W &\doteq W(1) \oplus \left(\bigoplus_{i \in I_0} \langle b_i^0 \rangle \right) \\
&\doteq W(2) \oplus \left(\bigoplus_{i \in I_0} \langle b_i^0 \rangle \oplus \bigoplus_{i \in I_1} \langle b_i^1 \rangle \right) \\
&\vdots \\
W &\doteq W(m) \oplus \left(\bigoplus_{i \in I_0} \langle b_i^0 \rangle \oplus \ldots \oplus \bigoplus_{i \in I_{m-1}} \langle b_i^{m-1} \rangle \right)
\end{aligned}$$

This decomposition is incredibly useful and is used throughout the remainder of this paper.

Continuing, a valuated vector space W is *bounded* if there exists an $m < \omega$ such that $W(m) = 0$. We have the following notable corollary:

Corollary 17. *A bounded valuated vector space is free.*

Proof. Let W be a valuated vector space. If $W(m) = 0$ for $m < \omega$, then the above decomposition ends at some finite point, leaving a valuated direct sum of a basis that implies W is free. That is, a bounded valuated vector space is free. □

We now turn to using this decomposition to classify valuated vector spaces.

5. Free Valuated Vector Spaces and Ulm Invariants

For a separable valuated vector space W, let us introduce some new definitions where $\{b_i^n\}$ spans $W(n)/W(n+1)$:

$$B^n := \bigoplus_{i \in I_n} \langle b_i^n \rangle \quad \text{and} \quad F := \bigoplus_{n < \omega} B^n.$$

The construction F is seen as a subspace of W through the sum map: it is clear by the decomposition of Section 4 that $\phi(F)$ is isometric to F. We know F is essentially a span of vectors and each element has finite support. Since each element has finite support, it is easy to see that the sum of two elements in F is also in F, the zero vector is in F and any multiple of an element in F is also in F. By the isometry, this is also true of $\phi(F)$.

Note that F is free. A subspace of the form $F \subseteq W$ is called a *basic* subspace of W. This leads us to a fundamental invariant of W. Define $f(W, n) := |I_n|$ as the *nth Ulm invariant* of W, which is the dimension of $W(n)/W(n+1)$.

The Ulm invariants of a valuated vector space provide a way to determine when certain vector spaces are isometric.

Theorem 18. *If V and W are free valuated vector spaces, then they are isometric if and only if $f(V, n) = f(W, n)$.*

Proof. In the forward direction, isometry implies that $V(n) \doteq W(n)$, and so we have $V(n)/V(n+1) \doteq W(n)/W(n+1)$. Each factor group is indexable by the same set I_n for each n, and therefore $f(V, n) = f(W, n)$.

Define $B_V^n := V(n)/V(n+1)$ and define B_W^n similarly for W. In the reverse direction, knowing that $f(W, n) = f(V, n)$ implies that the same I_n can be used to index each B_W^n and B_V^n. A natural isomorphism from B_W^n to B_V^n is that which swaps the basis vectors indexed by the same $i \in I_n$. Since B_W^n and B_V^n are homogeneous valuated vector spaces of the same value, value is certainly preserved and the isomorphism is also an isometry. Denote it by ϕ_n. Since V is free, each $x \in V$ has finite support; we have that $x = (x_n)_{n<\omega}$ for certain $x_n \in B_V^n$. It is clear that $\phi(x) := (\phi_n(x_n))_{n<\omega}$ is an isometry from V to W. $\square$

The importance of this section to our group classification is illustrated in the following corollary:

Corollary 19. *Let V be free, then $V \doteq \bigoplus_{n<\omega} V(n)/V(n+1)$.*

Proof. Immediate from Theorem 18. $\square$

Free valuated vector spaces are not the limit in terms of what Ulm invariants can classify, as we will show.

Before departing from free valuated vector spaces, we prove a result about them that will be useful later on.

Theorem 20. *Let V be a separable valuated vector space, then V is free if and only if it is the union of a countable chain of bounded subspaces.*

Proof. Let V be free. Define $B^k : V(k)/V(k+1)$. Note that B^k is a homogeneous subspace of V. Let $U_n := \bigoplus_{k<n} B^k$, which is isometric to a subspace of V and is bounded. Finally, we have that U_n is a subspace of U_{n+1}, and $V \doteq \bigcup_{n<\omega} U_n$.

Next, the reverse direction. Assume that $V \doteq \bigcup_{n<\omega} U_n$, with U_n bounded subspaces of V. Since U_n is bounded, we know it is free by Corollary 17. In addition to this, the subspace U_{n-1} is closed in U_n: we know that $U_n = U_{n-1} \oplus B^{n-1}$ and it is clear that summands of a free valuated vector space are closed. Also, the separable valuated vector space U_n/U_{n-1} is bounded, so it is free as well. By Theorem 15, we find that $U_n \doteq U_{n-1} \oplus U_n/U_{n-1}$, so we choose bases X_n of U_n such that $X_{n-1} \subseteq X_n$ and such that X_n forms the basis of a valuated decomposition of U_n. Define $X := \bigcup_{n<\omega} X_n$.

We show that $V \doteq \bigoplus_{b \in X} \langle b \rangle$: certainly the sum map is injective, as X was designed as independent. Furthermore, any element in V must be in some U_n, and is therefore within the span of those vectors in X under the sum map. The value of that element is preserved in the sum map because of the selection of the basis vectors in X_n. Therefore, we conclude that V is free. $\square$

Notably, we have the following corollary:

Corollary 21. *A subspace of a free valuated vector space is free.*

Proof. Let $S \subseteq V$ be a subspace of free valuated vector space V. By Theorem 20, we know that there exists $\{U_n\}$, a countable chain of bounded subspaces such that V is their union. Consider $\{S \cap U_n\}$, which is also a countable chain of bounded subspaces, and its union is S. We conclude that S is free. $\square$

This corollary finds use in Section 7. Now, we build from our results on free valuated vector spaces.

6. Complete Valuated Vector Spaces and Ulm Invariants

We start to expand on the results of Section 5 with a definition: a *dense* subspace $D \subseteq W$ is such if for every $w \in W$ and $\epsilon > 0$, there exists an $x \in D$ such that $d(w, x) < \epsilon$. Remember that since W is separable, so is D.

Example 22. *A basic subspace F is a dense subspace.*

Proof. Given any element $w \in W$ and any $k > 0$, we show that we may always find an $x \in F$ such that, equivalent to the restriction on distance, we have $|w + x| \geq k$. Since the vector spaces we are concerned with are separable, we can let $|w| = n$. By Theorem 15, we know that $W = W(k) \oplus W/W(k)$. So, we have that $w = w' - x$ for some $w' \in W(k)$ and $-x \in W/W(k)$. Note that x may be considered as in F since $W/W(k)$ is isometric to a subspace of F. We therefore have that $|w + x| = |w'| \geq k$. $\square$

Valuated vector spaces and their dense subspaces are related through their Ulm invariants, as the following lemma will help show.

Lemma 23. *Let D be a dense subspace of separable valuated vector space W. For all $n < \omega$, we have $D/D(n) \cong W/W(n)$.*

Proof. The natural map is $d+D(n) \to d+W(n)$. Since this map clearly preserves the group operation, we shall prove injectivity and surjectivity to show the isomorphism.

To the end of injectivity, suppose that $d + W(n) = d' + W(n)$. This implies that $d - d' \in W(n)$. Since $D(n) = D \cap W(n)$, and $d - d' \in D$, we must have that $d' + D(n) = d' + (d - d') + D(n) = d + D(n)$.

Now, to show that the map is surjective, first consider $w+W(n) \in W/W(n)$. We know that since D is dense in W, there exists a $d \in D$ such that $n < |d - w|$. This implies that $d-w \in W(n)$, leading us to see that $w+W(n) = w+(d-w)+W(n) = d + W(n)$. Therefore, $d + D(n)$ would map to $w + W(n)$.

□

Theorem 24. *Let D be a dense subspace of separable valuated vector space W, then $f(D, n) = f(W, n)$.*

Proof. Lemma 23 can be applied to show $D(n)/D(n + 1) \doteq W(n)/W(n + 1)$:

We require that $D(n)$ is dense in $W(n)$, then the same argument of Lemma 23 would work under a different valuation, that of V with n subtracted from the values of all nonzero elements. This would imply an isomorphism that is also an isometry, for all elements have the same value.

We now show $D(n)$ is dense in $W(n)$. If $w \in W(n)$ and $k > 0$, then we know there is a $d \in D$ such that $n, k \leq n + k < |w + d|$. If $|d| < n$, we would have $|w + d| = |d|$ and a contradiction. This implies that $d \in D(n)$ with $k < |w + d|$, and hence $D(n)$ is dense in $W(n)$. □

This result indicates that any separable valuated vector space has the same Ulm invariants as one of its basic subspaces. The equality of Ulm invariants between any valuated vector space would then naturally imply that their basic subspaces are isometric by the discussion of Section 5. This isometry is fundamental to our work. We will extend this isometry to classify other valuated vector spaces. Since dense subspaces are crucial in this effort, we dedicate the following section to developing basic results around them.

6.1. Basic Properties Regarding Dense Subspaces

This section lists basic results around dense subspaces that supplement the following section.

Theorem 25. *Let D be a dense subspace of W and V be another separable valuated vector space. If $g : W \to V$ is a valuated homomorphism with $D \subseteq \ker(g)$, then $W = \ker(g)$.*

Proof. Let $w \in W$, and $x_k \in D$ for all $k < \omega$ such that $k < |w - x_k|$. Then we know that

$$k < |w - x_k| \leq |g(w - x_k)| = |g(w) - g(x_k)| = |g(w)|.$$

This implies that $|g(w)| = \infty$ and that $g(w) = 0$. $\square$

Corollary 26. *Let D be a dense subspace of W and V be another separable valuated vector space. If $g_1, g_2 : W \to V$ are valuated homomorphisms with $g_1(x) = g_2(x)$ for all $x \in D$, then $g_1(w) = g_2(w)$ for all $w \in W$.*

Proof. Consider $g(w) := g_1(w) - g_2(w)$, which is valuated in that

$$|g(w)| = |g_1(w) - g_2(w)| \geq \min(|g_1(w)|, |g_2(w)|) \geq |w|.$$

We know that $D \subseteq \ker(g)$, so by Theorem 25, we have $W = \ker(g)$. This implies that $g_1(w) = g_2(w)$ for all $w \in W$.

$\square$

Lemma 27. *Let W and V be separable valuated vector spaces with $g : W \to V$ valuated. If $\{x_n\}_{n<\omega}$ is a sequence in W, then*

$$\lim_{n\to\infty} x_n = L \quad \textit{implies} \quad \lim_{n\to\infty} g(x_n) = g(L)$$

Proof. Let $k > 0$. We know there exists an $N < \omega$ such that $k < |x_n - L|$ for all $n \geq N$. Since g is valuated, we have that

$$k < |x_n - L| \leq |g(x_n - L)| = |g(x_n) - g(L)|.$$

This proves the implication, as the inequality holds for all $n \geq N$. $\square$

Lemma 28. *Let W and V be separable valuated vector spaces with $g : W \to V$ valuated. If $\{x_n\}_{n<\omega}$ is a Cauchy sequence in W, then $\{g(x_n)\}$ is a Cauchy sequence in V.*

Proof. Proving this is identical to proving Theorem 27, only that $L \to x_m$. $\square$

6.2. Extending the Isometry to Complete Valuated Vector Spaces

We seek to classify *complete* separable valuated vector spaces, one inside which every Cauchy sequence converges. As will be shown, such valuated vector spaces look like $\prod_{n<\omega} B^n$ for homogeneous B^n. Utilizing the accumulated results, this section extends the isometry of basic subspaces of complete valuated vector spaces with equal Ulm invariants to the complete valuated vector spaces themselves.

To this end, we start by considering a dense subspace D of a valuated vector space W and a complete valuated vector space C. Let there be a $g : D \to C$ that is valuated. We will extend g to all of W.

First note that we can define a function $h : W \to C$ such that,

$$h(w) = \lim_{n\to\infty} g(x_n),$$

where $\{x_n\}$ is the sequence in D with $\lim_{n\to\infty} x_n = w$. This function makes sense because we know that $\{x_n\}$ converges, so we know it is Cauchy. By Lemma 28, we know that $\{g(x_n)\}$ is a Cauchy sequence in C, which we know converges.

We must also check that two different sequences that converge to the same w are mapped to the same element of C, i.e. that the function is well defined. Let $\{x_n\}$ and $\{x'_n\}$ be such sequences. Then we know that

$$0 = h(0) = h(w - w) = \lim_{n\to\infty} g(x_n) - g(x'_n) = \lim_{n\to\infty} g(x_n) - \lim_{n\to\infty} g(x'_n).$$

The last equality follows in that we know the limits exist separately, and thus we see that the two are mapped to the same element in C.

We know that h is an extension of g. If $x \in D$ with $\{x_n\}$ converging to x, then

$$g(x) = \lim_{n\to\infty} g(x_n) = h(x)$$

by Lemma 27.

We can surmise that h is a vector space homomorphism: the familiar proof demonstrates that the limit of the sum is the sum of the limits, as long as the limits exist separately. Also, we know that $h(0) = g(0) = 0$.

In addition to this, we show that h is valuated as long as g is valuated, and preserves values if g preserves values. Let $w \in W$, and see that for valuated g,

$$\begin{aligned}|h(w)| &= |\lim_{n\to\infty} g(x_n)| \\ &= \lim_{n\to\infty} |g(x_n)| \\ &\geq \lim_{n\to\infty} |x_n| \\ &= |w|.\end{aligned}$$

Note that the value of the limit is the limit of the respective values. If g preserves value, then the above inequality becomes equality, and so h also preserves value. Summarizing to this point: if $g : D \to C$ is valuated, we have extended it to a valuated homomorphism h from W to C.

Consider if we had two valid h_1 and h_2 that satisfied the above. Since $h_1(x) = g(x) = h_2(x)$ for all $x \in D$, by Corollary 26, we then know that $h_1(w) = h_2(w)$ for all $w \in W$. Therefore we know that h is unique, the only valuated extension of g.

When is h an isometry? Certainly isometry requires that g preserves values. If g preserves values, then it is also injective: consider if $g(a) = g(b)$, then $\infty = |g(a) - g(b)| = |g(a - b)| = |a - b|$, which implies $a = b$. We have the same result for h.

To have surjectivity, we need to be able to form an inverse to h. We can add to our assumptions about the codomain what we are assuming about the domain, and vice versa. To this effect, if $g(D)$ is dense in C and W is also complete, we should be able to extend g^{-1} to an $h^{-1} : C \to W$, which would demonstrate isometry.

Explicitly, the extension is $h^{-1}(c) := \lim_{n\to\infty} g^{-1}(x'_n)$, where $\{x'_n\}$ is the sequence in $g(D)$ converging to $c \in C$. If $c \in C$, then $h^{-1}(c) \in W$, so there is a sequence in D that converges to $h^{-1}(c)$. Then, it must be that $h \circ h^{-1}(c) = c$, so that h is isometric with inverse h^{-1}.

Since we know that h is unique, in the case of isometry we can view it as simply the identity mapping between the two isometric spaces.

The main takeaway from this discussion is that we can extend the isometry between the dense subspaces of complete spaces to the complete spaces themselves.

6.3. Classifying Complete Valuated Vector Spaces by Ulm Invariants

We now apply the results of Section 6.2 to understand more about the structure of valuated vector spaces.

For each $n < \omega$, let B^n be defined as a homogeneous (value n) valuated vector space. Consider the subspace

$$F := \bigoplus_{n<\omega} B^n \quad \text{of} \quad \bar{B} := \prod_{n<\omega} B^n.$$

Note that each B^n is free since it is homogenous, so F is basic in $\bar{B}$. By previous discussion, we know that F is therefore dense in both W and in $\bar{B}$.

As mentioned prior, we have that $\bar{B}$ is complete: if $\{x_n\}$ is a Cauchy sequence in $\bar{B}$, then let $k > 0$ and we know there exists an N_k such that $k < |x_n - x_m|$ for all $n, m \geq N_k$. The value of each x_n is the minimum value of the elements in its tuple. Also, by construction of $\bar{B}$, we can assume that each tuple is ordered according to ascending value. Therefore, we know that $(x_n - x_m)_k = 0$ for all $n, m \geq N_k$. This implies that $(x_n)_k$ is fixed for all $n \geq N_k$.

We can thus define $x := \{(x_{N_k})_k\}_{k<\omega}$, which is an element of $\bar{B}$. Then, we must have that $k < |x - x_n|$ for $n \geq N_k$, and therefore $\{x_n\}$ converges to x.

Since $\bar{B}$ is complete, if $g : F \to \bar{B}$ is the natural map, then we extend the value-preserving g to a valuated map $W \to \bar{B}$. Call this extension h, which must be an isometry between W and $h(W)$ by the discussion of Section 6.2. Therefore, there is always a valuated embedding of an arbitrary W in the respective $\bar{B}$.

Intuitively, we can view any separable valuated vector space W as between a basic subspace and the corresponding valuated direct product of that basic subspace.

We also have the following crucial result for complete vector spaces:

Theorem 29. *Two complete, separable valuated vector spaces W and V are isometric if and only if they have the same Ulm invariants.*

Proof. Let F_V be a basic subspace of V. Since F_V is dense in V, we know it has the same Ulm invariants as V. If ϕ is the corresponding isometry, we know that $\phi(F_V)$ is both free and dense in W and therefore has the same Ulm invariants as W. Since F_V and $\phi(F_V)$ are both free and isometric, they also have the same Ulm invariants. Therefore, we have the forward direction.

If W and V have the same Ulm invariants, then we know that $F_V \doteq F_W$. Since each is dense in V and W respectively, and each V and W is complete, by the discussion of Section 6.2 we can extend the natural map $g : F_V \to W$ (with $g(F_V) \doteq F_W$) to an isometry between V and W. $\square$

We have shown so far that equality of Ulm invariants is equivalent to isometry for both free and complete vector spaces. Generalizing on our results a final time, we classify valuated vector spaces that are the direct sum of free and complete spaces in the next section.

7. Classification of Valuated Vector Spaces with Complete Cofree Subspaces

This section contains the culminating theorem of our work on valuated vector spaces. A valuated vector space V has a *cofree* subspace $P \subseteq V$ if V/P is free.

Note that a complete subspace of a valuated vector space is closed. Every convergent sequence is also a Cauchy sequence, which by the definition of completeness, must converge in the subspace.

Using Theorem 15, if V has a complete cofree subspace $P \subseteq V$, then $V \breve{=} P \oplus V/P$. One can view these valuated vector spaces as exactly those valuated vector spaces that are the direct sum of a free subspace and a complete subspace. Before classifying them, we need some basic lemmas regarding cofree subspaces.

Lemma 30. *Let V be a separable valuated vector space.*

1. *A subspace $P \subseteq V$ is cofree if and only if there is a free valuated vector space F and a valuated homomorphism $\phi : V \to F$ such that $P = \ker(\phi)$.*
2. *If both P and P' are cofree in V, then $\bar{P} := P \cap P'$ is also cofree.*

Proof.

1. The forward direction is seen through the map $V \to V/P$. In the reverse direction, using Theorem 20, let $\{U_n\}$ be the chain of bounded subspaces with its union being F. Say $\phi : V \to F$ is the assumed valuated homomorphism and suppose that $\bar{\phi} : V/P \to F$ is the isomorphism produced by modding out the kernel. Note that $\bar{\phi}$ is valuated: recall for $v + P \in V/P$, there exists a proper $s \in v + P$ with maximum value and $|v + P| := |s|$. Since $\phi(s) = \bar{\phi}(v + P)$, then

$$|v + P| = |s| \leq |\phi(s)| = |\bar{\phi}(v + P)|,$$

implying $\bar{\phi}$ is valuated. Consequentially, since U_n is bounded, then $\bar{\phi}^{-1}(U_n)$ is bounded. Also, since $\{U_n\}$ unions to F, then $\bigcup_{n<\omega} \bar{\phi}^{-1}(U_n) = V/P$. Seeing that $\{\bar{\phi}^{-1}(U_n)\}$ is a chain of bounded subspaces that unions to V/P allows us to conclude that V/P is free and P is cofree.

2. Using part 1 above, note that if ϕ and ϕ' are the respective homomorphisms,

then $\bar{\phi} : V \to F \oplus F'$ defined by $\bar{\phi}(x) := (\phi(x), \phi'(x))$ has kernel $\bar{P}$. This implies that $\bar{P}$ is cofree.

$\square$

Now we have all the results necessary for our classification.

Theorem 31. *Let V and W be separable valuated vector spaces with complete cofree subspaces P and P' respectively, then V is isometric to W if and only if $f(V, n) = f(W, n)$ for all $n < \omega$, and there exists an $N < \omega$ with $f(P, m) = f(P', m)$ and $f(V/P, m) = f(W/P', m)$ for all $m > N$.*

Proof. Let $F := V/P$ and $F' := W/P'$. In the forward direction, let V be isometric to W by $\phi : V \to W$. This isometry implies that $V(n)/V(n+1) \doteq W(n)/W(n+1)$, i.e. $f(V, n) = f(W, n)$. Consider that by ϕ, we have an embedding of P in W and P' in V. Therefore we let $\bar{P} := \phi^{-1}(P') \cap P$ and $\bar{P}' := \phi(P) \cap P'$. By Lemma 30.2, we know that $\bar{P}$ and $\bar{P}'$ are cofree, i.e. with $\bar{F} := V/\bar{P}$ and $\bar{F}' := W/\bar{P}'$, we have that

$$V \doteq \bar{P} \oplus \bar{F} \doteq \bar{P}' \oplus \bar{F}' \doteq W.$$

It is clear that $\bar{P} \doteq \bar{P}'$ and $\bar{F} \doteq \bar{F}'$. Note that $P/\bar{P} \subseteq \bar{F}$ and $P'/\bar{P}' \subseteq \bar{F}'$. As both are subspaces of a free vector space, they are also free by Corollary 21.

Since $\bar{P}$ is closed in P and $\bar{P}'$ is closed in P', we have by Theorem 15 that $P \doteq \bar{P} \oplus P/\bar{P}$ and similarly for P'. We show that for P and P' to be complete, it is required that $P/\bar{P}$ and $P'/\bar{P}'$ be bounded at some value N:

Since $P/\bar{P}$ is free, it looks like $P/\bar{P} \doteq \bigoplus_{n<\omega} B^n$ for homogeneous B^n of value n. Thus $P \doteq \bar{P} \oplus \bigoplus_{n<\omega} B^n$. For all n, let $b_n \in B^n$ with $b_n = 0$ exactly when $B^n = 0$. Define x_n such that that its kth element is given by

$$x_{n,k} = \begin{cases} b_k, & \text{if } k < n \\ 0, & \text{otherwise,} \end{cases}$$

an example being $x_3 = (b_0, b_1, b_2, 0, 0, \ldots)$.

The sequence $\{x_n\}_{n<\omega}$ is certainly Cauchy with each element in P, but it does not converge in P unless it is fixed after a finite $N < \omega$, corresponding to when $B^m = 0$ for all $m > N$. We have the same result for P'. Thus $P/\bar{P}$ and $P'/\bar{P}'$ are both bounded by some $N < \omega$, implying that P and P' have the same Ulm invariants for all $m > N$.

Furthermore, we note that $V \doteq \bar{P} \oplus (P/\bar{P}) \oplus F$, implying $\bar{F} = V/\bar{P} \doteq P/\bar{P} \oplus F$.

We can decompose $\bar{F}'$ similarly. Recalling our isometries, we have

$$\bar{F} \doteqdot P/\bar{P} \oplus F \doteqdot P'/\bar{P}' \oplus F' \doteqdot \bar{F}'.$$

Due to the bounded condition, this implies that $f(F, m) = f(F', m)$ for all $m > N$. We have proven the forward direction.

In the reverse direction, we seek an isometry between V and W. We know

$$V(N+1) \doteqdot P(N+1) \oplus F(N+1) \doteqdot P'(N+1) + F'(N+1) \doteqdot W(N+1),$$

where we show the second isometry through the assumption that $f(P, m) = f(P', m)$ and $f(F, m) = f(F', m)$ for all $m > N$:

Since $P(N+1)$ is closed in P it is also complete. In addition to this, we know $F(N+1)$ is a subspace of a free vector space, making it free. Our assumption implies that the Ulm invariants are equal between $P(N+1)$ and $P'(N+1)$ and between $F(N+1)$ and $F(N+1)$, implying their respective isometries through Theorems 29 and 18.

Recall the consequence of Corollary 16 that $B := V/V(N+1)$ is free with $f(F, n) = f(V, n)$ for all $n \leq N$, and that have $V \doteqdot V(N+1) \oplus B$. Define $B' := W/W(N+1)$. Note that $f(B, n) = f(B', n)$ for all $n < \omega$ by our assumption and the fact that $f(B, m) = f(B', m) = 0$ for $m > N$. We have that $B \doteqdot B'$. Combining this with the fact that $V(N+1) \doteqdot W(N+1)$, we see

$$V \doteqdot V(N+1) \oplus B \doteqdot W(N+1) \oplus B' \doteqdot W,$$

giving us the isometry of the forward direction.

$\square$

Using this theorem, we answer Kaplansky's test questions for a subclass of $p^{\omega+1}$-projective groups in the next section.

8. Classifying Some Infinite Abelian Groups and Answering Kaplansky's Test Questions

Recall that a group G is called $p^{\omega+1}$-projective if it has a subgroup $P \subseteq G[p]$ such that G/P is a direct sum of cyclic groups.

We assume the following result on these groups:

Theorem 32 (Irwin, Fuchs, [1] Theorem 3)**.** *If G and H are $p^{\omega+1}$-projective groups whose socles are isometric, then G and H are isomorphic.*

A subclass of these $p^{\omega+1}$-projective groups are separable and have a *complete cofree subsocle*, meaning that, for such a group G, there is a complete subsocle $P \subseteq G[p]$ such that $G[p]/P$ is free.

Combining Theorem 32 with Theorem 31, we will turn Kaplansky's test questions into questions about cardinal numbers. First, we need a lemma about cardinal numbers. For convenience, let $|\cdot|$ for a valuated vector space denote its dimension.

Lemma 33. *Let α, β and γ be cardinal numbers. Then,*

1. $\alpha = \alpha + \beta + \gamma$ *implies* $\alpha = \alpha + \beta$.
2. $\alpha + \alpha = \beta + \beta$ *implies* $\alpha = \beta$,

Proof. 1. In the finite case, both β and γ are 0, and the result is clear. If some cardinals are infinite, then we use cardinal arithmetic: $\kappa + \mu = \max(\kappa, \mu)$. Our assumption implies $\beta \leq \gamma \leq \alpha$, so $\alpha + \beta + \gamma = \alpha + \beta$.

2. This is immediate in the finite case. If some are infinite, then cardinal arithmetic shows $\alpha + \alpha = \alpha$ and similarly for β, and thus we have $\alpha = \beta$.

□

We are now in the position to answer Kaplansky's test questions, each in the affirmative.

Theorem 34. *Let G and H be separable $p^{\omega+1}$–projective groups with complete cofree subsocles. If G is isomorphic to a direct summand of H, and H is isomorphic to a direct summand of G, then G and H are isomorphic.*

Proof. By assumption, for some A and B, we have $G \cong H \oplus A$ and $H \cong G \oplus B$. This implies that $G[p] \doteqdot H[p] \oplus A[p]$, and it implies a similar isometry for $H[p]$. Let $G[p] \doteqdot P \oplus F$, where P is complete and cofree, with $F := G[p]/P$. Likewise, say that $H[p] \doteqdot P' \oplus F'$.

Let $P_A := P \cap A[p]$ and $P_H := P \cap H[p]$. Consider that $P \doteqdot P \cap H[p] \oplus P \cap A[p]$. We can see that

$$F \doteqdot (H[p] \oplus A[p])/(P_H \oplus P_A) \doteqdot H[p]/P_H \oplus A[p]/P_A.$$

Since $F_A := A[p]/P_A$ is a subspace of F, it is free. We have the same results regarding B, so define F_B accordingly.

We have the decompositions:

$$G[p] \doteq P \oplus F \doteq P' \oplus F' \oplus F_A$$
$$H[p] \doteq P' \oplus F' \doteq P \oplus F \oplus F_B$$

By applying Theorem 31 to one of these decompositions, we get that there exists an $N < \omega$ such that $f(P, m) = f(P', m)$ for all $m > N$. Using both of the decompositions, we know that there exists an N such that

$$f(F, m) = f(F', m) + f(F_A, m)$$
$$f(F', m) = f(F, m) + f(F_B, m),$$

for all $m > N$ (we are treating N as the generic maximum of all N). Rearranging, we see that

$$f(F, m) = f(F, m) + f(F_A, m) + f(F_B, m)$$
$$f(F', m) = f(F', m) + f(F_A, m) + f(F_B, m)$$

By Lemma 33.1, this implies that

$$f(F, m) = f(F, m) + f(F_B, m)$$
$$f(F', m) = f(F', m) + f(F_A, m)$$

We thus have that $f(F, m) = f(F', m)$ for all $m > N$. Finally, we also know by Theorem 31 that

$$f(G[p], n) = f(H[p], n) + f(A[p], n)$$
$$f(H[p], n) = f(G[p], n) + f(B[p], n).$$

Applying the same argument as for F and F' shows that $f(H[p], n) = f(G[p], n)$ for all $n < \omega$. We have the conditions that imply $H[p] \doteq G[p]$. Ultimately, by Theorem 32, we conclude $G \cong H$. □

Theorem 35. *Let G and H be separable $p^{\omega+1}$–projective groups with complete cofree subsocles. If $G \oplus G$ and $H \oplus H$ are isomorphic, then G and H are isomorphic.*

Proof. Immediately from the assumption, we get that

$$G[p] \oplus G[p] \doteq H[p] \oplus H[p],$$

and so,

$$f(G[p], n) + f(G[p], n) = f(H[p], n) + f(H[p], n).$$

Lemma 33.2 implies $f(G[p], n) = f(H[p], n)$ for all $n < \omega$.

Let $P \subseteq G[p]$ be complete and cofree, with $F := G[p]/P$. Define P' and F' similarly for H. Consider that $P \oplus P$ is complete and cofree in $G[p] \oplus G[p]$; it is cofree because $(G[p] \oplus G[p])/(P \oplus P) \doteq F \oplus F$ by the natural map. We have a similar result for $H \doteq P' \oplus F'$.

Therefore, we have that

$$P \oplus P \oplus F \oplus F \doteq P' \oplus P' \oplus F' \oplus F'.$$

By Theorem 31, for some N

$$f(P, m) + f(P, m) = f(P', m) + f(P', m)$$

for all $m > N$, implying that $f(P, m) = f(P', m)$ by Lemma 33.2 again. We also see that $f(F, m) = f(F', m)$, and by Theorem 31, we get $G[p] \doteq H[p]$. Applying Theorem 32, we get that $G \cong H$. □

Ultimately, our classification allowed answers to Kaplansky's questions to be derived quite naturally.

9. A Relevant Example

What is the scope of our results? Although we know the name of the groups we classified: separable $p^{\omega+1}$-projective groups with complete cofree subsocles, we would also like to know how this subclass is situated among p-groups. Unfortunately, fully mapping out their structure among general groups is beyond this analysis. We shall discuss a short result that shows exactly what groups have free socles, and then expand on this to give an example of a group with a socle that is the direct sum of free and complete parts.

We can combine Theorem 20 with the following:

Theorem 36 (Kaplansky [2] Theorem 12)**.** *Let G be a p-group, then G is a direct* *sum of cyclic groups if and only if $G[p]$ is the union of a chain of subgroups of bounded height.*

Connecting the equivalencies, we see that G is a direct sum of cyclic groups if and only if $G[p]$ is free. We therefore witness the free end of our classification's scope in exactly those groups that are the direct sum of groups of the form $\mathbb{Z}_{p^n}$. We also have the following relevant corollary:

Corollary 37. *If H is a direct sum of cyclic groups and G is a subgroup of H, then G is also a direct sum of cyclic groups.*

Proof. By Theorem 20, $H[p]$ is free. So too is $G[p] \subseteq H[p]$ by Corollary 21. Thus, we conclude that G is a direct sum of cyclic groups. $\square$

Now we can give an example. Assume that we have a complete separable valuated vector space C. We shall build a $p^{\omega+1}$-projective group G such that $G[p]$ is isometric to $C \oplus F$, where F is free.

Let B be a basic subspace of C. We can find an H that is a direct sum of cyclic groups such that $H[p]$ is isometric to B: we know that H has a free socle, so it is just a matter of getting the Ulm invariants to match. In other words, this involves enforcing that H has the same cardinality of summands of $\mathbb{Z}_{p^{n+1}}$ as the cardinality of $f(B,n)$ for all n. So, if H_n is a direct sum of copies of $\mathbb{Z}_{p^{n+1}}$ with $|H_n| = f(B,n)$, then we are letting $H := \bigoplus_{n<\omega} H_n$.

Define $\bar{H}$ as the torsion subgroup of $\prod_{n<\omega} H_n$. The torsion subgroup of a product group is called *torsion-complete*. Note that $\bar{H}[p]$ is complete and has the same Ulm invariants as C, so we have an isometry $C \doteq \bar{H}[p]$.

We now consider $X := \bigoplus_{x\in\bar{H}} \langle x \rangle$. Note that X is a direct sum of cyclics, while $\bar{H}$ is not in general. This group is certainly much larger than $\bar{H}$, with the sum map $\phi : X \to \bar{H}$ onto.

Let K be the kernel of ϕ. We will show that $G := X/K[p]$ has the desired property. We can find a subspace of $G[p]$ that is intuitively a complete valuated vector space isometric to C, namely $C' := X[p]/K[p] \subseteq G[p]$. This intuition will be proven correct:

The subspace C' inherits a valuation from the height function of G. Also, we have that ϕ restricted to $X[p]$ is a surjection to $\bar{H}[p]$. We know that this surjection induces an isomorphism $C' = X[p]/K[p] \cong \bar{H}[p] \doteq C$. Denote the isomorphism by ψ. To show ψ is an isometry, we must make sure that it preserves value with respect to the heights of elements as in G and $\bar{H}$.

To this end, if $y \in \bar{H}[p]$ with $|y|_{\bar{H}} = k$, we can find an $x \in \bar{H}[p]$ such that $p^k x = y$. This x is also in X as the generator of $\langle x \rangle$, and since $y \in \bar{H}[p]$, it must be that $p^k x \in X[p]$. We can thus let $z := p^k x + K[p] \in C'$. We know

$$\psi(z) = \phi(p^k x) = p^k x = y.$$

Since ϕ is a surjection, we know it restricts to a valuated transform between socles by Example 13. Since ψ is simply a restriction of ϕ, it is also valuated. Noting that $k \leq |z|_G$, we combine all this to see that

$$k \leq |z|_G \leq |\psi(z)|_{\bar{H}} = |y|_{\bar{H}} = k.$$

Since C' has valuation inherited from G, we have shown that $|z|_{C'} = |z|_C = k$, and may conclude that ψ is an isometry.

Through all this, we can now surmise that $C' \subseteq G[p]$ is complete and isometric to C.

Looking at G/C', we see that

$$G/C' = (X/K[p])/(X[p]/K[p]) \cong X/X[p] \cong pX.$$

Since $pX \subseteq X$ and X is a direct sum of cyclic groups, so too is G/C' by Corollary 37. Therefore, we surmise that G is $p^{\omega+1}$-projective. Furthermore, we find that C' is cofree in $G[p]$, and therefore $F := G[p]/C'$ is free. Ultimately, we have that $G \doteq C' \oplus F \doteq C \oplus F$.

To summarize, from a separable complete valuated vector space C we get a $p^{\omega+1}$-projective G with a complete cofree subsocle: we first built a group X that is a direct sum of cyclics and is an expansion of the corresponding torsion-complete realization of C: $\bar{H}$ (realization in this sense means that its socle is C). Considering the sum map from X to $\bar{H}$, we took G as X mod the socle of the sum map's kernel $K[p]$. We had that $X[p]/K[p] \subseteq G[p]$ is just C in disguise. Finally, we used C (rather, an embedding of C in G) to show that G was $p^{\omega+1}$-projective, and that thus C was cofree in $G[p]$.

One may consider the relationship between G and $\bar{H} \oplus W$, where W is a realization of F. It may seem that such a group would be isomorphic to G, for its socle is also $C \oplus F$, but it can be shown that $\bar{H} \oplus W$ is not even $p^{\omega+1}$-projective, and thus no isomorphism exists.

This concludes our brief foray into the general theory of the groups to which our results apply. Ideally, this example would give motivation to pursue in-depth the study of $p^{\omega+1}$-projectives. Recall the interpretation of arbitrary separable valuated vector spaces as being "between" their basic subspaces and complete realizations of their basic subspaces. It can be shown that such intuition is extended; the separable $p^{\omega+1}$-projective groups can be embedded in the corresponding torsion-complete group and hence viewed as between their basic subgroup and the torsion-complete group. This satisfactorily reinforces and demonstrates the notion that these groups are entirely characterized by their socles.

10. Conclusion

As discussed, the study of abelian groups generally follows three paths: classifying torsion groups, classifying torsion-free groups, and classifying groups that come from combining torsion and torsion-free groups. Kaplansky posed two questions that were designed to test the effectiveness of classification theorems that took a step along one of these paths. To restate the questions, they are:

1. If G is isomorphic to a direct summand of H, and H is isomorphic to a direct summand of G, are G and H necessarily isomorphic?

2. If $G \oplus G$ and $H \oplus H$ are isomorphic, are G and H isomorphic?

These simple questions can generate incredibly complex answers depending on the class of groups you ask them about. We asked them for $p^{\omega+1}$-projective groups with complete cofree subsocles, a subclass of torsion groups. Theorem 32 of Irwin and Fuchs shows that these groups are characterized by their socles. To answer the questions, we used this theorem to rephrase them in terms of socles. This rephrasing is not enough to allow answers, so we sought to classify the socles so the questions may be rephrased into a more workable form.

This required building up results around the separable valuated vector space, a generalization of the socle. Sections 2 through 4 built up basic results. We then turned to the problem of classifying separable valuated vector spaces, and used Ulm invariants to classify free separable valuated vector spaces in Section 5. We extended this classification to complete separable valuated vector spaces in Section 6.

The final extension to classifying the socles of these groups was made in Section 7 Theorem 31. This culminating theorem allowed us to translate Kaplansky's test questions again, this time into questions about cardinal numbers (the groups' Ulm invariants). Section 8 answered the translated questions both in the affirmative for the groups with which we are concerned.

Unfortunately, the general theory regarding these groups is beyond the scope of this analysis. Section 9 gave an example and motivated the significance of the groups to which our results apply.

The motivated reader may be left wondering where to turn next. The general theory of these groups is discussed in [1]. This source gives results beyond that $p^{\omega+1}$-projective groups are characterized by their socles. It also motivates the relation of the separable and non-separable cases, gives existence results as to when a group exists that is a realization of a given socle and expands on the example given in this paper, alongside other relevant discussion.

Ultimately, this work gave a hint at the vastness of abelian group theory while making a small but firm step into its depths. We answered Kaplansky's test questions for one subclass of groups, but there are still multitudes of others about which these questions may be asked.

11. Acknowledgments

I would like to thank Professor Patrick Keef of Whitman College's mathematics department. His guidance and expertise made this project possible and also incredibly interesting. The project was absorbing throughout the semester and I always looked forward to working on it and with him.

I would also like to thank Carl Felstiner and Taka Olds for their reviewing of the work and assistance in editing.

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