FORBIDDEN GRAPH MINORS

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Abstract. By identifying forbidden graph minors, families of graphs with a certain property can be characterized in an additional manner. We will review Kuratowski's identification and proof of the forbidden topological minors of planar graphs. In addition, Robertson and Seymour's Graph Minor Theorem will be examined in relation to sets of forbidden minors. By applying these concepts to two well-studied graph properties, we will gain some insight into the significance of these famous results.

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1. Introduction

Discussing graphs can be an arduous task due to the diverse forms they can take. It is helpful to characterize graphs into families, or sets of graphs with a shared property in order to narrow the scope of discussion and allow for substantive results to be uncovered. The method of characterizing graphs discussed in this paper will be the identification of the *forbidden minors* of a desired property. Graph minors are subgraphs with an additional allowed operation: edge contraction. We will formalize this definition in Section 2. Characterizing a graph property through its forbidden minors is extremely useful; instead of thinking about a set of graphs that has possibly infinite size, we can point to a set of graphs whose presence as minors guarantee exclusion from the set of graphs with the property. Now, having a set of forbidden minors with the same size as the original family is near useless. Thus this method is only noteworthy if the set of forbidden minors is smaller than the set of graphs with the desired property.

Planarity is an example of a graph property with interesting results and has been a subject of great interest since the inception of the study of graphs. Planar graphs are those that can be drawn in a plane with no crossing edges. The origin of the study of planar graphs stems from multiple famous mathematical puzzles whose solutions are made simple by utilizing graph theoretical concepts. One is the "Utilities Problem," first published in a book by Henry Ernest Dudeney in 1917. The problem was described as follows: three houses need to be connected to water, gas, and electricity sources using pipes that do not overlap. When thought of as a graph with each utility and each house represented by nodes, the solution is simple. As shown in Figure 1, the corresponding graph is $K_{3,3}$, which is nonplanar and thus cannot be drawn with no edges crossing. The problem is so famous that $K_{3,3}$ is often nicknamed the "utility graph".

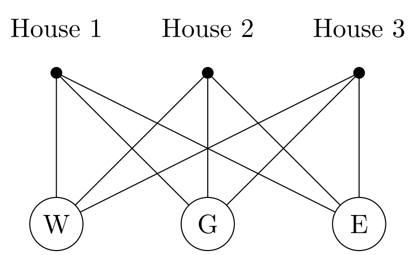

Figure 1. The Utility Problem

Another very famous problem involving planar graphs is the Four Color Theorem. The problem is often described in terms of coloring a map: what is the maximum number of colors needed to color all countries, with no two adjacent countries with the same color? In this case, each country corresponds to a node in a graph, and two nodes are connected by an edge if the two countries share a border. With respect to planarity, the solution translates to the maximum colors needed to color a planar graph. It was proved in the late 19$^{\text{th}}$ century that any planar graph could be colored with at most five colors. After many failed attempts and false proofs, the Four Color Theorem was proved in the late 20$^{\text{th}}$ century with the help of computers.

There was a breakthrough in characterizing all planar graphs in the late 1920s. Kazimierz Kuratowski found that the planar graphs are only those that do not contain the complete graphs K_5 and $K_{3,3}$, shown in Figure 2. We will examine the details of this result in Section 3.

FIGURE 2. Complete graphs K_5 (left) and $K_{3,3}$ (right).

Though very important, planarity is just one of many interesting graph properties. Does this method of identifying forbidden minors translate to other graph properties? Neil Robertson and Paul Seymour proved that it does. With a proof that took two decades and twenty papers, Robertson and Seymour proved that in any infinite sequence of graphs there must be one graph that is a proper minor of another. Named the "Graph Minor Theorem," the proof is beyond the scope of this paper; however, it has many significant consequences. Detailed in Section 4, the Graph Minor Theorem implies that any graph property that is minor closed has a finite set of forbidden minors. We knew this was true for planarity by Kuratowski's Theorem; that this is true for any minor closed property is remarkable, and we will explore some additional graph properties as examples of this result.

We will begin with a walk-through of the proof of Kuratowski's Theorem outlined by Douglas West in [4]. We will then generalize our focus to Robertson and Seymour's Graph Minor Theorem and the important results that stem from it. Finally we will explore examples of the Graph Minor Theorem through toroidal and outerplanar graphs, using Kuratowski's characterization as a basis to identify graphs in the set of forbidden minors.

2. Preliminaries

Notation and definitions will mainly be from [4] and any additional concepts will be introduced as necessary. The graphs in this paper will all be finite, simple, and undirected. The set of vertices in a graph G will be represented by $V(G)$ and the set of edges will be $E(G)$. As in [4], we will use the notation $n(G)$ to represent the size of $V(G)$ and $e(G)$ to represent the size of $E(G)$. A vertex is *adjacent* to another vertex if they are the endpoints of a shared edge. An edge is *incident* to a vertex, and vice versa, if the vertex is an endpoint of that edge.

2.1. Additional Definitions

The following are additional definitions needed to begin discussing the topic of graphs and minors.

Definition 2.1. *A graph H is a **subgraph** of G if $E(H) \subseteq E(G)$ and $V(H) \subseteq V(G)$. All endpoints of each $e \in E(H)$ must be in $V(H)$. A subgraph H is **induced** if $E(H)$ consists of all edges in $E(G)$ with endpoints in $V(H)$.*

Definition 2.2. *A graph has **connectivity k** if there is a set of k vertices that disconnects the graph and no smaller set. A graph is $k-$connected if the connectivity is at least k.*

If a graph has connectivity one, any vertex whose removal disconnects the graph is called a **cutpoint**. We say a graph has connectivity zero if it is not connected.

Definition 2.3. *A graph H is a **subdivision** of G if it is created by adding new vertices on the edges of G.*

Figure 3. K_3 (left) and a subdivision of K_3 (right).

Definition 2.4. *A graph H is a **topological minor** of G if G contains a subdivision of H as a subgraph.*

In other words, a topological minor is formed by suppressing vertices of degree two and removing vertices and edges.

Definition 2.5. *A subgraph H of G is a **minor** if it can be obtained by contracting edges and removing vertices and edges.*

Definition 2.6. *A **Kuratowski subgraph** is a subgraph that is a subdivision of K_5 or $K_{3,3}$.*

Definition 2.7. *A graph property is* ***minor closed*** *if the property is preserved through vertex deletion, edge deletion, and edge contraction. That is, if a graph has a minor closed property $\mathcal{P}$, then its minors will also have $\mathcal{P}$. For example, planarity is minor closed: any planar graph will remain planar after minor operations.*

Definition 2.8. *A* ***plane graph*** *is a drawing of a graph in a plane in which no edges cross. A graph is* ***planar*** *if there exists such a drawing.*

We will use $\mathcal{PL}$ to denote the planarity property. Additionally, in this paper, an **embedding** of a graph will refer to a drawing where no edges are crossing.

2.2. Euler's Formula

The fact that K_5 and $K_{3,3}$ are non planar is a well-known result that follows from a theorem first proved by Leonhard Euler. Known as "Euler's Formula", this result shows that any plane graph with n vertices, e edges, and f faces satisfies the following equation: $n - e + f = 2$. As a result, any planar graph G with at least three vertices must satisfy one of the following inequalities: (1) if G contains K_3 as a subgraph and (2) if not. The proofs of these results can be found in [4].

$$e(G) \leq 3 \cdot n(G) - 6 \tag{1}$$

$$e(G) \leq 2 \cdot n(G) - 4 \tag{2}$$

Examining K_5 in the context of these results, $e(K_5) = 10$ and $n(K_5) = 5$, which does not satisfy (1). Since $K_{3,3}$ does not contain K_3 as a subgraph, it must satisfy (2), but with $e(K_{3,3}) = 9$ and $n(K_{3,3}) = 6$, it does not. Therefore K_5 and $K_{3,3}$ must be nonplanar.

3. Kuratowski's Theorem

Now that the preliminary graph theory concepts have been established, we will proceed by proving Kuratowski's Theorem. The outline of the proof will follow that of the proof found in [4]. By introducing the following lemmas, we will set up a contradiction that will lead to the desired result.

Lemma 3.1. *Let G be a plane graph. If F is a set of edges bounding any face, then G has an embedding in which F bounds the infinite face.*

Proof. If F is the set of edges bounding the infinite face, we have the desired embedding of G. Suppose F bounds a finite face of G. We will proceed by embedding G onto the surface of a sphere.

Consider a sphere resting on the xy plane. For each point (x, y) on the plane, let L_{xy} be the line that goes through (x, y) and the top-most point of the sphere.

The point (x, y) will be mapped to the intersection of L_{xy} and the surface of the sphere. We can find the preimage of the points on the surface of the sphere by extending a line from the top of the sphere through the desired point, and finding the intersection with the xy plane. Note that the point at the top of the sphere with a horizontal tangent plane, named the *point at infinity*, is not mapped to by this procedure. In this case, the point at infinity will take the mapping of any non-edge point.

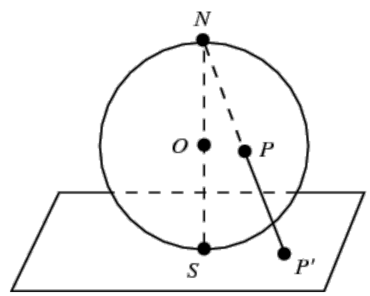

FIGURE 4. Projection from plane onto sphere.

Using this method, we will embed G onto a sphere. This embedding preserves planarity, since if two edges cross on the plane at a point (a, b), they will cross on the sphere at the point mapped to by (a, b). Let F' be the region on the sphere bounded by the projections of the elements of F. If we take a non-edge point x in the face bounded by F' and rotate the sphere such that x becomes the point at infinity, we can embed the graph back into the xy plane, and the point at infinity now corresponds to the infinite face of the graph. We now have an embedding of G such that F is the set of edges bounding the infinite face. $\square$

Definition 3.2. *A graph is* ***minimal nonplanar*** *if it is nonplanar and all proper subgraphs are planar.*

Definition 3.3. *An* ***S-lobe*** *is an induced subgraph of G whose vertex set consists of S and the vertices of some component of $G - S$.*

For a separating set S, the S-lobes would consist of each of the disconnected components of the graph along with the vertices and edges of S. Note that for each separating set S of a graph G, the union of the S-lobes is the original G.

Lemma 3.4. *Every minimal nonplanar graph is 2-connected.*

Proof. Let G be a minimal nonplanar graph. If G were unconnected, there would be a component of G that would be nonplanar, and thus G would not be minimal. Therefore G must be connected.

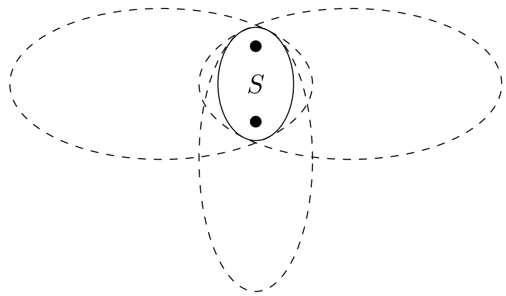

FIGURE 5. A separating set S and its corresponding S-lobes.

By way of contradiction suppose G has connectivity one, i.e. suppose G has a cutpoint. Let v be a vertex whose removal disconnects G and consider the $v-$lobes in G corresponding to the disconnected components. Let L be such a $v-$lobe and H be $G - L$. By definition, L and H are both planar. We will show that when adding L back to H to get G the resulting graph will be planar. It may be clear to see that combining two planar graphs at a vertex will result in a planar graph, but we will cover the two cases that complicate the process.

The first case is when v is located in the interior of the embedding of L. In this case, adding the two components may result in some edges crossing. However, using Lemma 3.1, we can find an embedding of the desired $v-$lobe with v incident to an edge on the infinite face. Using this embedding we can add the rest of G back to get a planar graph.

The second case to consider occurs when the embeddings of one of the subgraphs has v located on the infinite face, but in a location relative to other vertices where adding edges could feasibly create a nonplanar embedding. Figure 6 is an example of such a situation. In these embeddings, the edges incident to v would create an angle on the infinite face of 180° or less. Utilizing the flexibility of an embedding on a plane, we can arrange the vertices of each subgraph to create an angle of more than 180°, allowing for the combination of two subgraphs without intersecting edges.

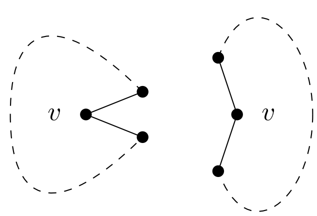

FIGURE 6. Case 2

It follows that combining all of the v-lobes will produce G, which would be planar. That is a contradiction, thus G must be 2-connected. $\square$

Lemma 3.5. *Suppose $S = \{x, y\}$ is a separating set of G. If G is nonplanar, then adding the edge (x, y) to some S-lobe produces a nonplanar graph.*

Proof. By way of contradiction suppose adding $e = (x, y)$ to all S-lobes produced only planar subgraphs. Since removing edges cannot make a planar graph nonplanar, all of the S-lobes must be planar as well.

We will show through induction that combining all such S-lobes will result in a planar graph. Let H, I be the S-lobes in a graph with two S-lobes. Using Lemma 3.1, find embeddings of H and I such that when adding e it bounds the infinite face. Since $H \cap I = \{x, y\}$, combining them will not cause any edges to cross between H and I, resulting in a planar graph.

Now suppose that we can combine all S-lobes into a planar graph for all graphs with $i = 2, \ldots, n-1$ number of S-lobes. Let G be a graph with n S-lobes. Removing one S-lobe, we can combine the rest of the $n - 1$ S-lobes to produce a planar subgraph of the original graph. Using Lemma 3.1, we can find an embedding that has e bounding the infinite face. We can do the same with the final S-lobe, and combine them to produce a planar graph. Since these are all of the S-lobes, combining them must produce the original graph, which must be planar.

It follows that combining all of the planar S-lobes in our original graph G produces a planar graph, which is a contradiction. Therefore there is some S-lobe that produces a nonplanar graph when (x, y) is added. $\square$

Lemma 3.6. *If G is a graph with the fewest edges among all nonplanar graphs without Kuratowski subgraphs, then G is 3-connected.*

Proof. We know that G is 2-connected from Lemma 3.4. By way of contradiction suppose that G has connectivity two and let $S = \{x, y\}$ be a separating set. By Lemma 3.5 we know that adding (x, y) to some S-lobe will produce a nonplanar graph. Let H be such an S-lobe. The graph $H + (x, y)$ must contain a Kuratowski subgraph by hypothesis. Let I be another S-lobe in G. Since both x and y are in the separating set and I is connected, there must be a path from x to y in I. As seen in Figure 7, we can replace (x, y) in H with that path, leaving us with a Kuratowski subgraph in $H \cup I$. It follows that G contains a Kuratowski subgraph, leading to a contradiction. Therefore, the connectivity of G must be at least three. $\square$

The notation $G \cdot e$ denotes a **contraction** of edge e in the graph G. The resulting graph replaces the endpoints of e with a single vertex adjacent to each of the vertices adjacent to the original endpoints.

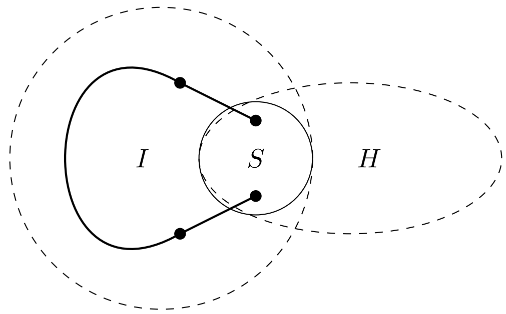

FIGURE 7.

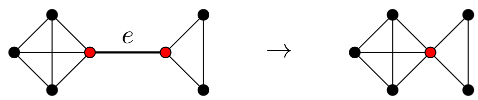

FIGURE 8. Contracting an edge, e.

Lemma 3.7. *Every 3-connected graph G with at least five vertices has an edge e such that $G \cdot e$ is 3-connected.*

Proof. We will proceed with a proof by contradiction. Let $e = (x, y)$ be an edge in G. The graph $G \cdot e$ cannot have a cutpoint; if it did then there would be a separating set of size at most two in G. Thus if $G \cdot e$ is not 3-connected then it must have a separating set S of size 2. If S did not contain the vertex produced by contracting the edge e, then S would be a separating set for G as well. However G is 3-connected, so S must contain the vertex produced by contracting e. Let z be the remaining vertex in S, and call it the *mate* of the adjacent pair x, y. It follows that $\{x, y, z\}$ is a separating set for G.

By way of contradiction, suppose $G \cdot e$ has connectivity two for all edges in G. All adjacent pairs will have a mate. Choose $e = (x, y) \in G$ and their mate z such that their removal yields a component H with the largest order. Let H' be an additional component in $G - \{x, y, z\}$. Since $\{x, y, z\}$ is a minimal separating set, each of x, y, and z must have a neighbor in both H and H'. Let u be a neighbor of z in H' and let v be the mate of z, u.

The graph $G - \{z, u, v\}$ is disconnected by definition. Let I be the induced subgraph on $V(H) \cup \{x, y\}$. It is connected, since x and y have neighbors in H. If $v \in H$, deleting it from I would not disconnect I, since that would mean $\{z, v\}$ is a separating set of size two in G. This means that the $I - v$ is contained in a component of $G - \{z, u, v\}$ that has more vertices than H. However, we chose H to have the largest order of components of G, so this is a contradiction. Therefore, there exists an edge e such that $G \cdot e$ is 3-connected. $\square$

Definition 3.8. *A vertex of degree at least three is called a* ***branch vertex****.*

Lemma 3.9. *If G has no Kuratowski subgraph, then any graph produced by contracting an edge also has no Kuratowski subgraph.*

Proof. We will proceed by proving the contrapositive. Suppose the graph resulting from contracting the edge $e = (x, y)$ contains a Kuratowski subgraph H. Let z be the vertex obtained by contracting e. If $z \notin H$, then H is a Kuratowski subgraph of G. Otherwise if $z \in H$ and z is not a branch vertex, we will replace z with either x, y, or the edge (x, y). We know that z must have degree two in H since it is part of a Kuratowski subgraph. Let a, b be the two edges incident to z. Without loss of generality, let a be the edge incident to x in G. The edge b must be incident to either x or y. If it is incident to x, then y would only be adjacent to x and thus y would have degree one in G, and we can replace z with x in G to obtain our Kuratowski subgraph. If b is adjacent to y, then replacing z with the edge (x, y) would act as a subdivision of H present in G, which would also be a Kuratowski subgraph.

In the case that z has degree three in H, if either of x or y was a pendant vertex, or a vertex with degree one, then it could not be in the Kuratowski subgraph. If either x or y are incident to at most one edge incident to z, then we can replace z with the edge (x, y). This adds a vertex on the path containing x and y, making the other vertex the corresponding branch vertex in the Kuratowski subgraph in G. If there are one or more vertices that are adjacent to both x and y, we can remove one of each pair of those edges, resulting in one of x or y having one edge with a counterpart incident to z.

The final case is when z has degree four. Since no vertices in a subdivision of $K_{3,3}$ have degree more than three, this case must happen when H is a subdivision of K_5. Similar to the previous case, if there are edges incident to z that were incident to both x and y we will consider only one. If all but one edge incident to z came from one of x or y, then, then the other vertex could be suppressed in G and we would have a Kuratowski subgraph.

Finally, if z is incident to two edges from each x and y, let u_1, u_2 be the branch vertices at the ends of the paths leaving z on the edges incident to x in G, and let v_1, v_2 be the corresponding branch vertices from the paths leaving z on edges incident to y in G. By deleting the path between u_1 and u_2 in addition to the path between v_1 and v_2, we can replace z with the edge (x, y) and obtain a subdivision of $K_{3,3}$ in G. As depicted in Figure 9, the vertices x, v_1, and v_2 make up one partition and y, u_1, u_2 make up the other.

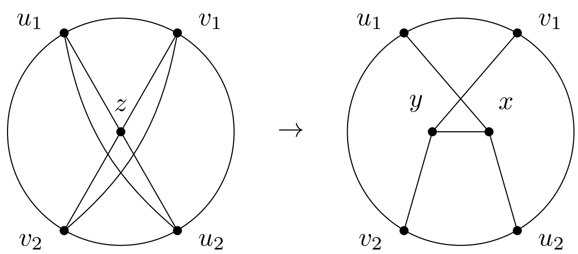

FIGURE 9. Subdivision of $K_{3,3}$ in G.

We have proved that if $G \cdot e$ has a Kuratowski subgraph, then G has a Kuratowski subgraph. It follows that if G has no Kuratowski subgraph then neither does $G \cdot e$. □

Definition 3.10. *A* ***convex embedding*** *of a graph is a planar embedding in which each face boundary is a convex polygon.*

Theorem 3.11 (Tutte, 1960)**.** *If G is a 3-connected graph with no subdivision of K_5 or $K_{3,3}$, then G has a convex embedding in the plane with no three vertices on a line.*

Proof. We will proceed by induction on the number of vertices, n, in G. The base case will be when $n \leq 4$. Note that all vertices in a 3-connected graph must have degree at least three. Therefore, the only 3-connected graph on four or fewer vertices is K_4, which clearly has an embedding with the desired characteristics.

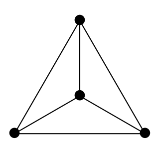

FIGURE 10. Embedding of K_4.

Now, suppose every graph with fewer than n vertices has a desired embedding. Let G be a 3-connected graph with n vertices and no Kuratowski subgraph. Let $e = (x, y)$ be an edge such that $H = G \cdot e$ is 3-connected, guaranteed by Lemma 3.7, and let z be the vertex obtained by contracting e. The graph H has no Kuratowski subgraph by Lemma 3.9, and therefore by the induction step we have a convex embedding of H with no three vertices on a line.

The subgraph of this embedding obtained by deleting edges incident to z has a face containing z, and since $H - z$ is 2-connected this face must be bounded by a cycle. The vertex z may be in the infinite face; in this case the cycle will consist of the edges bounding the infinite face. Let C represent this cycle. Since all neighbors of z are in C, they must be neighbors of x, y, or both, in G.

Let $x_1, \ldots, x_k$ be the neighbors of x in order on C. If all neighbors of y are located between x_i and x_{i+1} we can replace z with x and place y at a point close to x. Since G is 3-connected y must have at least two other neighbors in the cycle. Placing y such that it is within the triangle created by x and two consecutive neighbors of y in the cycle guarantees that all faces created are convex.

Now suppose the previous case does not occur, i.e. there are neighbors u and v of y that alternate with neighbors x_i, x_{i+1} of x, or y must be neighbors with at least three vertices $x_j, x_k,$ and x_l on the cycle. In the first case, as shown in Figure 11, there would be a subdivision of $K_{3,3}$, with the two partitions being $\{x_i, x_{i+1}, y\}$ and $\{u, v, x\}$. In the second case, let $x_j, x_k,$ and x_l be the shared neighbors. As shown in Figure 11, we have a subdivision of K_5. Since G does not have any Kuratowski subgraphs by hypothesis, we know that the case in the previous paragraph must be the only case. It follows that G has a convex embedding in the plane.

□

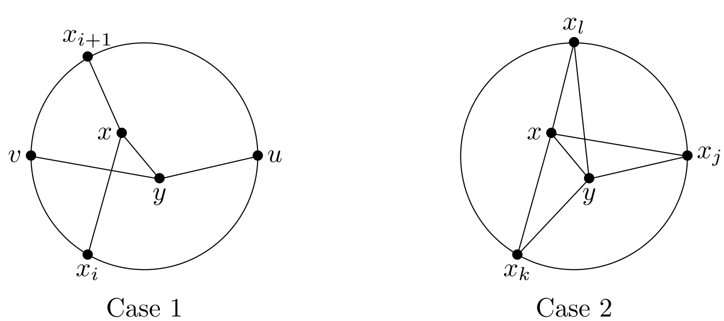

FIGURE 11.

We are now equipped with the results to set up a contradiction to prove Kuratowski's Theorem.

Theorem 3.12 (Kuratowski, 1930)**.** *A graph is planar if and only if it does not contain a subdivision of* K_5 *or* $K_{3,3}$.

Proof. Let G be a planar graph. Planarity is a minor closed property: all minors of a planar graph will be planar. Since K_5 and $K_{3,3}$ are nonplanar, they must not be topological minors of G.

Next, suppose for contradiction we have a nonplanar graph that does not contain a Kuratowski subgraph. If there is such a graph there must be one with fewest edges, call it G. By Lemma 3.6, G must be 3-connected. Then, by Tutte's Theorem, G

has a convex embedding and thus is planar. This contradicts our supposition so G must have a Kuratowski subgraph. $\square$

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4. The Robertson-Seymour Theorem

In 1970, German mathematician Klaus Wagner conjectured that for any infinite set of graphs, at least one graph must be a minor of another. He was not able to produce a proof for this conjecture, and it remained unproven until 2004 when Robertson and Seymour published [3]. The proof was very complex, taking a series of 20 papers on the topic to finish. Though the proof is much too complicated to address in this paper, we will explore important corollaries and examples. The theorem is stated as follows:

Theorem 4.1 (The Graph Minor Theorem)**.** *In any infinite sequence of graphs $G_1, G_2, \ldots$ there are indices $i \neq j$ such that G_i is a minor of G_j.*

In this form, the Graph Minor Theorem does not make characterizing forbidden graph minors any less daunting. However, the following corollary shows that the set of forbidden minors for a minor closed graph property is finite. This result is very significant and narrows the scope of possibilities for forbidden minor sets drastically.

Corollary 4.2. *Let $\mathcal{P}$ be a graph property that is minor closed. There is a finite set of forbidden minors, Forb($\mathcal{P}$), such that G has $\mathcal{P}$ if and only if it has no minor in Forb($\mathcal{P}$).*

Proof. Let S be the set of graphs without property $\mathcal{P}$. Let M be the set of graphs in S that do not have any proper minors in S. Suppose M had infinite order. Then, by the Graph Minor Theorem, some graph in M must have a minor in M, which brings us to a contradiction. Therefore M must have finite order. We now need to prove that all graphs in S must have a minor in M. We will proceed by inducting on the sum of the number of vertices and the number of edges in a graph.

Let n be the smallest integer such that a graph in S has n vertices and edges. Note that here may be more than one graph with the minimum number, e.g. the forbidden minors for planarity. We are not concerned with the actual graphs and thus we can proceed with solely the count. Any graph with n vertices and edges has no proper minor in S since any minor would have fewer vertices or edges. Now suppose all graphs whose sum of vertices and edges is fewer than i have a minor in M. Let $G \in S$ be a graph with i vertices and edges. If G has no proper minor in S, then $G \in M$. Otherwise, let H be a minor of G. Since it is a minor, H must have fewer vertices and/or edges and, by the induction hypothesis, has a minor in M. It follows that G has a minor in M, namely the minor of H in M. □

4.1. Wagner's Theorem

With the Graph Minor Theorem, Kuratowski's characterization of planar graphs has been extended to any minor closed graph property. While Kuratowski identified forbidden topological minors, Robertson and Seymour's theorem references graph minors. The counterpart of Kuratowski's Theorem referencing graph minors is Wagner's Theorem, published by Wagner in 1937, which is stated as follows:

Theorem 4.3 (Wagner, 1937). *A graph is planar if and only if it does not contain K_5 nor $K_{3,3}$ as a minor.*

It is simple to see that Wagner's Theorem is a direct result of Kuratowski's Theorem, since a topological minor can be obtained by normal minor operations. The suppression of a vertex with degree two can be done by contracting one of the edges incident to that vertex. It is also true that Kuratowski's Theorem directly implies Wagner's Theorem as well, though it is a little less clear to see. This is done by proving that if G has a minor of $K_{3,3}$ then it has a topological minor of $K_{3,3}$, and if it has K_5 as a minor, then it has one of K_5 or $K_{3,3}$ as a topological minor.

As an example observe the Petersen Graph in Figure 12. The Petersen Graph contains K_5 as a minor, and $K_{3,3}$ as a topological minor. This shows that having K_5 as a minor does not necessarily guarantee the existence of K_5 as a topological minor.

Figure 12. The Petersen Graph contains $K_{3,3}$ as a topological minor.

5. Forbidden Minors for Other Graph Properties

Kuratowski's Theorem and Wagner's Theorem are specific instances of Corollary 4.2, when $\mathcal{P}$ denotes planarity. Due to the significance of Kuratowski's result, the set of forbidden minors for a given property is often called the **Kuratowski set** for the property. There are many instances of minor closed graph properties other than planarity, though finding the graphs in a set of forbidden minors for a given property can be difficult. In the next sections we will explore elements of the Kuratowski sets for two new graph properties.

5.1. Outerplanar Graphs

We will proceed by identifying the Kuratowski set of a graph property that is closely related to planarity: outerplanarity. An **outerplanar** graph is a graph that can be embedded in a plane with all vertices on one face. We will denote the property as $\mathcal{OPL}$. When discussing outerplanar graphs it is often useful to identify and keep consistent the face incident to all vertices. Since outerplanar graphs are planar, we can utilize Lemma 3.1 to redefine outerplanar as having all vertices on the infinite face. In the following discussion we will assume that outerplanar graphs have all of the vertices incident to the infinite face, unless otherwise noted.

In order to guarantee a finite set of forbidden minors we need to know that $\mathcal{OPL}$ is minor closed. Since planarity is minor closed, it is sufficient to show that a minor of an outerplanar graph will have all vertices incident to one face. It is clear that deleting edges and vertices will not change which faces are incident to each vertex: it can only expand already adjacent faces. In the case of an edge contraction, note that the vertices incident to the contracted edge must be incident to the infinite face, and thus when contracted the new vertex will be incident to the infinite face as well. Any of the minor operations on an outerplanar graph produce an outerplanar minor. Therefore $\mathcal{OPL}$ is minor closed.

We may now use Corollary 4.2 to show that there is a finite set of forbidden minors for outerplanar graphs; however, we do not yet know how large this set is, let alone which graphs are contained within it. While the Graph Minor Theorem guarantees that the size of the set of forbidden minors is finite, it does not provide a method of identifying the specific graphs that make up the set. Luckily Forb($\mathcal{OPL}$) is rather small and its elements can be identified following a few key points.

The set Forb($\mathcal{OPL}$) must consist of graphs that are minor minimal non-outerplanar, i.e., all proper minors must be outerplanar. To get started finding the elements of Forb($\mathcal{OPL}$) we will once again look back at Kuratowski's Theorem. Outerplanarity is planarity with an additional constraint, so it may be helpful to start our search with the elements of Forb($\mathcal{PL}$): K_5 and $K_{3,3}$. These graphs are not outerplanar, but are also not minor minimal. The minors produced by removing a vertex from each graph are also not outerplanar. Since these graphs, K_4 and $K_{2,3}$, are relatively small, it does not take much work to see that they are minor minimal non-outerplanar.

We now have two elements of Forb($\mathcal{OPL}$): K_4 and $K_{2,3}$. It turns out these are the only elements. In order to prove this we will utilize the following result. First, a new definition:

Definition 5.1. *A **spanning cycle** in a graph G is a path containing all vertices that starts and ends on the same vertex.*

FIGURE 13. Minor Minimal Outerplanar Graphs K_4 (left) and $K_{2,3}$ (right).

Proposition 5.2. *The boundary of the infinite face of a 2-connected outerplane graph is a spanning cycle.*

A proof of Proposition 5.2 can be found in [4]. We will now show that Forb($\mathcal{OPL}$) consists solely of K_4 and $K_{2,3}$.

Theorem 5.3. *A graph is outerplanar if and only if it does not contain K_4 nor $K_{2,3}$ as a minor.*

Proof. Since outerplanarity is minor closed, the forward implication is clear: an outerplanar graph may not contain K_4 nor $K_{2,3}$.

We will prove the reverse direction through a proof by induction, inducting on the number of vertices, $n(G)$. For the base case consider the graph with one vertex. This graph is clearly outerplanar.

Now, suppose that all graphs with fewer than m vertices that do not contain K_4 nor $K_{2,3}$ are outerplanar. Let G be a graph that does not contain K_4 nor $K_{2,3}$ and $n(G) = m$. If G is disconnected, each of the components of G has fewer than m vertices and are outerplanar by the induction hypothesis. Thus all vertices are incident to the infinite face and G is outerplanar.

If G has connectivity one, let x be a cutpoint. The x-lobes of G have fewer than m vertices and are thus outerplanar. We can combine these lobes at x in a way such that each x-lobe retains its outerplanarity. It follows that G is outerplanar.

In the case that G has connectivity two or more, consider two adjacent vertices x and y. Contracting the edge (x, y) to a vertex z produces a minor of G. Since this minor is outerplanar, by Proposition 5.2, all vertices are on a cycle. Let

$$c_1 = z, c_2, \ldots, c_{m-1}$$

be the vertices in the cycle in order. This minor is outerplanar by the induction hypothesis. When reversing the contraction to obtain the original G, there are a few cases of how the edges incident to x and y may be oriented.

In Case 1, pictured in Figure 14, an edge incident to x may cross an edge incident to y. Formally, if $x = c_1$ and $y = c_m$, x is adjacent to some c_i and y to some c_j

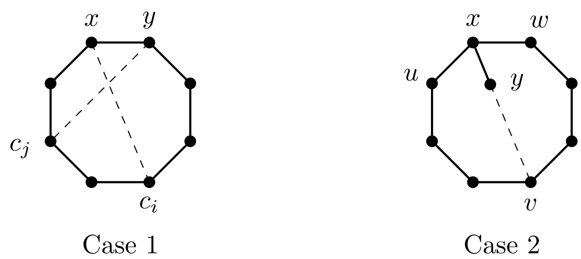


Figure 14.

where $i > j$. In this case we have a minor of K_4 with x, y, c_i and c_j as the vertices with degree three. This brings us to a contradiction.

In Case 2, y is not located on the cycle. Since G is 2-connected y must be adjacent to another vertex v on the cycle. The vertex y cannot be adjacent to u nor w, since y is not on the cycle. In this case we have a minor of $K_{2,3}$, with x and v as the set of size two and y along with the neighbors of x on the cycle as the set of size three. This brings us to a contradiction as well.

Both cases lead to a contradiction. Therefore, G must be outerplanar.

□

Another interesting method of exploring forbidden minors is through exploring embeddability in different surfaces. Finding the forbidden minors of graphs embeddable in the sphere is relatively simple: we can use the method utilized in Lemma 3.1 to see that they would be the same as Forb($\mathcal{PL}$). What do these forbidden minor sets look like for embedding on surfaces with higher genera?

5.2. Toroidal Graphs

Consider the graphs embeddable on a torus, which has genus one. We will refer such graphs as **toroidal graphs**, denoting the graph property with $\mathcal{T}$. Due to its unique orientation it is not unreasonable to think that the Kuratowski set for toroidal graphs may look very different from the Kuratowski sets for planarity and outerplanarity. In order to begin discussing Forb($\mathcal{T}$) we must have a method of depicting toroidal embeddings so that they are easy to visualize on a page.

We will use the method used in [4]. We will depict a torus by imagining two cuts along its surface: a vertical cut which results in a cylindrical surface, and a horizontal cut which results in a plane. See Figure 15 for a visual representation. The resulting rectangular surface can be used to depict embeddings in the plane: opposing sides represent the cuts made in the torus. In this depiction, edges and vertices can cross along corresponding sides of the rectangle.

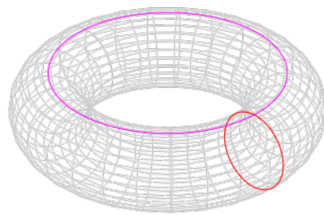


FIGURE 15. Visualizing embeddings in a torus, in a plane.

Using this method, Figure 16 shows that we are able to embed K_5 and $K_{3,3}$ in a torus. Note that edges and vertices may cross between sides with matching arrows.

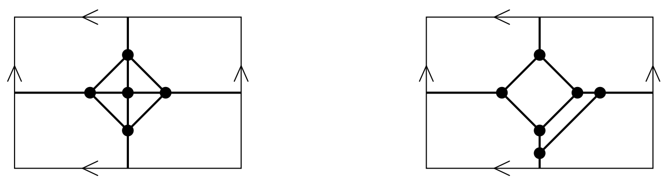


FIGURE 16. K_5 (left) and $K_{3,3}$ (right) embedded in a torus.

Once again we run into the issue that there is no consistent way to identify each of the graphs in the Kuratowski set for a given property. According to [5] there are at least 16 thousand known forbidden minors for toroidal graphs. It would be unreasonable and perhaps uninteresting to attempt to list them all in a paper like this. With the sheer number of forbidden minors, it is much more reasonable to explore restrictions to the graphs and trim the set accordingly. For example, there are only four of the 16 thousand that do not have $K_{3,3}$ as a minor. In addition, when restricting the connectivity there are only 68 with connectivity two. We restrict our search in a similar fashion and identify those forbidden minors of toroidal graphs with connectivity less than two.

In order to proceed we will cover a few key concepts regarding embedding graphs in surfaces of varying genera. We will be using notation and definitions from [1].

Definition 5.4. *The **genus** of a graph G, denoted by $\gamma(G)$, is the smallest genus surface in which G can be embedded.*

Since K_5 and $K_{3,3}$ cannot be embedded in a sphere but can be embedded in a torus, they both have genus one. Since they are the only minor minimal nonplanar graphs, they are the only minor minimal graphs with genus one. The graphs in Forb($\mathcal{T}$) must have genera at least two.

Definition 5.5. *A **block** of G is a subgraph B of G that is maximal with respect to the property that removing any single vertex of B does not disconnect it.*

Note that the blocks of G must be 2-connected. The following corollaries will help identify the elements of Forb($\mathcal{T}$) that have connectivity less than two. They follow from Theorem 1 of [1].

Corollary 5.6. *The genus of any graph is the sum of the genera of its blocks.*

Corollary 5.7. *The genus of a graph is the sum of the genera of its components.*

These results simplify the calculation of the genus of a given graph, and help narrow our search. We will consider the graphs in Forb($\mathcal{T}$) with connectivity one and those with connectivity zero separately.

Elements of Forb($\mathcal{T}$) that have connectivity one must have at least two blocks, since a graph with only one block would have connectivity two by definition. Removing blocks of genus zero from a graph would not change the genus of the graph, and therefore each of the blocks in our desired graphs must have a genus of at least one.

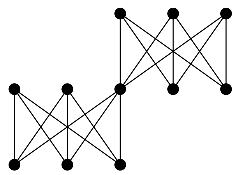
It is important to note that we are looking for graphs of genus exactly two. A graph with connectivity less than two and with genus three or more must have a proper minor of genus two or more, and thus the orginal graph of genus three or more is not minor minimal non-toroidal.

It follows that there are two blocks and both have genus one. The only minor minimal graphs with genus one are the elements of Forb($\mathcal{PL}$). Therefore we can conclude that if a graph G is in Forb($\mathcal{T}$), then it must have two connected blocks, each of which are either K_5 or $K_{3,3}$. It follows that the graphs in Figure 17 are the elements of Forb($\mathcal{T}$) with connectivity one.

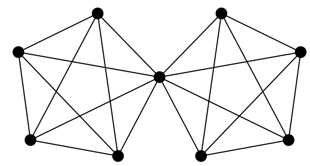
Now that we have found the forbidden minors of connectivity one, we will identify the elements in Forb($\mathcal{T}$) that are not connected.

Suppose we a have a graph $G \in$ Forb($\mathcal{T}$) that has connectivity zero. By definition it must have at least two disconnected components. By Corollary 5.7 the sum of the genera of these components must be two. However, similar to the previous case of graphs with connectivity one, any components in G with genus zero can be removed and the genus of G will be preserved. Therefore if G is to be minor minimal, each component must have genus at least one.

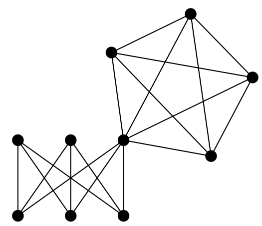
If there was a component in G with genus two or greater, then we could remove the other components and the resulting subgraph would still have genus two, and thus would not be embeddable in the torus. Therefore each component must have genus one, and it follows that there must be only two disconnected components. We know that the minor minimal graphs with genus one are K_5 and $K_{3,3}$, so we can conclude that the graphs in Forb($\mathcal{T}$) with connectivity zero have two components, each of which are either K_5 or $K_{3,3}$. These graphs are shown in Figure 18.



$K_{3,3}$ and $K_{3,3}$

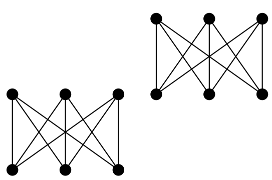


K_5 and K_5

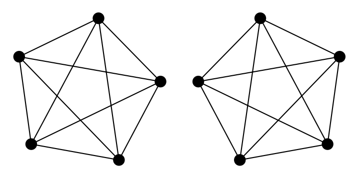


$K_{3,3}$ and K_5

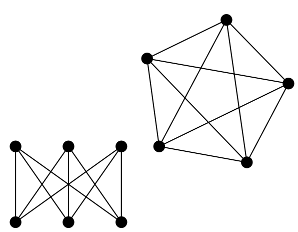
FIGURE 17. Elements of Forb($\mathcal{T}$) with connectivity one.



$K_{3,3}$ and $K_{3,3}$



K_5 and K_5



$K_{3,3}$ and K_5

FIGURE 18. Elements of Forb($\mathcal{T}$) with connectivity zero.

When dealing with such a large set of forbidden minors it can be daunting and extremely difficult to identify them all. However as we have shown, adding conditions to the graphs can help tame the set and produces interesting results.

6. Conclusion

We have explored parts of of the Kuratowski sets for two minor closed graph properties: outerplanar and toroidal. Of course, there are many other properties that would be fascinating to explore with respect to their Kuratowski sets. Properties may be quite simple; bounding the number of edges or the number of vertices is minor closed and would fall under the Graph Minor Theorem. Other restrictions may be added to already established properties as well. For example, [5] defines a way to modify any property $\mathcal{P}$ by finding graphs G where there exists a vertex v such that $G - v$ has $\mathcal{P}$. Applying this to planarity, they find the Kuratowski set for graphs that are "strongly almost planar."

While the search for forbidden minors started with Kuratowski's Theorem in 1930, it is still a fascinating problem that is getting attention. Robertson and Seymour's grand result in the Graph Minor Theorem confirmed Wagner's suspicion and proved that any set of forbidden minors for a minor closed property must be finite. As in the case of toroidal graphs, the categorization 'finite' does not give any indication at the size or or difficulty in finding the elements of these sets. There are still countless graph properties with Kuratowski sets that are not yet completely known.

In addition, [5] shows that even if a property $\mathcal{P}$ is not minor closed, the list of graphs that are minor minimal $\mathcal{P}$ is finite. Therefore, with a property that is not minor closed, while the Kuratowski set may not be finite, the minor minimal graphs with $\mathcal{P}$ could feasibly all be found.

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