The Monty Hall Problem

Yurixy Lopez Martinez

A project presented for the degree of
Bachelors of Arts, Mathematics

Mathematics Department
Whitman College
Walla Walla, WA
May 10, 2019

Abstract

This paper begins by offering a detailed explanation of the solution to the Monty Hall Problem utilizing decision trees and mathematical concepts of conditional probability, mainly Bayes' Theorem. We will proceed to investigate the various versions of the problem that have risen throughout the years among scholars, mainly focusing on the benefits of a particular strategies. We will conclude by briefly discussing some applications of the Monty Hall Problem to other disciplines, mainly the probabilistic aspects.

Contents

1 Introduction

The Monty Hall Problem is a recognized probability problem. The canonical, classical version of the problem is described as follows:

> *A contestant is shown three identical doors. Behind one of them is a car. The other two conceal goats. The contestant is asked to choose, but not open, one of the doors. After doing so, Monty, who knows where the car hides, opens one of the two remaining doors. He always opens a door he knows to be incorrect (goat-concealing doors will be referred to as the incorrect doors), and randomly chooses which door to open when he has more than one option (which happens on the occasion where the contestant's initial choice conceals the car). After opening an incorrect door, Monty gives the contestant the option of either switching to the other unopened door or sticking with their original choice. The contestant then receives whatever is behind the door they choose. What should the contestant do?*[1]

This problem was first coined by Universtiy of California, Berkeley mathematician Steve Selvin in 1975. He proposed and solved it in a letter to *Scientific American*, which is a popular American academic journal. Selvin based the Monty Hall Problem off a 70's television game show called "Let's Make A Deal," hosted by Monty Halparin. This game show had various formats but in general, a player had to decide between winning a small prize and gambling on some probability of winning a greater prize.

Even though this problem lacks mathematical notation and may seem simple at first, it caused quite a discussion among intelligent people, primarily professional statisticians. Steve Selvin, proposed in the letter that the contestant should switch to the remaining door when given the option because their chances of winning are two times greater. On the contrary, many people reasoned that there is no advantage to switching or sticking when given the opportunity because the player is deciding between a door that hides a car and a door that hides a goat; therefore, they have 50-50 chance of winning if they decide to switch or stick. In the years to come, Steve Selvin received and responded to numerous letters of disbelief. For the most part, the problem remained under the radar.

It was not until 1990 that the Monty Hall Problem became famous. It was formulated as a question to Marilyn vos Savant's for her "Ask Marilyn" column in *Parade* magazine. Marilyn vos Savant at the time held the highest IQ in the world. She responded to the question that the contestant should switch because they will have a $\frac{2}{3}$ chance of winning the car, while sticking would only give the contestant a $\frac{1}{3}$ chance of winning the car. Readers of this well-known magazine refused to believe her answer. Even prolific mathematician Paul Erdos was unconvinced, until he was shown computer simulation of the predicted result. Approximately 1,000 PhD accredited readers wrote to the magazine claiming that vos Savant was wrong, even after being given explanations, simulations, and formal mathematical proofs.

One of the early probability puzzles related to the Monty Hall Problem dates back to Joseph Bertrand's Box Paradox, posed in 1889. In this paradox, there are three boxes: a box containing two gold coins, a box with two silver coins, and a box with one of each coin. After choosing a box at random and withdrawing one coin at random that happens to be a gold coin, the question that is arises is,what is the probability that the other coin is gold? Now, the intuitive answer is $\frac{1}{2}$, but the probability that the other coin is gold is actually $\frac{2}{3}$, just like with the Monty Hall Problem. In fact, its reasoning and mathematical interpretation is just as with what we will be endeavouring with the Monty Hall Problem.

The Three Prisoner's Problem, published in Martin Gardner's *Mathematical Games* column in *Scientific American* in 1959, is also very similar to the Monty Hall Problem. It says that there are three prisoners, A, B and C, in separate cells and sentenced to death. The governor has selected one of them at random to be pardoned. The warden knows which one is pardoned, but is not allowed to tell. Prisoner A begs the warden to let him know the identity of one of the others who are going to be executed, "If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, flip a coin to decide whether to name B or C."

The warden tells A that B is to be executed. Prisoner A is pleased because he believes that his probability of surviving has gone up from $\frac{1}{3}$ to $\frac{1}{2}$, as it is now between him and C. Prisoner A secretly tells C the news, who is also pleased, because he reasons that A still has a chance of $\frac{1}{3}$ to be the pardoned one, but his chance has gone up to $\frac{2}{3}$. What is the correct answer? It was concluded that prisoner A would still have a $\frac{1}{3}$ chance of being pardoned but the unnamed prisoner would have a $\frac{2}{3}$ chance, so prisoner C's reasoning is correct.

It is evident that the Monty Hall problem was a groundbreaking problem of its time. In the sections to come we will continue with a detailed solution and discussion of the Monty Hall Problem and discuss its diverse variations that have spurred throughout the years among mathematicians and statisticians.

2 Monty Hall Problem

We will begin by breaking down the Monty Hall problem in pieces. After the player selects a door, Monty opens one of the remaining two doors and reveals what the door is hiding. Monty though, opens his door following this strategy:

1. Monty always opens a door that hides a goat
2. Monty never opens the door the player selects

3. If Monty can open more than one door, following strategies 1 and 2, then he has opened the door at random. This strategy happens when the contestant selects the winning door.

So, after Monty opened his door and reveals what it was hiding, he offers the contestant the option to switch doors. This is where this famous problem arises; the contestant wants to know whether it is a better option to stick with their initial door selection or whether they should switch to the door that has not been opened.

At first glance, it may seem that the probability of winning the prize does not change regardless of switching or sticking. Often the contestant may reason that if they select door 1 and Monty opens door 2, then they are left with equal probability ($\frac{1}{2}$), that the prize is either behind door 1 or door 3.

Therefore, many may argue that it does not matter whether the contestant switches doors or not, that the probability of winning is the same if the contestant keeps their original option or if they switch. Nonetheless, we can show that theoretically and practically that the probability of winning the prize is actually higher if the player were to switch. We will examine this further in the following sections.

3 Why Switch?

It is important to take into consideration that Monty always opens a losing door, and the door is different from the contestant's selection. Each door has a $\frac{1}{3}$ probability that the car is behind it, so the contestant will select a door with a $\frac{1}{3}$ probability of winning if they stick with it throughout the game. The remaining two doors will have a probability of $\frac{2}{3}$ so that the total probability among the doors is 1. When Monty opens one of the remaining doors, we become aware of an important piece of information. This door contains a goat, so we disregard it for the remainder of the game. Now, we still have the contestant selection which has a $\frac{1}{3}$ probability, the remaining $\frac{2}{3}$ probability now belongs only to the door that has not been opened. The probability that the prize is behind the remaining door is two times greater; therefore, it is advantageous to switch.

Another way to visualize the problem is through the use of **decision trees**. Decision trees are useful tools to help decide between several courses of action by providing an effective structure that lays out options in order to investigate possible probabilistic outcomes.

The following decision tree, Figure 1, lays out the different scenarios of the Monty Hall Problem. We will assume that the contestant initially chooses door 1 (without loss of generality):

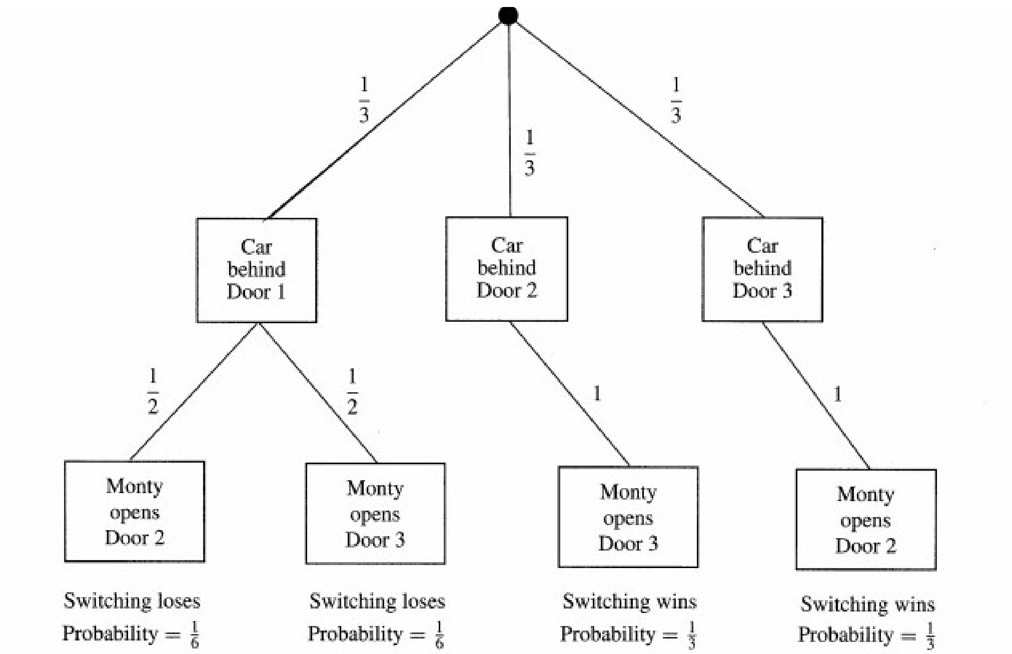

Figure 1: Contestant initially chooses Door 1. [1]

We can see from the decision tree, that there are four different scenarios for Monty after the contestant opens door 1:

1. Monty opens door 2, car is behind door 1
2. Monty opens door 3, car is behind door 1
3. Monty opens door 2, car is behind door 3
4. Monty opens door 3, car is behind door 2

It is intuitive to say that each scenario has the same probability of happening, but that can not be the case, because the car is behind door 1 only $\frac{1}{3}$ of the time, not $\frac{1}{2}$ of the time. In the first two scenarios Monty is opening either door 2 or door 3 at random. It does not matter which door he opens because there is a goat behind each one. This leaves for each of these scenarios to have a probability of $\frac{1}{6}$.

In the next two scenarios, Monty is not choosing his door at random, he is using strategies number 1 and 2. For scenario 3, there is probability of $\frac{1}{3}$ that the car is behind door three, the same with scenario 4, it has probability of $\frac{1}{3}$.

In conclusion, scenario 1 and 2, will leave a probability $\frac{1}{6}$ of winning if the contestant decides to switch, but remember that both of these scenarios together happen $\frac{1}{3}$ of the the time. On the other hand, for the remaining two scenarios, respectively, there is a probability $\frac{1}{3}$ of winning if the contestant decides to switch. Together, these two events happens $\frac{2}{3}$ of the time. We see again with the implementation of a decision tree that switching is advantageous.

4 Conditional Probability: Bayes' Theorem

The Monty Hall Problem is rooted in the concepts of statistics and probability theory, the mathematical concepts that will help endeavour a more clear understanding for the solutions to its different variations and its classical, canonical version. We will rely heavily on Bayes' Theorem, which is a derivation of conditional probability. This theorem was first formulated by an eighteenth-century British mathematician known as Thomas Bayes, and it is very important for our work. We will begin by defining conditional probability.

Definition 4.1. ***Conditional Probability*** Let A and B be events in a probability space. Then we define the conditional probability of A given B, denoted by $P(A|B)$, by the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

for $P(B) \neq 0$.

The intersection $A \cap B$ of these two events is the event in which both A and B occur; therefore, $P(A \cap B)$ is the probability in which both A and B occur.

With the Monty Hall Problem, we are interested in the following probabilities:

- M_j, the event that Monty opens door j, for $j = 1, 2, 3, ...$
- C_i, the event that the prize is behind door i, for $i = 1, 2, 3, ...$

We say that two events, A and B, are independent if,

$$P(A \cap B) = P(A)P(B).$$

It follow that,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A).$$

From this computation we get that for independent events A and B, $P(A|B) = P(A)$.

Bayes' theorem connects $P(A|B)$ and $P(B|A)$, which is key to solving different variations of the Monty Hall Problem.

Theorem 4.1. ***Bayes' Theorem*** For independent events A and B, and $P(B) \neq 0$,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

Proof. We start from the definition of conditional probability. Recall that

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Likewise, the probability of event B given event A is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Rearranging and combining the two equations we find that

$$P(A|B)P(B) = P(A \cap B) = P(B|A)P(A).$$

Dividing both sides by $P(B)$, we obtain Bayes' Theorem:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

□

To find $P(B)$, we can look at a partition of the sample space, and add the amount of $P(B)$ that falls in each partition, that is $P(B) = P(A \cap B) + P(A^C \cap B) = P(B|A)P(A) + P(B|A^C)P(A^C)$, where A^C is the complementary event of A. So the theorem can be restated as

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^C)P(A^C)}.$$

More generally, when A_i forms a partition of the event space, we have

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_n)P(A_n)$$

for $i = 1, 2, 3, \cdots, n$.

Therefore, when A_i forms a partition of the event space, we have

$$P(A_i|B) = \frac{P(B|A_j)P(A_j)}{\sum_{n=i} P(B|A_i)P(A_i)},$$

for any A_i in the partition. This is result is known as the **Law of Total Probability**.

5 Bayesian Monty

We can now approach the Monty Hall Problem through these concepts of Conditional Probability which will give us a way to numerically understand that it is advantageous for the contestant to switch.

Recall C_1, C_2, C_3, are the events that the car is behind door 1, 2, and 3, respectively. Also recall that M_1, M_2, M_3 are the events that Monty opens door 1, 2, and 3, respectively. We will assume, from the decision tree in the previous section that the contestant initially chose door 1 and Monty then opens door 2. So, we must evaluate $P(C_3|M_2)$, the probability that the car is behind door 3, assuming that Monty has opened door 2. We know,

$$P(C_1) = \frac{1}{3},\ P(C_2) = \frac{1}{3},\ \text{and } P(C_1) = \frac{1}{3}.$$

The following notation, $P(C_i|M_j)$, is defined as the probability that the car is behind door i, for $i = 1, 2, 3$ given that Monty opens door j, for $j = 1, 2, 3$. The Law of Total Probability says that

$$P(M_j) = P(M_j|C_1)P(C_1) + P(M_j|C_2)P(C_2) + \cdots + P(M_j|C_n)P(C_n).$$

We can use this result and apply it to Bayes' Theorem, which states

$$P(C_i|M_j) = \frac{P(C_i)P(M_j|C_i)}{P(M_j)}$$

where C_i and M_j are independent events and $P(M_j) \neq 0$.

Therefore,

$$P(C_3|M_2) = \frac{P(M_2|C_3)P(C_3)}{P(M_2|C_1)P(C_1) + P(M_2|C_2)P(C_2) + P(M_2|C_3)P(C_3)}.$$

We assume further that Monty will not open the door concealing the car or the door that the contestant initially chose, so we have

$$P(M_2|C_2) = 0 \text{ and } P(M_2|C_3) = 1.$$

Plugging the results to the formula we get

$$P(C_3|M_2) = \frac{1\left(\frac{1}{3}\right)}{P(M_2|C_1)(\frac{1}{3}) + 0(\frac{1}{3}) + 1(\frac{1}{3})} = \frac{1}{P(M_2|C_1) + 1}.$$

Recall that Monty chooses his door randomly when he has more than one option, which implies that $P(M_2|C_1) = \frac{1}{2}$. So,

$$P(C_3|M_2) = \frac{1}{P(M_2|C_1) + 1} = \frac{1}{(\frac{1}{2}) + 1} = \frac{2}{3}.$$

This is the result that we deduced from the decision tree in the previous section. There is a $\frac{2}{3}$ probability that the car is behind door 3; therefore the contestant should be inclined to switch.

6 Variations of the Monty Hall Problem

Up to this point, we have looked at the classical, canonical version of the Monty Hall Problem. This problem does not stop there. Over the years, this problem has gained much admiration and has been modified with different caveats. This paper will look at some of those variations of the Monty Hall Problem, mainly those found in [1]. We will analyze these variations as we did with the classical version, with decision trees and Conditional Probability.

6.1 Monty Chooses Randomly

Recall that in the original Monty Hall Problem, Monty only chooses at random when the contestant has chosen the door that conceals the car. That is the only instance in which the contestant benefits from sticking, but that scenario only happens with probability $\frac{1}{3}$ in comparison to the other scenarios where switching is beneficial, which happen with probability $\frac{2}{3}$. In the case where the contestant has initially selected a door that conceals a goat, Monty is forced to open the remaining door that conceals the goat.Monty is able to make this decision because he knows the location of the car. What if Monty does not know the location of the car, he would have to choose randomly from the doors different from the contestant's selection.

We are going to start off assuming that the contestant initially chooses door 1. We previously looked at the case where the prize is behind door one in the original version of the Monty Hall Problem. For the other cases we will make use of Figure 2, which will help us to visualize the different scenarios.

In the case that the car is not behind door 1, meaning that the contestant has selected a goat-concealing door, the game will end (the contestant will loose) if Monty opens a car concealing door. This happens $\frac{1}{3}$ of the time. So putting everything together, we conclude that there is a probability of $\frac{1}{3}$ that the contestant will loose by switching, a probability of $\frac{1}{3}$ that the contestant will win by switching, and a probability of $\frac{1}{3}$ that the game will end, in other words, the contestant loses. Still, even for this version of the Monty Hall Problem we see that the contestant will loose $\frac{2}{3}$ of the time and will win only $\frac{1}{3}$ of the time, which only happens if they switch.

Although it may seem a bit redundant, this variation of the Monty Hall Problem enhances the importance of Monty to choose randomly only when the contestant selects the door that conceals the prize.

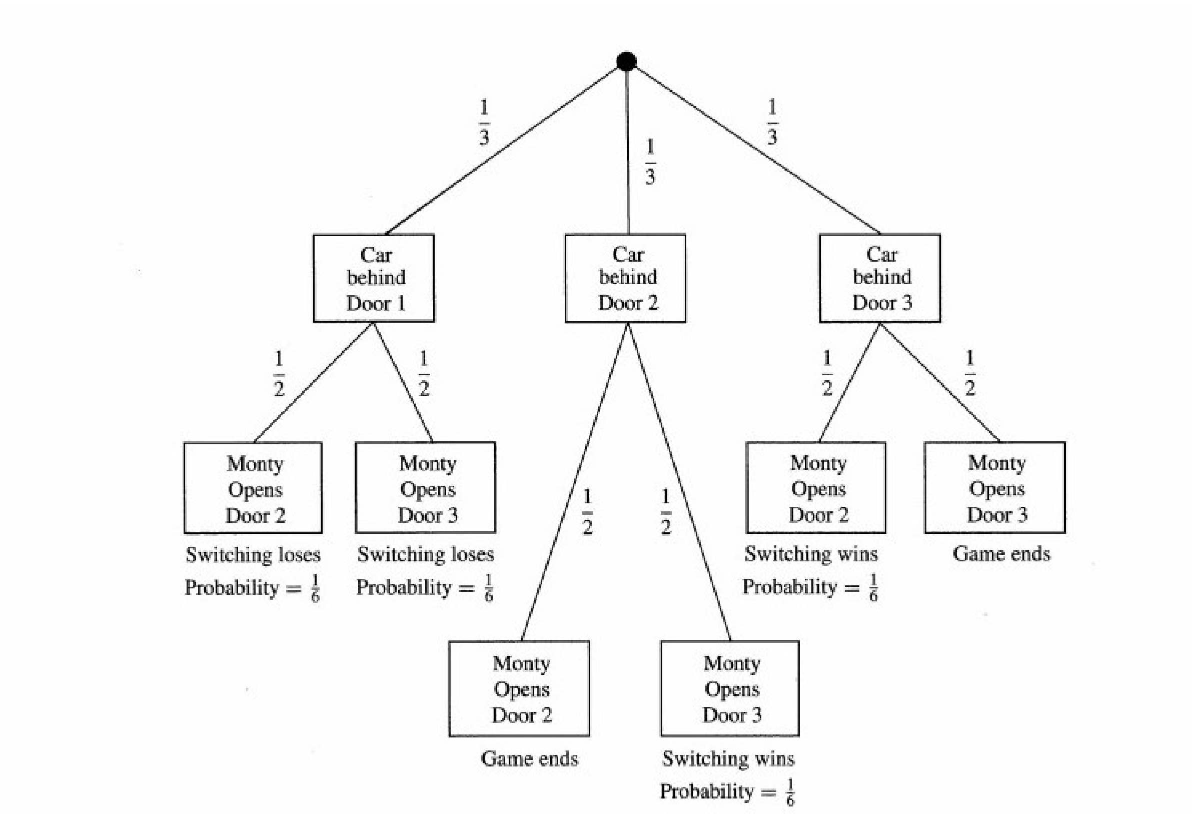

Figure 2: Contestant initially chooses Door 1. [1]

6.2 Highest-Ordered Selection

Let us now consider another simple variation of the Monty Hall problem.
The contestant chooses one of the three equally likely doors. Monty then opens a door he knows to be empty, this time, however, we assume that he opens the highest-numbered door available to him with probability p and therefore picks the lower numbered door with probability $1 - p$.

We will begin to look at this variation with specific numerical values. Suppose, without loss of generality, that the contestant initially chooses door 1 and assume that the car is behind door 1. Monty, in this scenario has a choice between opening door 2 or opening door 3. Suppose that the probability that he opens door 2 is $\frac{1}{4}$ and the probability that he opens door 3 is $\frac{3}{4}$. Knowing Monty's strategy we can construct the following decision tree, see Figure 3:

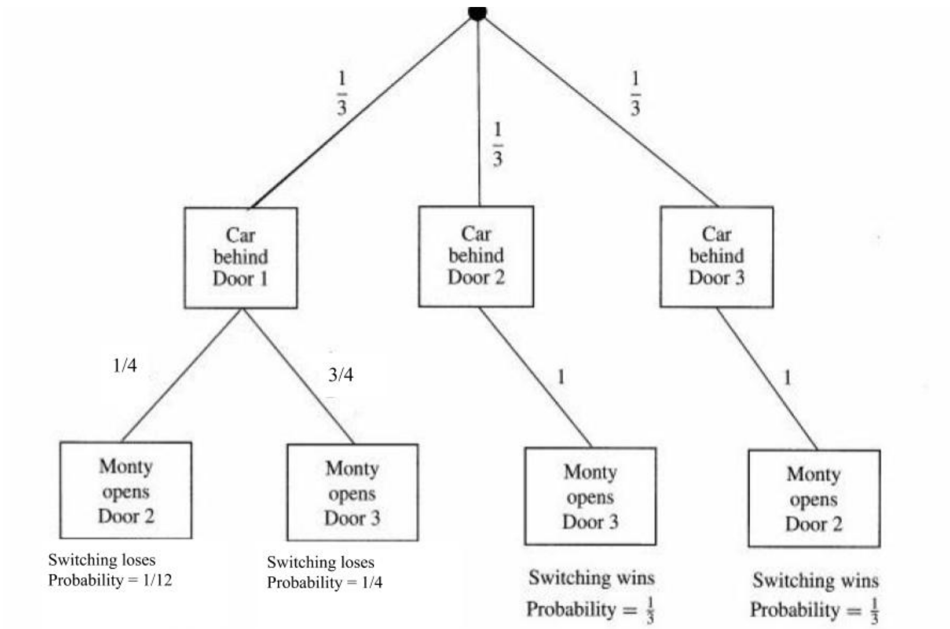

Figure 3: Contestant initially chooses Door 1. [1]

We know that the prize is behind door 1 only $\frac{1}{3}$ of the time. Monty will open door 2 only $\frac{1}{4}$ of the time, and if the contestant switches, they will loose; this scenario happens with probability $\frac{1}{12}$ Now, let us look at when Monty opens door 3 with probability of $\frac{2}{3}$. In this scenario, switching will not be advantageous for the contestant; this scenario happens with probability $\frac{1}{4}$.

On the other hand, if the car were to be behind door 2, Monty has no choice but to open door 3. This scenario will happen $\frac{1}{3}$ of the time. If the car were to be behind door 3, Monty has no choice but to open door 2. This scenario will also happen $\frac{1}{3}$ of the time. In total, the scenario where it will be advantageous for the contestant to switch is $\frac{2}{3}$. Since the total probability has to sum to 1, the probability where it is not advantageous for the contestant to switch is $\frac{1}{3}$. So the contestant has a better chance to win switching when given the opportunity.

Now, let us look at a general case, where Monty, given the opportunity to choose between door 2 and door 3, has probability p and $1-p$, respectively of opening each door. Consider the decision tree below, Figure 4:

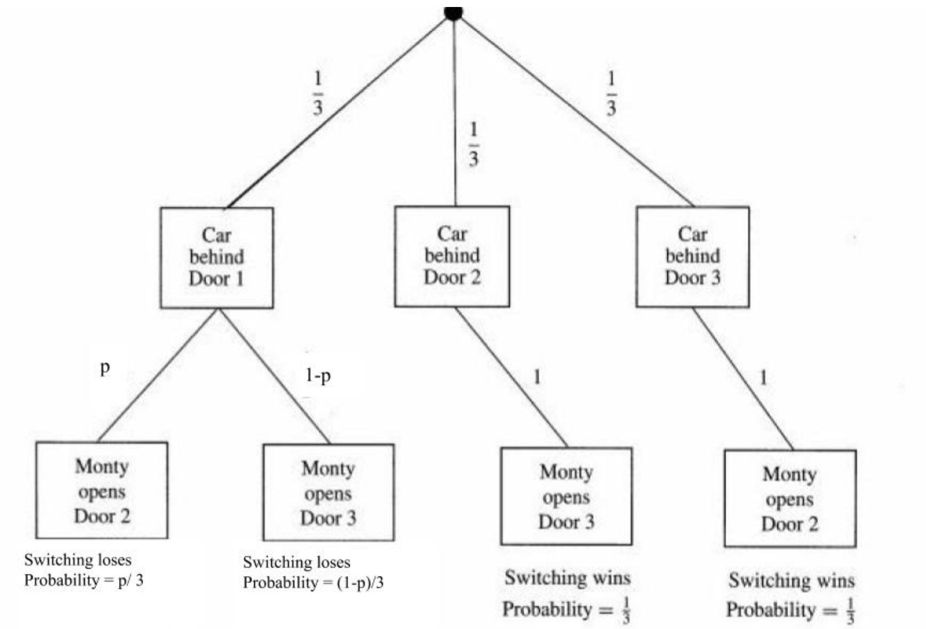

Figure 4: Contestant initially chooses Door 1. [1]

We can use Bayes' Theorem and the Law of Total Probability to obtain the probability of the event where the contestant chooses the wrong door, assuming that they initially chose door 1.

$$P(C_3|M_2) = \frac{P(C_3)P(M_2|C_3)}{P(C_1)P(M_2|C_1) + P(C_2)P(M_2|C_2) + P(C_3)P(M_2|C_3)}$$

$$= \frac{\frac{1}{3}\cdot 1}{\frac{1}{3}\cdot p + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1} = \frac{\frac{1}{3}}{\frac{p}{3}+\frac{1}{3}} = \frac{1}{p+1}$$

From the result, we can conclude that switching is good $\frac{1}{p+1}$ of the time, which implies that switching is not good $1 - \frac{1}{p+1} = \frac{p}{p+1}$ of the time. Therefore, the contestant gains an advantage by switching than sticking with their original choice.

The reasoning is similar if Monty opens the lowest-numbered door instead, that is, if the probability is higher that he opens door 2. In this situation however, if the car is behind door 1, then Monty opens door 3 with probability $1-p$ and opens door 2 with probability p. Consequently, door one now has probability $\frac{1-p}{2-p}$ of concealing the car, and door 2 or 3 have probability $\frac{1}{2-p}$ of concealing the car.

Note that in the case where $p = \frac{1}{2}$, which corresponds to the classical Monty Hall Problem, the formulas give the correct answer.

6.3 Many doors

Let us now consider a variation of the Monty Hall Problem where there are many doors to choose from. This version of the Monty Hall Problem states the following [1]:

We assume there are n identical doors, where n in an integer satisfying $n \geq 3$. One door conceals a car, the other $n-1$ doors conceal goats. The contestant chooses one door at random and does not open it. Monty then randomly opens a door he knows to conceal a goat. He gives the contestant the option of either sticking with their original choice or switching to to one of the remaining doors. The contestant makes a decision and Monty randomly opens another goat-concealing door, and again he gives the contestant the option of switching or sticking. This process continues until only two doors remain in play.

After close reading of this version of the Monty Hall Problem, we notice the following approaches that that contestant can follow:

- Select a door at random and stick with it throughout.
- Switch doors randomly at every opportunity.
- Stick with the first choice until only two doors remain.

We notice that the first approach that the probability of the initial choice is not going to change if the contestant sticks with it throughout. The reason for this is because Monty is always going to open a goat-concealing door and he is going to open it at random from among his options. Therefore, if the contestant decides to follow this strategy, they will win with probability $\frac{1}{n}$.

For the second approach, we see that it will become the original problem after $n-3$ cases, mainly because we will be left with only one car-concealing door, the door that Monty has revealed, and the unopened door.

For the last strategy, we can employ Bayesian analysis, we will go more in depth in section to follow. We begin by denoting A as the event in which the contestant's initial door choice conceals a car and $\bar{A}$ the event in which it does not conceal the car. So we have, $P(A) = \frac{1}{n}$ and $P(A) = \frac{n-1}{n}$. Therefore, switching last minute wins with probability $\frac{n-1}{n}$.

We will prove the result for the third strategy in a more rigorous manner.

Suppose that we have $n \geq 3$ doors. Again, without loss of generality, the contestant initially chooses door 1. This door has $\frac{1}{n}$ probability that is it a car-concealing door. Monty, according the the game rules, has to open one of the remaining doors to reveal a goat, let us say that he opens door i, with $2 \geq i \geq n$. We must show that the probability of door 1 remains the same throughout the game.

We will denote C_i the event that the car is behind door i and M_i the event that Monty opens door i. So according to Bayes' Theorem:

$$P(C_1|M_i) = \frac{P(C_1)P(M_i|C_1)}{P(M_i)}.$$

There are $n-1$ door that Monty can choose from once the contestant chooses door 1, so $P(M_i|C_1) = \frac{1}{n-1}$ and the probability that the car is behind the door Monty chooses is given by $P(C_i) = \frac{1}{n}$. To get the value of $P(M_i)$ we will utilize the Law of Total Probability. We know that the car can be either behind door 1 which happens with probability $\frac{1}{n}$, which means that Monty can open any of the remaining $n-1$ door with equal probability. It can also be behind door i, which Monty can not open, or either behind any of the other $n-2$ doors different from door 1 and door i. So,

$$\begin{aligned} P(M_i) &= P(M_i|C_1)P(C_1) + P(M_i|C_i)P(C_i) + P(M_i|\bar{C}_1 \cap \bar{C}_i)P(\bar{C}_1 \cap \bar{C}_i) \\ &= \frac{1}{n-1}\left(\frac{1}{n}\right) + 0\left(\frac{1}{n}\right) + \frac{1}{n-2}\left(1 - \frac{2}{n}\right) \\ &= \frac{1}{n-1}. \end{aligned}$$

So, the probability of door 1, after plugging the results into Bayes' Theorem, reveals that

$$P(C_1|M_i) = \frac{P(C_1)P(M_i|C_1)}{P(M_i)} = \frac{(\frac{1}{n})(\frac{1}{n-1})}{\frac{1}{n-1}} = \frac{1}{n}.$$

This result is the base case of a proof by induction.

Proof. Assume Monty has eliminated x doors, $0 \leq x \leq n-3$, also the contestant has stuck with door 1 throughout the game. By inductive hypothesis, assume $P(C_1) = \frac{1}{n}$. Monty will open another goat-concealing door, j. We again want to show that the probability of door 1 does not change. By Bayes' theorem, this is equivalent to showing that

$$P(M_j|C_1) = P(M_j).$$

There are $n-x-1$ doors remaining in play, their probabilities sum to $\frac{n-1}{n}$. Monty chooses randomly among the empty doors, so:

$$P(C_j) = \frac{n-1}{n(n-x-1)} \text{ and } P(M_j|C_1) = \frac{1}{n-x-1}.$$

We use the Law of Total Probability as we did before,

$$\begin{aligned}
P(M_j) &= P(M_j|C_1)P(C_1) + P(M_j|\bar{C_1} \cap \bar{C_j})P(\bar{C_1} \cap \bar{C_j}) \\
&= \frac{1}{n-x-1}\left(\frac{1}{n}\right) + \frac{1}{n-x-2}\left(1 - \frac{1}{n} - \frac{n-1}{n(n-x-1)}\right) \\
&= \frac{1}{n-x-1}.
\end{aligned}$$

Therefore, we have $P(M_j|C_1) = P(M_j)$, as desired. □

The effect of Monty's random door opening is to redistribute the entire $\frac{n-1}{n}$ probability equally over the remaining doors different from our initial choice, which will remain with probability $\frac{1}{n}$ throughout the game space.This means that there is a probability of $\frac{n-1}{n}$ that the car is somewhere else. Therefore, there is an advantage to switching.

6.3.1 Alternative Solution

The following proof shows that the probability of the contestant's initial selection of door 1 does not change throughout the game if the contestant decides to stick with the original door. That is, Monty's action of opening some door other than the contestant's original choice will not affect the contestant. Therefore, we seek to prove that

$$P(C_1|M_i) = \frac{P(C_1)P(M_i|C_1)}{P(M_i)} = \frac{1}{n}$$

for $n \geq 3$, by showing that the events C_1 and M_i are independent, that is, $P(M_i|C_1) = P(M_i)$.

Proof. Assume that x doors remain, and that the contestant sticks with their initial choice, door 1. We have to show that $P(M_i|C_1) = P(M_i)$, where $i \neq 1$. According to the Law of Total Probability, we need all the terms to be in the form $P(M_i|C_j)P(C_j)$, except where $i = j$ because Monty does not open a door that conceals a car.

We have the case where $j = 1$,

$$P(M_i|C_1) = \frac{1}{x-1}.$$

Let p be that probability of door 1 and assume that the remaining doors all have the same probability. When $j \neq 1$, we have,

$$P(M_i|C_j) = \frac{1-p}{x-1}.$$

There are $x - 1$ doors different from door 1, which collectively have probability of $1 - p$, therefore,

$$P(C_j) = \frac{1-p}{x-1}.$$

It follows, from the Law of Total Probability that,

$$P(M_i) = \frac{p}{x-1} + (x-2)\left(\frac{1}{x-2}\right)\left(\frac{1-p}{x-1}\right) = \frac{1}{x-1} = P(M_i|C_1).$$

Therefore, since $P(M_i) = P(M_i|C_1)$, door 1 has probability $\frac{1}{n}$ at any time of the game and the remaining doors also have the same probability. □

From this proof, we find again that the probability of the initial choice does not change at any stage of the game, unless the contestant decides to switch to a different door.

6.4 Many Doors, Many Cars

This following variation of the Monty Hall Problem expands on the situation in which there are more than three doors [1]. This scenario follows the same process as in the previous subsection. However, now there are $n \geq 3$ doors concealing $1 \leq j \leq n-2$ cars and $n-j$ goats. The contestant's initial choice conceals a car with probability $\frac{j}{n}$. From our previous section, we know that this probability will not change if the contestant decides to follow a the sticking strategy.

Let us now look at the probability of winning as a result of switching. Denote F_c and F_g the events that the contestant's first choice is a car or a goat, respectively. Denote S_c and S_g as the contestant's second choice being a car or goat, respectively. By the Law of Total Probability,

$$P_{switch} = P(F_g)P(S_c|F_g) + P(F_c)P(S_c|F_c).$$

We had already established that $P(F_c) = \frac{j}{n}$, which means that $P(F_g) = \frac{n-j}{n}$.

To find the conditional probabilities of the remaining events is slightly more complicated since it depends on whether Monty reveals a car or a goat.

If Monty reveals a goat, then there will still be $n-2$ doors available, of which j of them conceal cars and $n-j-1$ of them conceal goats. So,

$$P(S_c|F_g) = \frac{j}{n-2} \text{ and } P(S_c|F_c) = \frac{j-1}{n-2}.$$

Plugging those results to the Law of Total Probability,

$$P_{switch} = \frac{n-j}{n}\left(\frac{j}{n-2}\right) + \frac{j}{n}\left(\frac{j-1}{n-2}\right).$$

We notice that $\frac{n-1}{n-2} > 1$, meaning that the chances of winning are increased by switching.

If Monty reveals one of the j cars, for $j \geq 2$, we have

$$P(S_c|F_g) = \frac{j-1}{n-2} \text{ and } P(S_c|F_c) = \frac{j-2}{n-2}.$$

Again, we plug our results to the Law of Total Probability,

$$P_{switch} = \frac{n-j}{n}\left(\frac{j-1}{n-2}\right) + \frac{j}{n}\left(\frac{j-2}{n-2}\right).$$

Through some equation manipulation we find that this quantity is smaller than $\frac{j}{n}$, which was the probability of winning by sticking. In this case the contestant would do better by sticking to their original choice.

6.5 Random Car Placement

Another variation of the Monty Hall Problem that we can consider is the random placement of the car, that is, all doors are equiprobable, with probability summing to 1. Let us now suppose that Monty's strategies remain the same to the classical Monty Hall problem, but the car is place behind door 1 with probability p_1, behind door 2 with probability p_2, and behind door 3 with probability p_3. Without loss of generality, $p_1 \leq p_2 \leq p_3$. Assume that the contestant opens door i, the probability that the contestant is correct can be figured out using Bayes' theorem and the law of total probability,

$$\begin{aligned}
P(C_i|M_j) &= \frac{P(C_i)P(M_j|C_i)}{P(M_j)} \\
&= \frac{p_i \cdot \frac{1}{2}}{p_i(\frac{1}{2}) + (1-p_i)(\frac{1}{2})} \\
&= p_i.
\end{aligned}$$

That is, the probability of winning if the contestant does not switch depends on the door that they initially choose, it increases from door to door, $p_1 \leq p_2 \leq p_3$, so the contestant has advantage if they choose door 3. That means that the probability of winning if the contestant switches is $1-p_1 \geq 1-p_2 \geq 1-p_3$. This implies that if the contestant switches, they have better advantage of winning if they initially choose door 1 and then switch.

Suppose that the contestant chooses door 3, then Monty shows door 2, and the contestant decides to stick with door 3. Then they have a probability p_3 of winning. If the contestant had decided to switch, suppose that they have chosen door 1 initially, then they will have a probability of $1-p_1$ of winning if they do switch. We again have to figure out the best approach for this problem. So far, we have seen that there has been an advantage to switching, so let us say that $1-p_1 \geq p_3$. We know that

$$p_1 + p_2 + p_3 = 1$$

when $p_2 = 0$, then it is the case that $p_1 = 0$, so in this case, the probability of switching from door 1 still wins with probability 1, so there is not an advantage of switching or sticking. Now consider the case where $p_2 \neq 0$, so

$$p_1 + p_3 < 1$$

$$p_3 < 1 - p_1.$$

As we can see from the above inequality, it is strictly a greater chance of choosing door 1 and switching, than choosing door 3 and sticking.

6.6 Two Contestants

Let us now consider the case in which we have two contestants participating in the game. In this variation, we will still have three doors, but now, we will have contestant 1 and contestant 2. The game proceeds as follows:

> *There are three doors and two contestants. Contestant 1 chooses a door, and then contestant 2 chooses a different door. If both contestants choose goats, then one is eliminated at random. If one chose a door with the car behind it, then the other is eliminated. Monty then opens the door chosen by the eliminated contestant [1].*

The following are the different scenarios to this version of the problem:

- Contestant 1 selects the car. Monty eliminates contestant 2. Switching loses.
- Contestant 2 selects the car. Monty eliminates contestant 1. Switching loses.
- Neither contestant selects the car. One contestant is eliminated at random. Switching wins.

We know that these different scenarios each happen $\frac{1}{3}$ of the time. Switching loses $\frac{2}{3}$ of the time. Therefore this is a case where switching is not advantageous. This makes sense because, thinking back to a single-contestant game, the only door that remained with $\frac{1}{3}$ probability of concealing the car is the door that cannot be opened by Monty. Therefore, the $\frac{2}{3}$ probability of concealing the car is for those doors that can be opened by Monty, mainly the door that the contestants choose. Therefore, the contestant will win $\frac{2}{3}$ of the time if they stick with their initial choice.

7 Applications of Monty Hall Problem

The Monty Hall Problem has not only sparked interest among mathematicians and statisticians, it has also presented itself intriguing and applicable to other disciplines. We will discuss some applications that have gained recognition in the fields of philosophy, physics, economy, and cognitive science and psychology. These non-mathematical variations were, for the most part, influenced by the fact that the Monty Hall Problem became very famous and int served as a way for other realms of academics to become part of the dialogue that was caused by this paradox.

7.1 Philosophy

"Philosophers found connections between the Monty Hall Problem and various long-standing problems in their own discipline" [1]. One of those "long-standing" problems that Rosenhouse is referring to is the single-case probability. Single-case probabilities are conceived as logical constructs, rather than physical realities in which probability statements apply directly to individual events [3]. Baumann Peter, a professor of philosophy, notices that the single-case probability, which was coined by K.R. Popper in 1957, could be exemplified utilizing Monty Hall Problem [3].

Baumann basically proposed that there is no answer to the question of what the rational player should do in an isolated case, at least no probabilistic answer [3]. He argues that for there to be an answer there has to be a series of games, in other words, looking at a single case does not do much in terms of whether the contestant should switch or not. Baumann even tries to make it clear to other philosophers, who responded to one of his writing about the Monty Hall Problem, that switching would only be admissible in a series of games, not in a single-case as many argue [3].

7.2 Physics

On a different realm, physicists devised a quantum mechanical version of the Monty Hall Problem. In [4] they quantize the parts of the problem that can be quantized, the prize, the contestant, and their choice of door. They start off by lying the the main quantum variable, the position of the prize, in a 3-dimensional space, H, and call it the *game space*. So, opening a door would correspond to a measurement along a one-dimensional projection on H. The game proceeds closely analogous to the classical version of the Monty Hall Problem:

- The game space system is prepared quantum mechanically in another system called notepad, and denote Monty as Q.
- The contestant chooses a one dimensional projection p on H.
- Monty, Q, chooses a one dimensional projection q, but recall that Monty chooses another door different from p and he must not reveal the prize. That gives the two-dimensional space $(1 - q)H$.
- The contestant is now given the opportunity to chose a one-dimensional projection p' on $(1 - q)H$.

In this quantized version of Monty Hall Problem, the goal or main question is how the player should choose the projection p' in order to maximize the chances of winning.

7.3 Economics

Economists compared the relevance of the Monty Hall Problem to the problems of human decision-making in competitive environments. In [5], it claims that failures observed in decision-making tasks all have the same roots. Competitive decision-makers tend to fail to properly consider all the information needed to solve a problem. Using what economist call "protocol analysis," they show that competitive decision-makers tend to focus on their own goals, to the exclusion of other parties, in the Monty Hall problem, the rules of the game, and the interaction among the parties in light of these rules [5].

Economists contributed to the study of how people understand competitive environments by exploring how negotiators' limited focus of attention can lead to systematic errors in competitive contexts [5]. As was discussed earlier in this paper that many highly educated individuals were incredulous of the correct answer to the Monty Hall Problem, they were not able to attain grasp of the information given to them. The answer to the Monty Hall Problem is systematic, but many just thought of their "own goals," disregarding a protocol analysis [5].

7.4 Cognitive Science and Psychology

Cognitive scientists and psychologists were also interested by the Monty Hall Problem. They tried to determine why, exactly, people have so much trouble understanding this problem [1]. There have been numerous experiments done in regards to human interaction with the Monty Hall Problem, and the dilemma that they find themselves when given the option of switching or sticking from their original chosen door.

In a particular study, researchers Burns and Wieth [6] found that about 14.5 percent of study participants chose to switch when given the opportunity. They blame this on the "well-known human proclivity" that a negative consequence incurred by inaction hurts less than the same negative consequence incurred through some definite action [2]. It is to no surprise why many renowned mathematicians, professors, and many others were unable to comprehend the mathematical, more technical answer to the Monty Hall Problem.

8 Conclusion

The Monty Hall Problem is a very applicable problem not only in mathematics (statistics and probability theory), but in different realms of academia. Rosenhouse [1] continues the list of Monty Hall Problem applications to game theory, computer scientist, law, and of course, education. Our research and understanding of this problem is just the surface, like mentioned before, there are many variations, it is a matter of time for more investigating to be able to dive into them. There is probably still room for more variations among all the work done with this problem, specially the cases where it is not advantageous to switch.

9 Acknowledgements

This paper would not have been possible without the immense support of Professor Patrick Keef. Thank you for helping me choose this interesting topic and guiding me through the writing process. I would also like to thank my peer editor Sarah Rothschild, for going far and beyond with meticulous commentary on each of my drafts. Again, thank you both both for your support.

References

[1] Jason Rosenhouse. *The Monty Hall Problem: The Remarkable Story of Math's Most Contentious Brain Teaser*. Oxford University Press, Inc., New York, New York, 2009.

[2] Stephen Lucas, Jason Rosenhouse, and Andrew Schepler. *The Monty Hall Problem, Rediscovered*. Mathematics Magazine, 82(5):332-342, 2009.

[3] Peter Baumann. "Single-case probabilities and the case of Monty Hall: Levy's View," *syntheses*, Vol. 162, No. 2, May 2008. pp. 265-273.

[4] G. M. D'Arioano, R.D. Gill, M. Keyl, R.F. Werner, B. Kummer, H. Maassen, "The Quantum Monty Hall Problem," *Quantum Information and Computation*, Vol. 2, 2002, pp. 355-366.

[5] A. Tor, M. Bazerman, "Focusing Failures in Competitive Environments: Explaining Decision Errors in the Monty Hall Game, the Acquiring a Company Problem, and Multiparty Ultimatums," *Journal of Behavioral Decision Making*, Vol. 16, 2003, pp. 353-374.

[6] B. Burnsand M. Wieth. "The collider principle in casual reasoning: why the Monty Hall Problem is so hard," *Journal of Experimental Psychology, Genral*, Vol. 103, No. 3, 2004, pp. 436-449.