Bounded Derivatives Which Are Not Riemann Integrable

by

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Certificate of Approval

This is to certify that the accompanying thesis by Elliot M. Granath has been accepted in partial fulfillment of the requirements for graduation with Honors in Mathematics.

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Abstract

In an elementary calculus course, we talk mostly, or exclusively, about integrating continuous, real-valued functions. Since continuous functions on closed intervals are integrable, the Fundamental Theorem of Calculus gives us a method to calculate these integrals (given that we can find an antiderivative). Furthermore, the Fundamental Theorem of Calculus states that the integral can be used to define an antiderivative of a continuous function. In this paper, we will present a method for establishing the existence of antiderivatives of continuous functions without using any integration theory. In addition, we will explore the potentially counter-intuitive topic of derivatives which are not Riemann integrable. It is easy to find a function whose derivative is unbounded, and thus not Riemann integrable; what is more surprising is that even bounded derivatives are not necessarily Riemann integrable. We will present two classes of functions, one conceived by Volterra and one by Pompeiu, which are differentiable on closed intervals, and whose derivatives are not Riemann integrable. Finally, we will develop the Henstock integral as a tool which integrates all derivatives.

1 Finding an antiderivative of a continuous function without integration

When first learning about differentiation and integration, it is tempting to think of "antiderivative" and "integral" as synonymous. In some sense, differentiation "undoes" integration, and *vice versa*. Of course, the Fundamental Theorem of Calculus canonizes this relationship: if $f\colon [a,b] \to \mathbb{R}$ is integrable and has an antiderivative F, then we can calculate its integral as $F(b) - F(a)$. Moreover, the Fundamental Theorems states that, if f is continuous, then $\int_a^x f$ is an antiderivative of f. This shows that functions such as e^{-x^2} indeed have antiderivatives, even if they are impossible to write using elementary functions.

Given the intimate relationship between integrals and derivatives, it is perhaps interesting that we can establish the existence of antiderivatives of continuous functions without relying on any integration theory. Lebesgue gave a proof of this fact [2]; it is recorded below in a modern format.

1.1 Lebesgue's method of antidifferentiation

We prove that any continuous function $f\colon [a,b] \to \mathbb{R}$ has an antiderivative by estimating f using continuous piecewise linear functions. With some care, we can find an antiderivative of each of these functions without invoking the Fundamental Theorem of Calculus. Finally, we can show that these piecewise estimations, together with their antiderivatives, can be used to construct an antiderivative of f. First, it requires proof that continuous piecewise linear functions have antiderivatives. Figure 1 gives us an intuition for how to construct such an antiderivative.

First, we begin with some useful definitions.

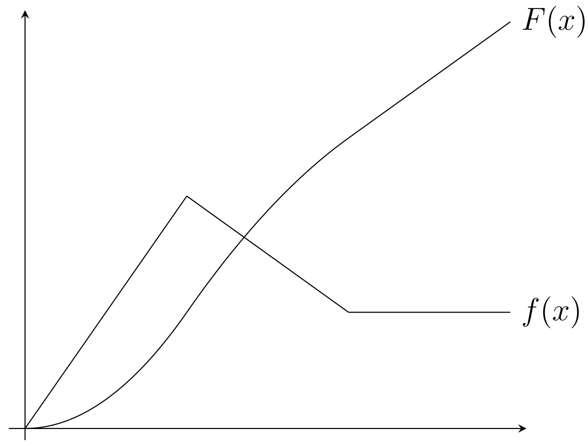

Figure 1: A piecewise linear, continuous function together with an antiderivative we can find without using the Fundamental Theorem of Calculus. Note that F is still differentiable at the points where f has a cusp.

Definition 1.1. A *partition* of $[a, b]$ is a set $P = \{x_0, x_1, \ldots, x_n\}$ such that $x_i < x_{i+1}$, $a = x_0$, and $b = x_n$. The *norm* of a partition is denoted $||P||$, where $||P|| = \max\{x_i - x_{i-1} : 1 \le i \le n\}$. □

Definition 1.2. A function $f\colon [a, b] \to \mathbb{R}$ is piecewise linear if there is some partition $\{x_0, \ldots, x_n\}$ of $[a, b]$ such that, for any $1 \le i \le n$, f is of the form $a_i x + b_i$ for all $x \in [x_{i-1}, x_i]$, where a_i, b_i are real numbers. □

Theorem 1.3. *If $f\colon [a, b] \to \mathbb{R}$ is continuous and piecewise linear on $[a, b]$, then there exists a function $F\colon [a, b] \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in [a, b]$ (one-sided limits are used at the endpoints).*

Proof. We proceed by induction on the number of linear segments of f. If f has one linear segment, i.e. if $f(x) = c_1 x + c_2$ for $x \in [a, b]$, then the function $F(x) = c_1 x^2/2 + c_2 x$ is an antiderivative for f on $[a, b]$. Our induction hypothesis is that a continuous, piecewise linear function defined on a closed interval with k linear segments has an antiderivative on that closed interval.

Consider the function

$$f(x) = \begin{cases} g(x), & \text{if } a \le x < c; \\ d_1x + d_2, & \text{if } c \le x \le b, \end{cases}$$

where d_1, d_2 are constants, $g(c) = d_1c + d_2$, and g is a continuous, piecewise linear function with k linear segments on $[a, c]$. By the induction hypothesis, g has an antiderivative G on $[a, c]$. We claim that the function

$$F(x) = \begin{cases} G(x), & \text{if } a \le x < c, \\ \frac{1}{2}d_1x^2 + d_2x - \left(\frac{1}{2}d_1c^2 + d_2c\right) + G(c), & \text{if } c \le x < b, \end{cases}$$

is an antiderivative for f on $[a, b]$. This is clearly true on $[a, b] \setminus \{c\}$, so it remains to show that $F'(c) = f(c)$. Note that

$$\begin{aligned} \lim_{h\to 0^+} \frac{F(c+h) - F(c)}{h} &= \lim_{h\to 0^+} \frac{\frac{1}{2}d_1(c+h)^2 + d_2(c+h) - \frac{1}{2}d_1c^2 - d_2c}{h} \\ &= \lim_{h\to 0^+} \frac{d_1ch + \frac{1}{2}d_1h^2 + d_2h}{h} \\ &= d_1c + d_2, \end{aligned}$$

and on the other hand

$$\begin{aligned} \lim_{h\to 0^-} \frac{F(c+h) - F(c)}{h} &= \lim_{h\to 0^-} \frac{G(c+h) - G(c)}{h} \\ &= G'(c) \\ &= d_1c + d_2. \end{aligned}$$

Thus F is indeed an antiderivative for f on $[a, b]$. By induction, any continuous piecewise linear function on a closed interval has an antiderivative. $\square$

Having established that continuous piecewise linear functions have antiderivatives, we claim that any continuous function on a closed interval can be estimated with a sequence of such functions. Figure 2 shows a simple continuous function, together with a linear approximation in six parts. As our intuition suggests, we can choose increasingly fine linear approximations that will converge to the desired function. Using the Mean Value Theorem, we can show that the corresponding sequence of antiderivatives converges to an antiderivative of f.

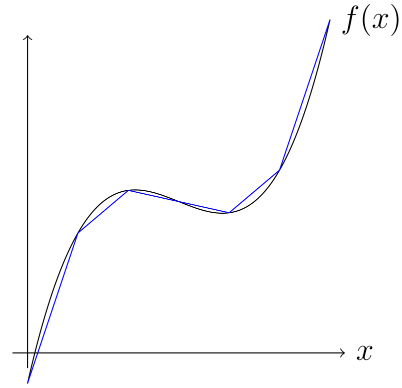

Figure 2: We may approximate any continuous function using a piecewise linear continuous function.

Theorem 1.4. *If $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then there exists a sequence $\{f_n\}$ of continuous piecewise linear functions that converges uniformly to f on $[a,b]$.*

Proof. Since f is continuous on a closed and bounded interval, f is also uniformly continuous on $[a,b]$. Thus, for each positive integer n, there exists $\delta_n > 0$ such that if $|x-y| < \delta_n$, then $|f(x) - f(y)| < 1/n$ for all $x, y \in [a,b]$. Without loss of generality, $\{\delta_n\}$ is decreasing. Let $P_n = \{x_0, \dots, x_{k_n}\}$ be a partition of $[a,b]$ with norm less than δ_n and, for each n, define f_n as a piecewise linear function joining the points $(x_i, f(x_i))$ where $0 \leq i \leq k_n$.

We claim that $\{f_n\}$ converges uniformly to f on $[a,b]$. Let $\epsilon > 0$ and pick N such that $1/N < \epsilon/2$. Let $x \in [a,b]$, and identify an interval $[x_i, x_{i+1}]$ containing x. Then $|x_i - x| < \delta_n$, and without loss of generality, for all $n \geq N$, $\delta_n < \delta_N$. Also note that, for all $1 \leq i \leq n$, if $x, y \in [x_{i-1}, x_i]$, then $|f(y) - f(x)| \leq \max\{x : x \in [x_{i-1}, x_i]\} - \min\{x \in [x_{i-1}, x_i]\} = |f(x_i) - f(x_{i-1})|$. Using this fact together with the uniform continuity of f,

$$\begin{aligned}
|f_n(x) - f(x)| &\leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f(x)| \\
&= |f_n(x) - f_n(x_i)| + |f(x_i) - f(x)| \\
&\leq |f_n(x_{i+1}) - f_n(x_i)| + |f(x_i) - f(x)| \\
&= |f(x_{i+1}) - f(x_i)| + |f(x_i) - f(x)| \\
&< 1/N + 1/N \\
&< \epsilon.
\end{aligned}$$

It follows that $\{f_n\}$ converges uniformly to f on $[a,b]$. $\square$

In general, it is not necessarily true that $\lim\limits_{x \to x_0} \lim\limits_{y \to y_0} f(x,y) = \lim\limits_{y \to y_0} \lim\limits_{x \to x_0} f(x,y)$. For example,

$$1 = \lim_{x \to 0} \lim_{y \to 0} \frac{x+2y}{x+y} \neq \lim_{y \to 0} \lim_{x \to 0} \frac{x+2y}{x+y} = 2.$$

Before continuing, we will establish Lemma 1.5, which shows that, in the right circumstances, we may simply interchange limits. In the same way that uniform continuity preserves continuity, it also preserves limits.

Lemma 1.5. *Let $\{f_n\}$ be a sequence of functions that converges uniformly to f on $[a,b] \setminus \{c\}$ where $c \in [a,b]$. Then, provided each function f_n and f has a limit at c,*

$$\lim_{x \to c} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to c} f_n(x).$$

Proof. Denote $y_n = \lim_{x \to c} f_n(x)$. Since $\{f_n\}$ converges uniformly, given $\epsilon > 0$, there exists N such that if $n, m > N$, then $|f_n(x) - f_m(x)| < \epsilon$ for all points $x \in [a, b] \setminus \{c\}$ (this is the Cauchy Criterion for Uniform Convergence). Then, since $|f_n(x) - f_m(x)| < \epsilon$, we have $|y_n - y_m| = |\lim_{x \to c}(f_n(x) - f_m(x))| \leq \epsilon$. Thus, $\{y_n\}$ is a Cauchy sequence converging to some $y \in \mathbb{R}$.

Let $\epsilon > 0$. By the uniform convergence of f, pick N_1 such that $n \geq N_1$ implies $|f(x) - f_n(x)| < \epsilon$. Pick $N_2 \geq N_1$ such that if $n \geq N_2$, we have $|y_n - y| < \epsilon$. Since $y_{N_2} = \lim_{x \to c} f_{N_2}(x)$, pick $\delta > 0$ such that $|x - c| < \delta$ implies $|f_{N_2}(x) - y_{N_2}| < \epsilon$. If $|x - c| < \delta$, we see that

$$\begin{aligned} |f(x) - y| &\leq |f(x) - f_{N_2}(x)| + |f_{N_2}(x) - y_{N_2}| + |y_{N_2} - y| \\ &< 3\epsilon. \end{aligned}$$

Thus $\lim_{x \to c} f(x) = y = \lim_{n \to \infty} y_n$, as desired. $\square$

Perhaps contrary to our intuition, it is possible to give a sequence of functions $\{F_n\}$ which converges uniformly to F, and whose derivatives f_n converge pointwise to f, but where F is not an antiderivative of f.

Example 1.6. This example is adapted from Gordon [4]. For each $n \in \mathbb{Z}^+$, define $F_n \colon [0, 1] \to \mathbb{R}$ where

$$F_n(x) = \begin{cases} x, & \text{if } x \geq \frac{1}{n}, \\ \frac{3}{2}nx^2 - \frac{1}{2}n^3x^4, & \text{if } x < \frac{1}{n}. \end{cases}$$

It is clear that each F_n is differentiable away from $1/n$. At $x = 1/n$, the right-sided limit of F_n is $1/n$ and the left-sided limit is $\frac{3}{2}\left(\frac{1}{n}\right) - \frac{1}{2}\left(\frac{1}{n}\right) = \frac{1}{n}$.

Furthermore, the right-sided derivative is 1 and the left-sided derivative is $3n\left(\frac{1}{n}\right) - 2n^3\left(\frac{1}{n}\right)^3 = 1$, so F_n is differentiable on $[0,1]$. For all $n > 1$, if $x \in [0, 1/n]$,

$$\begin{aligned}
|F_n(x) - x| &\le \left|\frac{3}{2}nx^2\right| + \left|\frac{1}{2}n^3x^4\right| + |x| \\
&\le \frac{3}{2n} + \frac{1}{2n} + \frac{2}{2n} \\
&= \frac{3}{n}.
\end{aligned}$$

Since $3/n$ converges to 0, it follows that $\{F_n\}$ converges uniformly to x on $[0,1]$. Furthermore, for each n, $F_n'(0) = 0$. If $x \in (0,1]$, then there is some N such that, if $n > N$, then $F_n'(x) = x$. Thus $\{F_n'\}$ converges pointwise to the function $f\colon [0,1] \to \mathbb{R}$ where $f(0) = 0$ and $f(x) = 1$ for all $x \in (0,1]$. However, $F'(x) = 1$ for all $x \in [0,1]$, and thus $F' \neq f$. (Additionally, since f does not have the intermediate value property, it has no antiderivative). $\square$

It thus requires proof that our sequence of functions which estimate an antiderivative for a function indeed converges to the desired antiderivative.

Theorem 1.7. *Let $\{F_n\}$ be a sequence of functions that are differentiable on $[a,b]$, and suppose that $\{F_n(c)\}$ converges for some $c \in [a,b]$. If $\{F_n'\}$ converges uniformly to f on $[a,b]$, then $\{F_n\}$ converges uniformly to a function F on $[a,b]$. Furthermore, F is differentiable on $[a,b]$ and $F'(x) = f(x)$ for all $x \in [a,b]$.*

Proof. (Rudin [5]) Let $\epsilon > 0$. Choose N such that if $n, m \ge N$, then we have $|F_n(c) - F_m(c)| < \epsilon$ and $|F_n'(x) - F_m'(x)| < \epsilon$ for all $x \in [a,b]$. Let $x, y \in [a,b]$

where $x < y$. By the Mean Value Theorem, for some $d \in (x, y)$,

$$|(F_n - F_m)(x) - (F_n - F_m)(y)| = |x - y| \left| \frac{(F_n - F_m)(x) - (F_n - F_m)(y)}{x - y} \right| \tag{1}$$

$$= |x - y||(F_n - F_m)'(d)|$$

$$< |x - y|\epsilon$$

$$\leq \epsilon(b - a).$$

Thus, if $m, n \geq N$, for any $x \in [a, b]$,

$$\begin{aligned} |F_n(x) - F_m(x)| &\leq |(F_n - F_m)(x) - (F_n - F_m)(c)| + |F_n(c) - F_m(c)| \\ &< \epsilon(b - a) + \epsilon \\ &= \epsilon(b - a + 1). \end{aligned}$$

Thus $\{F_n\}$ converges uniformly to some F on $[a, b]$. Fix $c \in [a, b]$ and define the functions

$$\phi_n(x) = \frac{F_n(x) - F_n(c)}{x - c}; \qquad \phi(x) = \frac{F(x) - F(c)}{x - c},$$

where $x \neq c$. Similar to (1), for any $n, m \geq N$,

$$|\phi_n(x) - \phi_m(x)| = \left| \frac{(F_n - F_m)(x) - (F_n - F_m)(c)}{x - c} \right| < \epsilon$$

whenever $x \neq c$. Since $\{\phi_n\}$ converges pointwise to ϕ on $[a, b] \backslash \{c\}$, it converges uniformly to ϕ on $[a,b]\setminus\{c\}$. By Lemma 1.5,

$$\begin{aligned}F'(c) &= \lim_{x\to c}\lim_{n\to\infty}\phi_n(x)\\ &= \lim_{n\to\infty}\lim_{x\to c}\phi_n(x)\\ &= \lim_{n\to\infty} f_n(c)\\ &= f(c).\end{aligned}$$

Since $c\in[a,b]$ was arbitrary, we have shown that $F'=f$ on $[a,b]$. □

We now proceed to establish the existence of antiderivatives of continuous functions without the use of the Fundamental Theorem of Calculus or integration theory.

Theorem 1.8. *If $f\colon[a,b]\to\mathbb{R}$ is continuous on $[a,b]$, then f has an antiderivative on $[a,b]$.*

Proof. Let f be a continuous function on $[a,b]$. By Theorem 1.4, there exists a sequence $\{f_n\}$ of continuous piecewise linear functions that converges uniformly to f on $[a,b]$. By Theorem 1.3, there exists a sequence $\{F_n\}$ of functions on $[a,b]$ such that $F_n'(x)=f_n(x)$ for all $x\in[a,b]$. For convenience, choose each F_n such that $F_n(a)=0$. Then by Theorem 1.7, $\{F_n\}$ converges uniformly to a function F on $[a,b]$, F is differentiable on $[a,b]$, and $F'(x)=f(x)$ for all $x\in[a,b]$. □

1.2 A version of the Fundamental Theorem of Calculus

Having established that we can find an antiderivative of a continuous function $f\colon[a,b]\to\mathbb{R}$ on $[a,b]$, we can now prove a version of the Fundamental Theorem of Calculus. A *tagged partition* of a closed interval $[a,b]$ is a finite set ${}^tP=\ \{(t_i,[x_{i-1},x_i]) : 1\le i\le n\}$, where $a=x_0$, $x_n=b$, and $x_{i-1}<x_i$ for

$1 \le i \le n$; additionally, $t_i \in [x_{i-1}, x_i]$ for $1 \le i \le n$. Denote the *norm* of a tagged partition ${}^tP$ by $||{}^tP|| = \max\{x_i - x_{i-1} : 1 \le i \le n\}$. Finally, define the *Riemann sum* of f with tagged partition ${}^tP$ by

$$S(f, {}^tP) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

We will use the following definition of what it means for a function to be Riemann integrable. A similar definition applies for functions $f\colon I \to \mathbb{R}$ where I is an interval, but we are primarily concerned with closed intervals here.

Definition 1.9. A function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ if there is a number L satisfying the following: for each $\epsilon > 0$, there is $\delta > 0$ for which, given any tagged partition ${}^tP$ of $[a,b]$ with $||{}^tP|| < \delta$, we have $|S(f, {}^tP) - L| < \epsilon$. We denote the Riemann integral of f as $\int_a^b f$. □

Theorem 1.10 (Fundamental Theorem of Calculus)**.** *If $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then f has an antiderivative F, f is Riemann integrable, and $\int_a^b f = F(b) - F(a)$.*

Proof. By Theorem 1.8, f has an antiderivative F on $[a,b]$. Let $\epsilon > 0$. Since f is uniformly continuous, there is $\delta > 0$ such that, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let the set ${}^tP = \{([x_{i-1}, x_i], t_i) : 1 \le i \le n\}$ be a partition of $[a,b]$ where $||{}^tP|| < \delta$. By the Mean Value Theorem, for each $i \in \{1, \ldots, n\}$, there is $s_i \in (x_{i-1}, x_i)$ satisfying $F(x_i) - F(x_{i-1}) = f(s_i)(x_i - x_{i-1})$. Then, for each $i \in \{1, \ldots, n\}$, we have

$$\begin{aligned} |f(t_i)(x_i - x_{i-1}) - (F(x_i) - F(x_{i-1}))| &= |f(t_i)(x_i - x_{i-1}) - f(s_i)(x_i - x_{i-1})| \\ &= |(f(t_i) - f(s_i))(x_i - x_{i-1})| \\ &< \epsilon(x_i - x_{i-1}). \end{aligned}$$

It follows that

$$\begin{aligned}|S(f,{}^tP) - (F(b) - F(a))| &= \left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - \sum_{i=1}^{n}(F(x_i) - F(x_{i-1}))\right| \\ &\leq \sum_{i=1}^{n} |f(t_i)(x_i - x_{i-1}) - (F(x_i) - F(x_{i-1}))| \\ &< \sum_{i=1}^{n} \epsilon(x_i - x_{i-1}) \\ &= \epsilon(b - a).\end{aligned}$$

Thus, f is Riemann integrable on $[a, b]$ and $\int_a^b f = F(b) - F(a)$. □

Remark. Although Theorem 1.10 allows us to integrate any continuous function on a closed interval, it is weaker than the version which states "if f is Riemann integrable on $[a, b]$ and has an antiderivative F on $[a, b]$, then $\int_a^b f = F(b) - F(a)$." The difference may seem negligible, but Theorem 1.10 does not allow us to integrate the classes of functions which are not continuous on $[a, b]$ but which have an antiderivative on $[a, b]$. As an example, consider the function $f(x) = x^2 \sin \frac{\pi}{x}$ if $x \in (0, 1]$ and $f(0) = 0$. Then, with the details omitted, f is differentiable on $[0, 1]$, f' is not continuous on $[0, 1]$, and f' is Riemann integrable on $[0, 1]$. In this case we have $\int_0^1 f' = 0$.

1.3 Use of the Mean Value Theorem

In the proof of Theorem 1.7, we used the Mean Value Theorem to show that, if a sequence of functions $\{f_n\}$ uniformly converges to f, then an appropriately chosen corresponding sequence of antiderivatives $\{F_n\}$ uniformly converges to F where $F' = f$. However, we can provide a similar version of Rudin's proof without using the Mean Value Theorem. We use instead the following result.

Theorem 1.11. *If f is a piecewise linear, continuous function on $[a, b]$ and*

F is an antiderivative for f on $[a,b]$, then

$$|F(b) - F(a)| \leq (b-a)\max\{|f(x)| : x \in [a,b]\}.$$

Proof. We proceed again by induction on the number of linear terms of f. First, suppose $f\colon [a,b] \to \mathbb{R}$ is linear and $f(x) = mx+n$. Then the maximum value of $|f(x)|$ on $[a,b]$ is either $|ma+n|$ or $|mb+n|$. Without loss of generality, assume $|f(x)| \leq |mb+n|$ on $[a,b]$. If F is an antiderivative of f on $[a,b]$, then $F(b) - F(a) = \frac{1}{2}m(b^2-a^2) + n(b-a)$. Then

$$\begin{aligned} |F(b) - F(a)| &= (b-a)\left|\frac{1}{2}m(a+b) + n\right| \\ &\leq (b-a) \cdot \frac{1}{2} \cdot (|ma+n| + |mb+n|) \\ &= (b-a)|mb+n| \end{aligned}$$

as desired. Our induction hypothesis is that, if g is a piecewise linear, continuous function on a closed interval, say, $[a,c]$, and G has k linear pieces, then

$$|G(c) - G(a)| \leq (c-a)\max\{|g(x)| : x \in [a,c]\}$$

where G is an antiderivative of g.

Now consider a function $f\colon [a,b] \to \mathbb{R}$, where $a < c < b$, with $k+1$ linear parts defined by

$$f(x) = \begin{cases} g(x), & \text{if } a \leq x \leq c; \\ mx+n, & \text{if } c < x \leq b. \end{cases}$$

Then

$$\begin{aligned}
|F(b) - F(a)| &\leq |F(b) - F(c)| + |F(c) - F(a)| \\
&\leq (b-c)\max\{|f(x)| : x \in [c,b]\} + (c-a)\max\{|f(x)| : x \in [a,c]\} \\
&\leq (b-c)\max\{|f(x)| : x \in [a,b]\} + (c-a)\max\{|f(x)| : x \in [a,b]\} \\
&= (b-a)\max\{|f(x)| : x \in [a,b]\}
\end{aligned}$$

which concludes the proof. □

Although the Mean Value Theorem is often considered an elementary theorem to be taught in any calculus course, avoiding the use of it has some advantages. Although the statement of the Mean Value Theorem is intuitive, its proof is somewhat roundabout. The use of Rolle's Theorem as a lemma is potentially confusing and tedious for some students. Of course, we cannot say whether the Mean Value Theorem or Theorem 1.11 is more understandable to students.

2 Bounded derivatives which are not Riemann integrable

It is tempting to think that the Fundamental Theorem of Calculus proves that every derivative is Riemann integrable. That is, given a differentiable function $f\colon [a,b] \to \mathbb{R}$, a calculus student might assume that the integral of f' over $[a,b]$ must be $f(b) - f(a)$. However, this is only true if f' is integrable in the first place. We can give some simple counterexamples. Take $f(x) = x^{1/3}$ on the interval $(0,1)$. Then, since $f'(x) = \frac{1}{3}x^{-2/3}$ is unbounded on $(0,1)$, it is not Riemann integrable. In a sense, it is false to write $\int_0^1 f' = f(x)|_0^1 = 1$, although we commonly abuse this notation to refer to the so-called "improper integral.” In this case, we might write

$$\int_0^1 f' = \lim_{a\to 0^+} \int_a^1 f' = 1$$

This example uses an open interval, and takes advantage of the fact that f' is unbounded at an endpoint. Suppose we restrict ourselves to functions which are continuous and differentiable on a closed interval $[a, b]$ (with one-sided derivatives at a and b). It turns out that the derivative may still be unbounded. Consider the function $g\colon [-1, 1] \to \mathbb{R}$ where

$$g(x) = \begin{cases} x^k \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

for $1 < k < 2$. Then, away from the origin, we can differentiate g using elementary methods. At the origin, we have

$$\begin{aligned} \lim_{h\to 0} \left(\frac{h^k \sin \frac{1}{h}}{h} \right) &= \lim_{h\to 0} \left(h^{k-1} \sin \frac{1}{h} \right) \\ &= 0. \end{aligned}$$

Thus

$$g'(x) = \begin{cases} kx^{k-1} \sin \frac{1}{x} - x^{k-2} \cos \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

On the domain $[-1, 1]$, g' is defined. However, while the term $kx^{k-1} \sin \frac{1}{x}$ is bounded, $x^{k-2} \cos \frac{1}{x}$ is not. Thus the derivative of g is not Riemann integrable, despite the fact that g is differentiable everywhere.

In both of the examples we just explored, we find a way to ensure that the derivative of a function is unbounded, which shows that the derivative is not Riemann integrable without further thought. What if we add an additional requirement that f' be bounded? Is it always integrable? The answer is still no. The most widely known counter-example was constructed in 1881 by Vito Volterra [6]. We will provide an overview of his function, before focusing rigorously on another example by Dimitrie Pompeiu with certain other properties.

2.1 Volterra's function (an overview)

In this section we will present a non-rigorous overview of Volterra's bounded derivative which is not Riemann integrable. Refer to Gordon for a simplified version of Volterra's function in full detail [3].

To make sense of Volterra's construction of a bounded derivative which is not Riemann integrable, it helps to start with a notion of exactly how we want the derivative to "misbehave." At the start of this section, we noted that it is fairly easy to write down a function whose derivative is unbounded, and thus not integrable. Another common requirement (which is also sufficient) for Riemann integrability is that a function be bounded and continuous almost everywhere. That is, if we can construct a derivative whose set of discontinuities does not have measure zero, then it cannot be integrable. But how can we easily introduce discontinuities in a derivative? All derivatives have the Intermediate Value Property, so we cannot have a derivative that looks like, for example, the indicator function $f\colon [0,1] \to \mathbb{R}$ mapping rationals to 1 and irrationals to 0. Inspired by our earlier example, consider the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then we have

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Note that f' is bounded on a finite interval containing the origin, but f' is not continuous at 0. Now recall the Cantor set, which is constructed by taking the interval $[0, 1]$ and removing the middle third, then removing the middle third of the remaining intervals, and so on. The resulting set has measure zero, so we will use a modified version of the Cantor set that begins with removing the middle fourth of the interval $[0, 1]$, etc. (the details are omitted here). Note that this so-called thick Cantor set does not have measure zero (the sum total of the lengths of intervals removed is $1/2$).

Now, construct a function h as follows. At each step of the creation of our thick Cantor set, we will place two copies of f in each deleted interval, so that h' has discontinuities at each of their endpoints. For example, the first removed interval has length $1/4$, so find the largest value of x in $[0, 1/8]$ such that $f'(x) = 0$ and call it x_0. Define the functions

$$a(x) = \begin{cases} 0, & \text{if } x = 0; \\ x^2 \sin \frac{1}{x}, & \text{if } 0 \leq x \leq x_0; \\ x_0^2 \sin \frac{1}{x_0}, & \text{if } x_0 < x \leq \frac{1}{8}. \end{cases}$$

and define the reflection of a across $x = 1/8$:

$$b(x) = \begin{cases} x_0^2 \sin \frac{1}{x_0}, & \text{if } 1/8 \leq x \leq \frac{1}{8} + x_0; \\ \left(\frac{1}{4} - x\right)^2 \sin \dfrac{1}{\frac{1}{4} - x}, & \text{if } \frac{1}{8} + x_0 < x < \frac{1}{4}; \\ 0, & \text{if } x = \frac{1}{4}. \end{cases}$$

Finally, define the first term in our sequence f_1. Remember that a and b are functions, not constants.

$$f_1(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq \frac{3}{8}; \\ a\left(x - \frac{3}{8}\right), & \text{if } \frac{3}{8} < x \leq \frac{1}{2}; \\ b\left(x - \frac{3}{8}\right), & \text{if } \frac{1}{2} < x \leq \frac{5}{8}; \\ 0, & \text{if } \frac{5}{8} \leq x \leq 1. \end{cases}$$

Although cumbersome to write, Figure 3 provides a clearer picture of what Volterra's function begins to look like. Note that f_1 is differentiable on $(0, 1)$, f_1' is bounded, and f_1' is discontinuous at $3/8$ and $5/8$. We continue this process for each step of the construction of the thick Cantor set, obtaining a limit function f with infinitely many discontinuities. As it turns out, f' is discontinuous at every point of the thick Cantor set, which, as we noted, does not have measure zero. Thus, the function f' is a bounded derivative which is not Riemann integrable.

2.2 Pompeiu's function

In addition to being cumbersome to write and relying on the notion of a perfect nowhere dense set which does not have measure zero, which may be inaccessible to an undergraduate student, Volterra's function is also integrable on

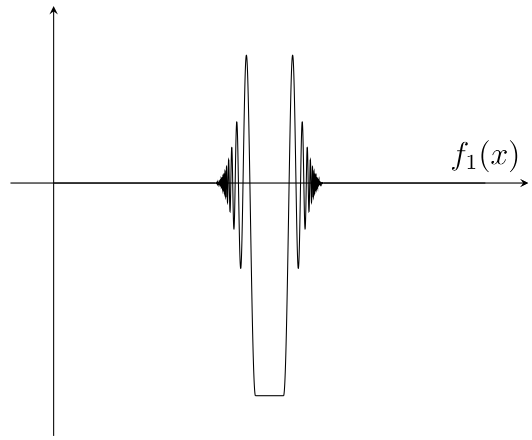

Figure 3: The first term in the construction of Volterra's function. At this scale, there appear to be two non-differentiable cusps, but function is differentiable everywhere.

many intervals. In particular, it is integrable on any interval removed during the construction of the thick Cantor set (more trivially it is integrable on the intervals where it is constant). In this sense, as a counter-example, Volterra's function "fails" on many subintervals of its domain. We turn our attention to another example of a bounded, non-integrable derivative.

In Volterra's function, the guiding idea was to find a function whose set of discontinuities does not have have measure zero. Consider the fact that, if a derivative f' is Riemann integrable on $[a, b]$, then its integral *must* be $f(b) - f(a)$. On the other hand, given a sequence of partitions of $[a, b]$ whose norm converges to 0, we must also have that respective sequence of upper and lower Riemann sums of f' converge to $f(b) - f(a)$. Pompeiu's example hinges on the construction of a positive, strictly increasing function h whose derivative vanishes on a dense subset. In this manner, if h' were integrable, then its integral must be the positive number $h(b) - h(a)$. However, given any partition of its domain, the lower Riemann sum of h' is zero, since $h'(x) = 0$ on a dense subset, and thus the integral must be 0. We conclude that h' is not

Riemann integrable.

The general idea in the construction of our function is to create a positive, strictly increasing function which has a "kink" at each point of a dense subset where the derivative is 0. To achieve this, we will begin with a function whose derivative is ∞ at every rational on $[0, 1]$, and then take its inverse (we define infinite derivative in Section 2.4). Such a function has a derivative which is both bounded and not Riemann integrable on any subinterval of its domain.

This function was originally developed by Pompeiu, and the proof we use here was outlined by Bruckner et al. [1]. Throughout Section 2.2, the meaning of the functions f and h, will not change, nor will the sequences $\{a_k\}$, $\{q_k\}$, and $\{f_k\}$.

2.2.1 Beginning our construction (defining f)

Consider the function referred to at the beginning of Section 2, $p(x) = x^{1/3}$. At the origin, we have

$$\lim_{h\to 0} \frac{p(h)}{h} = \lim_{h\to 0} \frac{h^{1/3}}{h} = \infty$$

and, away from the origin, $p'(x)$ is simply the positive number $\frac{1}{3}x^{-2/3}$. Now let $\{q_k\}$ be a listing of $\mathbb{Q} \cap [0, 1]$, and for each $k \in \mathbb{Z}^+$, define

$$f_k(x) = (x - q_k)^{1/3}.$$

Let $\{a_k\}$ be a sequence of positive numbers such that $\sum\limits_{k=1}^{\infty} a_k^{3/5}$ converges. The reason for the exponent $3/5$ becomes evident in the proof of Theorem 2.2. For now, we simply care that the sum $\sum\limits_{k=1}^{\infty} a_k$ converges. This will be sufficient to prove that the sum $\sum_{k=1}^{\infty} a_k f_k(x)$ converges uniformly on $[0, 1]$.

Lemma 2.1. *If* $\sum_{k=1}^{\infty} a_k$ *converges, then* $\sum_{k=1}^{\infty} a_k f_k(x)$ *converges uniformly on* $[0, 1]$.

Proof. Since each f_k is increasing, it is bounded above by its value at 1. Also, since $(x - q_k)^{1/3} \le x^{1/3}$, we have that $f_k(x) \le 1$ for all $x \in [0, 1]$. In fact, $|f_k(x)| \le 1$ on $[0, 1]$. Since $|a_k f_k(x)| \le |a_k|$ and $\sum_{k=1}^{\infty} a_k$ converges, it follows that $\sum_{k=1}^{\infty} a_k f_k(x)$ converges uniformly on $[0, 1]$ by the Weierstrass M-test. $\square$

We have shown that $\sum_{k=1}^{\infty} a_k f_k(x)$ converges uniformly to some limit function f. Note that since each f_k is continuous, we know that f is continuous on $[0, 1]$. This will be relevant later in Section 2.2.3. Figure 4 may provide some intuition for what f will look like.

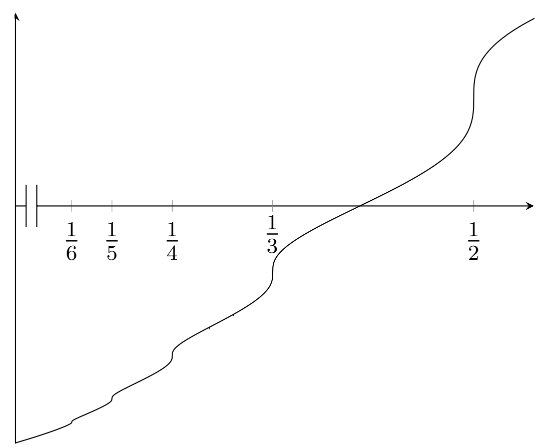

Figure 4: An example of what the fifth partial sum of f may look like. We have picked a convenient geometric series for $\{a_k\}$ and used some arbitrary rational numbers for the first few terms of $\{q_k\}$.

2.2.2 Differentiating our function (determining f')

The intention with the construction of f thus far was to ensure that the derivative of f is ∞ on a dense subset. Knowing that f' "blows up" on a dense subset, the following result is surprising.

Theorem 2.2. *Let $\{a_k\}$ be a sequence of positive numbers such that $\sum\limits_{k=1}^{\infty} a_k^{3/5}$ converges. Then $\sum\limits_{k=1}^{\infty} a_k f_k'(x)$ converges almost everywhere.*

Proof. Let D be the set of points $x \in [a,b]$ where $\sum\limits_{k=1}^{\infty} a_k f_k'(x)$ diverges. Let $\epsilon > 0$ and define the interval

$$A_k = \left(q_k - \epsilon a_k^{3/5}, q_k + \epsilon a_k^{3/5}\right).$$

Let $A = \bigcup\limits_{k=1}^{\infty} A_k$ and note that $\mathbb{Q} \cap [0,1] \subseteq A$. Denoting the length of A_k by $\ell(A_k)$, we have $\ell(A_k) = 2\epsilon a_k^{3/5}$ and

$$\sum_{k=1}^{\infty} \ell(A_k) = 2\epsilon \sum_{k=1}^{\infty} a_k^{3/5} = 2\epsilon L,$$

where L is a real number. Let $c \in [0,1] \setminus A$. Note that $|c - q_k| \geq \epsilon a_k^{3/5}$, so

$$\frac{a_k}{3|c-q_k|^{2/3}} \leq \frac{a_k}{3(\epsilon a_k^{3/5})^{2/3}} = \frac{a_k^{3/5}}{3\epsilon^{2/3}}.$$

Since $\sum\limits_{k=1}^{\infty} \frac{1}{3\epsilon^{2/3}} a_k^{3/5}$ converges and a_k is positive for $k \geq 1$, by the comparison test,

$$\sum_{k=1}^{\infty} \frac{a_k}{3(c-q_k)^{2/3}} = \sum_{k=1}^{\infty} a_k f_k'(c).$$

converges. We have shown that, for any $c \in [a,b] \setminus A$, $\sum\limits_{k=1}^{\infty} a_k f_k'(c)$ converges; thus, $D \subseteq A$. Since $\epsilon > 0$ was arbitrary and the sum of the lengths of the intervals in A is $2L\epsilon$, D has measure zero and $\sum\limits_{k=1}^{\infty} a_k f_k'(x)$ converges almost everywhere on $[0,1]$. □

Remark. It follows from Theorem 3.5 that we can use many different sequences for $\{a_k\}$. We can use any geometric sequence $\{r^k\}$, where $0 < r < 1$, since $(r^k)^{3/5} = (r^{3/5})^k$ and $0 < r^{3/5} < 1$. When outlining the basics of Pompeiu's function, then, we can simply pick $a_k = 1/2^k$. Additionally, we can use the sequence $\{1/n^p\}_{n=1}^{\infty}$, for any $p > 5/3$.

At this juncture, we would hope to simply differentiate f term by term to show that f' indeed is the series $\sum_{k=1}^{\infty} a_k f_k'(x)$. However, we noted that $f'(x) = \infty$ on a dense subset, so we cannot use this method. As it turns out, the function f' is indeed the function we would expect it to be if we could differentiate term-by-term. In other words, the naive method indeed gives the correct result, but proving this requires care. In this section, we will at times use the following definition.

Definition 2.3. Let $f\colon [a,b] \to \mathbb{R}$ be a function and let $c \in (a,b)$. We say that $f'(c) = \infty$ if, for all $M > 0$, there exists $\delta > 0$ such that

$$\frac{f(x) - f(c)}{x - c} \geq M$$

for all x satisfying $0 < |x - c| < \delta$. $\square$

Theorem 2.4. *The function f is differentiable on $[0,1]$ (its derivative is ∞ at rationals), and*

$$f'(x) = \sum_{k=1}^{\infty} a_k f_k'(x).$$

Proof. Let $c \in [0,1]$. Our proof consists of three claims:

(1) If $c \in \mathbb{Q}$, then both $f'(c)$ and $\sum_{k=1}^{\infty} a_k f_k'(c)$ are ∞.

(2) If $c \notin \mathbb{Q}$ and $\sum_{k=1}^{\infty} a_k f_k'(c)$ converges to ∞, then $f'(c) = \infty$. (The existence of points not in $\mathbb{Q}$ where $\sum_{k=1}^{\infty} a_k f_k'(x)$ converges to ∞ is not obvious and will be commented on later.)

(3) If $c \notin \mathbb{Q}$ and $\sum_{k=1}^{\infty} a_k f_k'(c)$ converges to a real number r, then $f'(c) = r$.

For claim (1), note that, for some $j \in \mathbb{Z}^+$, $c = q_j$. Since $f_j'(c) = \infty$, for any $M > 0$, there is $\delta > 0$ such that $0 < |x - c| < \delta$ implies

$$\frac{f_j(x) - f_j(c)}{x - c} \geq M.$$

Then, since each f_k is increasing,

$$\begin{aligned}\frac{f(x) - f(c)}{x - c} &= \sum_{k=1}^{\infty} \frac{f_k(x) - f_k(c)}{x - c} \\ &\geq \frac{f_j(c + h) - f_j(c)}{h} \\ &\geq M.\end{aligned}$$

This shows that $f'(c) = \infty$. Since each f_k is increasing, $f_k'(x)$ is positive for all $k \in \mathbb{Z}^+$ and $x \in [0, 1]$. Additionally, recall that each a_k is also positive. It follows that $\sum\limits_{k=1}^{\infty} a_k f_k'(c) = \infty$.

For claim (2), suppose that $c \notin \mathbb{Q}$ and $\sum\limits_{k=1}^{\infty} a_k f_k'(c)$ converges to ∞. Let $M > 0$. There exists a positive integer N such that $\sum\limits_{k=1}^{N} a_k f_k'(c) > M$. Then there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then

$$\sum_{k=1}^{N} a_k \frac{f_k(x) - f_k(c)}{x - c} > M.$$

Again, since each f_k is increasing,

$$\begin{aligned}\frac{f(x) - f(c)}{x - c} &= \sum_{k=1}^{\infty} a_k \frac{f_k(x) - f_k(c)}{x - c} \\ &\geq \sum_{k=1}^{N} \frac{f_k(x) - f_k(c)}{x - c} > M.\end{aligned}$$

Thus shows that $f'(c) = \infty$.

For claim (3), suppose that $\sum\limits_{k=1}^{\infty} a_k f_k'(c)$ converges to a real number r_k. Note that c must not be a rational number. Since c is neither 0 nor 1, we may talk about the two-sided derivative of f at c. Thus, there is some $\eta > 0$ such that $[c-\eta, c+\eta] \subseteq [0,1]$. Define the function $\phi\colon [-\eta, \eta] \to \mathbb{R}$ by

$$\phi(x) = \frac{f(c+x) - f(c)}{x} = \sum_{k=1}^{\infty} a_k \frac{(c+x-q_k)^{1/3} - (c-q_k)^{1/3}}{x}. \tag{2}$$

Note that, if $f'(c)$ exists, then $\lim\limits_{x\to 0} \phi(x) = f'(c)$. We will show that the sum in (2) converges uniformly on its domain and apply Lemma 1.5. Note that

$$x = (c+x-q_k) - (c-q_k)$$

and

$$\frac{a^{1/3} - b^{1/3}}{a-b} = \frac{1}{a^{2/3} + a^{1/3}b^{1/3} + b^{2/3}}$$

which follows from factoring a difference of cubes. Applying both of these facts, we have

$$\begin{aligned} \phi(x) &= \sum_{k=1}^{\infty} \frac{(c+x-q_k)^{1/3} - (c-q_k)^{1/3}}{(c+x-q_k) - (c-q_k)} \cdot a_k \\ &= \sum_{k=1}^{\infty} \frac{a_k}{(c+x-q_k)^{2/3} + (c+x-q_k)^{1/3}(c-q_k)^{1/3} + (c-q_k)^{2/3}}. \end{aligned} \tag{3}$$

For convenience, let

$$r_k = \frac{(c+x-q_k)^{1/3}}{(c-q_k)^{1/3}},$$

and then expression (3) becomes

$$\sum_{k=1}^{\infty} \frac{a_k}{(c-q_k)^{2/3}} \cdot \frac{1}{r_k^2 + r_k + 1}.$$

Now, since r_k is a real number, by considering $r_k^2 + r_k + 1$ as a polynomial in r_k, we see that $r_k^2 + r_k + 1$ has a global minimum of $3/4$. Thus, for all $k \in \mathbb{Z}^+$,

$$\left| \frac{a_k}{(c - q_k)^{2/3}} \cdot \frac{1}{r_k^2 + r_k + 1} \right| \leq \frac{4}{3} \left| \frac{a_k}{(c - q_k)^{2/3}} \right|.$$

Recall that our beginning assumption was that

$$\sum_{k=1}^{\infty} a_k f_k'(c) = \frac{1}{3} \sum_{k=1}^{\infty} \frac{a_k}{(c - q_k)^{2/3}}$$

converges. Since the terms are positive, the sum converges absolutely, and $\sum_{k=1}^{\infty} \frac{4}{3} \left| \frac{a_k}{(c - q_k)^{2/3}} \right| < \infty$. By the Weierstrass M-test,

$$\sum_{k=1}^{\infty} a_k \frac{(c + x - q_k)^{1/3} - (c - q_k)^{1/3}}{x}$$

converges uniformly to ϕ on $[c - \eta, c + \eta]$.

Since each term in (2) is continuous, ϕ is continuous. Thus, by Lemma 1.5,

$$\begin{aligned}
f'(c) &= \lim_{x \to 0} \phi(x) \\
&= \lim_{x \to 0} \lim_{N \to \infty} \sum_{k=1}^{N} a_k \frac{f_k(c + x) - f_k(c)}{x} \\
&= \lim_{N \to \infty} \lim_{x \to 0} \sum_{k=1}^{N} a_k \frac{f_k(c + x) - f_k(c)}{x} \\
&= \lim_{N \to \infty} \sum_{k=1}^{N} a_k f_k'(c) \\
&= \sum_{k=1}^{\infty} a_k f_k'(c)
\end{aligned}$$

as desired. We have shown that $f'(x) = \sum_{k=1}^{\infty} a_k f_k'(x)$ for all $x \in [0, 1]$. $\square$

Recall that our goal is to create a strictly increasing function whose derivative vanishes at some point in every interval. So far we have created a function whose derivative is infinite at some point on every interval. We turn now to the inverse of f.

2.2.3 Finding Pompeiu's function (exploring $h = f^{-1}$)

Following Lemma 2.1, we noted that f is continuous. By the Extreme and Intermediate Value theorems, the range of f is a closed interval. Call it $[a, b]$. Since f is a strictly increasing function, f has an inverse function. Call it h. Graphically, the inverse of a function $\mathbb{R} \to \mathbb{R}$ looks like a reflection across the line $y = x$. Thus, it is intuitive that, at points where $f'(c) = \infty$, we have $h'(f(c)) = 0$.

Theorem 2.5. *Let $c \in \mathbb{Q} \cap [0, 1]$ and let $d = f(c)$. Then $h'(d) = 0$.*

Proof. Let $\{w_n\}$ be any sequence in $[a, b] \setminus \{d\}$ converging to d and let $\{v_n\}$ be the corresponding sequence in $[0, 1]$ converging to c where $f(v_n) = w_n$. Then

$$\lim_{n\to\infty} \frac{h(w_n) - h(d)}{w_n - d} = \lim_{n\to\infty} \frac{v_n - c}{f(v_n) - f(c)}. \tag{4}$$

Since the limit of the reciprocal in (4) converges to infinity (since $f'(c) = \infty$), we have that $h'(d) = 0$. □

We already know our derivative h' vanishes at countably many points at least. However, it bears proof that this set is indeed dense in $[0, 1]$. Here, we call a set A *dense* in $[a, b]$ if, for every open interval $O \subset [a, b]$, $A \cap O$ is nonempty.

Theorem 2.6. *Let $S = f(\mathbb{Q} \cap [0, 1])$. Then S is dense in $[a, b]$.*

Proof. Let $c \in (a, b)$ and let V be an open interval in (a, b) containing c. Since f is continuous, the preimage of V is open. Call it U. Since the rationals are dense in $[0, 1]$, U contains a rational number q. Then $f(q)$ is in S and V, so S is dense in $[a, b]$. □

We next examine some properties of h'.

Theorem 2.7. *The function h is differentiable on $[a, b]$ and h' is bounded on $[a, b]$.*

Proof. It follows from properties of inverse functions that if f is differentiable at c then h is differentiable at $f(c)$, and that $h'(f(c)) = 1/f'(c)$. But f is strictly increasing, so $f'(x) > 0$ for all $x \in [0, 1]$, which establishes that h is differentiable at $f(c)$ whenever $f'(c)$ exists. If $f'(c) = \infty$, then by Theorem 2.5, $h'(f(c)) = 0$. Thus h is indeed differentiable on its domain $[a, b]$. Again, since f is strictly increasing, its inverse h is strictly increasing. Because h' is nonnegative, to show that h' is bounded, it suffices to find an upper bound. Since $h'(f(c)) = 1/f'(c)$, it suffices to find a lower bound for f'.

By Theorem 2.4, the k-th term in f' is

$$a_k f_k'(x) = \frac{a_k}{3(x - q_k)^{2/3}}.$$

But $(x - q_k)^{2/3}$ is a positive number bounded by 1, so

$$a_k f_k'(x) \geq \frac{a_k}{3}.$$

Since each a_k is positive, we may use, for example, $a_1/3$ as a lower bound for f'. This shows that h' is bounded. □

We are now ready to turn to the main result of Section 2.2.

Theorem 2.8. *The function h' is not Riemann integrable on any subinterval of its domain.*

Proof. Suppose that h' is Riemann integrable on $[c, d] \subseteq [a, b]$. By the Fundamental Theorem of Calculus, $\int_c^d h' = h(d) - h(c) > 0$ (recall that h is strictly increasing). On the other hand, h' vanishes on a dense set of $[c, d]$. Thus, for any partition of $[c, d]$, the lower Riemann sum of h' is 0. Thus, we must have $\int_c^d h' = 0$, a contradiction. Therefore h' is not integrable on $[c, d]$. $\square$

This concludes our main example of a bounded derivative which is not Riemann integrable. Although we cannot graph h, it seems that h is somewhat more intuitive to visualize than Volterra's function using copies of $x^2 \sin \frac{1}{x}$. If we start with any partial sum of f, then its inverse will be integrable, but we may picture h as an increasing function with a horizontal "kink" on a dense set (refer to Figure 4 for an idea of what f looks like, and recall that the plot of h is simply the reflection about $y = x$). Then h', for its part, will be a nonnegative function which, as we note in Section 2.2.4, is positive on a dense set and zero on another dense set. This gives some intuition for why, in some sense, it is difficult to ascribe a value to the "area" under h'.

2.2.4 Further notes on Pompeiu's derivative (h')

As noted in Theorems 2.5 and 2.6, $h'(x) = 0$ on a dense subset of $[a, b]$. However, the fact that h' is not integrable on any subinterval leads to the following observation.

Theorem 2.9. *The function h' attains the value 0 on a dense subset, attains a positive value on another dense subset, and, where h' is positive, it is also discontinuous.*

Proof. We have seen that $h'(x) = 0$ on a dense subset, and we know from the proof of Theorem 2.7 that h' is nonnegative. Suppose that $h'(x)$ is identically 0 on an interval. Then h' is integrable over that interval, which contradicts Theorem 2.8. Thus h' attains a positive value on every interval, so $h'(x)$ is positive on a dense subset of $[a, b]$.

Next we claim that, if $h'(c) > 0$, then h' is discontinuous at c. For all $n \in \mathbb{Z}^+$, there is some $x_n \in (c - 1/n, c + 1/n)$ such that $h'(x_n) = 0$. Then $\{x_n\}$ converges to c and $\{h'(x_n)\}$ converges to 0. Since $h'(c) > 0$, h' is discontinuous at c. $\square$

Recall that we constructed the function f with the intention of having an infinite derivative at every rational in $[0, 1]$. Furthermore, Theorem 2.2 together with Theorem 2.4 shows that f' exists (in a sense) almost everywhere on $[0, 1]$. It is perhaps surprising, then, that there are irrational points in $[0, 1]$ where f' is ∞. This fact is not obvious, and relies on some concepts that may not be taught in undergraduate analysis. A *Baire class one* function is any function which is the pointwise limit of a sequence of continuous functions. (Recall that the uniform limit of a sequence of continuous functions is continuous, but pointwise convergence need not preserve this property.) As it turns out, all derivatives are Baire class one functions.

Theorem 2.10. *If f is differentiable on $[a, b]$, then f' is a Baire class one function.*

Proof. First, extend f to a function $\tilde{f} \colon [a, b + 1] \to \mathbb{R}$ where

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } a \le x \le b; \\ f'(b)(x - b) + f(b), & \text{if } b < x \le b + 1. \end{cases}$$

Now define

$$\phi_n(x) = \frac{\tilde{f}(x+1/n) - \tilde{f}(x)}{1/n}.$$

Then each ϕ_n is continuous on $[a,b]$ since both f and $\tilde{f}$ are continuous on $[a,b]$. Since $\lim_{n\to\infty} \phi_n(x) = f'(x)$ for all $x \in [a,b]$, f' is a Baire class one function. $\square$

A well-known property of continuous functions is that the preimage of open sets under a continuous function is open. (In fact, this property is equivalent to continuity.) As it turns out, there is a similar result for Baire class one functions. First, we need some new terms. While any union of open sets is open, the same is not necessarily true for an infinite intersection of open sets. We will call a set S an F_σ set if S is a countable union of closed sets; we will call S a G_δ set if S is a countable intersection of open sets. The following basic properties of F_σ and G_δ sets will be useful, although we will not prove them here.

(1) The complement of an F_σ set is a G_δ set.

(2) The complement of a G_δ set is an F_σ set.

(3) A finite intersection or countable union of F_σ sets is an F_σ set.

(4) A finite union or countable intersection of G_δ sets is a G_δ set.

Theorem 2.11. *If $f\colon [a,b] \to \mathbb{R}$ is a Baire class one function, then the preimage of any open set under f is an F_σ set.*

Proof. Let $\{f_p\}$ be a sequence of continuous functions which converge pointwise to f on $[a,b]$. We will show that, for all $r \in \mathbb{R}$, the preimage of $(-\infty, r)$, and similarly (r, ∞), are F_σ sets. Namely, we will show that

$$f^{-1}((-\infty,r)) = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{p=n}^{\infty} f_p^{-1}((-\infty, r-1/k]). \tag{5}$$

Since each f_p is continuous, we know $f_p^{-1}((-\infty, r-1/k])$ is closed for every $p, k \in \mathbb{Z}^+$. Their intersection is closed, and the countable union of these closed sets is an F_σ set. That is, if equation (5) is true, then $f^{-1}((-\infty, r))$ is an F_σ set.

Suppose that $x \in f^{-1}((-\infty, r))$. Then $f(x) \in (-\infty, r)$, and, for some $K \in \mathbb{Z}^+$, we have $f(x) \in (-\infty, r-1/K)$. Since $\{f_p(x)\}$ converges to $f(x)$, there is some N such that, if $p \geq N$, we have $f_p(x) \in (-\infty, r-1/K)$. That is,

$$x \in \bigcap_{p=N}^{\infty} f_p^{-1}((-\infty, r-1/K)) \subseteq \bigcap_{p=N}^{\infty} f_p^{-1}((-\infty, r-1/K])$$

and this shows

$$x \in \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{p=n}^{\infty} f_p^{-1}((-\infty, r-1/k]). \tag{6}$$

Conversely, begin by supposing (6) is true. Then there exist positive integers K and N such that

$$x \in \bigcap_{p=N}^{\infty} f_p^{-1}((-\infty, r-1/K)).$$

This means that $f_p(x) \in (-\infty, r-1/K)$ for all $p \geq N$. Since $\{f_p(x)\}_{p=N}^{\infty}$ converges to $f(x)$, we have that $f(x) \leq r-1/K < r$, and thus $x \in f^{-1}((-\infty, r))$. This proves equation (5). As we noted, this establishes that $f^{-1}(-\infty, r))$, and similarly that $f^{-1}((r, \infty))$, are F_σ sets.

Now, given any open set U, we can write U as the countable union of open intervals (a_i, b_i) where $i \geq 1$, and thus

$$U = \bigcup_{i=1}^{\infty} \left((-\infty, b_i) \cap (a_i, \infty)\right). \tag{7}$$

Using some basic properties of inverse set functions, note the following:

$$f^{-1}\left(\bigcup_{i=1}^{\infty}(-\infty, b_i) \cap (a_i, \infty)\right) = \bigcup_{i=1}^{\infty}\left(f^{-1}((-\infty, b_i)) \cap f^{-1}((a_i, \infty))\right). \tag{8}$$

Now, since the intersection of two F_σ sets is an F_σ set, and the countable union of F_σ sets is an F_σ set, it follows from (7) and (8) that $f^{-1}(U)$ is an F_σ set. □

Before we can prove the existence of irrational points where f' converges to ∞, we need Theorem 2.12, which we will not prove here. A *limit point* of a set A is a number x such that, for all $r > 0$, the set $A \cap (x - r, x + r)$ is nonempty. The set A is *nowhere dense* if $A \cup \{x : x \text{ is a limit point of } A\}$ contains no open intervals.

Theorem 2.12 (Baire Category Theorem [4])**.** *If $\{E_n\}$ is a sequence of nowhere dense sets, then any interval $[a, b]$ contains a point x where x is not contained in any E_n.*

Recall that S is the set of points $x \in [a, b]$ where $f^{-1}(x)$ is rational. In Theorem 2.5, we showed that $h'(x) = \infty$ for all $x \in S$. We are now ready to prove the following.

Theorem 2.13. *There exists a point $x \notin S$ such that $h'(x) = 0$.*

Proof. By Theorem 2.10, we know that h' is a Baire class one function. Using Theorem 2.11, we see that $h'^{-1}((-\infty, 0) \cup (0, \infty))$ is an F_σ set, and thus $h'^{-1}(\{0\})$ is a G_δ set. As we noted previously (see Theorems 2.5 and 2.6), $h'(S) = \{0\}$, and so $S \subseteq h'^{-1}(\{0\})$. If we can show that S is not a G_δ set, then S is a proper subset of $h'^{-1}(\{0\})$, which guarantees some $x \notin S$ which maps to 0 under h'. By way of contradiction, suppose S is a G_δ set and thus that $[a,b] \setminus S$ is an F_σ set. There is a sequence $\{E_n\}$ of closed sets such that

$$[a,b] \setminus S = \bigcup_{n=1}^{\infty} E_n.$$

Now, since S is dense in $[a,b]$, none of the sets E_n may contain an open interval. Thus, $\{E_n\}$ is a sequence of nowhere dense sets. But then

$$[a,b] = ([a,b] \setminus S) \cup S = \left(\bigcup_{n=1}^{\infty} E_n \right) \cup \left(\bigcup_{n=1}^{\infty} \{f(q_n)\} \right).$$

However, the Baire Category Theorem guarantees a point $x \in [a,b]$ not contained in any of the nowhere dense sets E_n or $\{f(q_k)\}$. Thus, S is not a G_δ set. There must be a point $x \notin S$ such that $h'(x) = 0$. This corresponds to a point $t \notin [0,1] \cap \mathbb{Q}$ such that $f'(t) = \infty$. $\square$

3 Absolute continuity and Baire class one

We saw that h' vanishes on dense set which is at least countably infinite. Similarly, it is positive on a dense set. It turns out that h' is positive on a set which does not have measure zero. In order to prove that claim, we will need the following result.

Theorem 3.1. *If f' is bounded, then f maps any set of measure zero to a set of measure zero.*

Proof. Let f be a differentiable function and let S be a set of measure zero in the domain of f. Suppose $M > 0$ is a bound for f', and fix $\epsilon > 0$. Then there exists some sequence of intervals $\{(a_k, b_k)\}$ such that

$$S \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k) \qquad \text{and} \qquad \sum_{k=1}^{\infty} (b_k - a_k) < \frac{\epsilon}{M+1}.$$

Since each $x \in S$ is contained in some interval (a_j, b_j), which is in turn contained in $f((a_j, b_j))$,

$$f(S) \subseteq \bigcup_{k=1}^{\infty} f((a_k, b_k)).$$

Using the Mean Value Theorem, if we take any two points $x, y \in (a_k, b_k)$, we have

$$|f(y) - f(x)| \leq |y - x|M \leq (b_k - a_k)M$$

and thus, for each $k \in \mathbb{Z}^+$, we can cover $f((a_k, b_k))$ with an open interval of length $(M + 1)(b_k - a_k)$. Since

$$\sum_{k=1}^{\infty}(M + 1)(b_k - a_k) < \epsilon,$$

we have shown that $f(S)$ has measure zero. □

Now we can prove our desired result.

Theorem 3.2. *The set* $E = \{x : h'(x) \neq 0\}$ *does not have measure zero.*

Proof. Suppose that E has measure zero. Since h' is bounded, by Theorem 3.1, the image of E under h also has measure zero. Then, for any $x \in [0, 1]$, x is either in $h(E)$, or not in $h(E)$. If $x \notin h(E)$, then, since h is a bijection, there is some $y \in [a, b] \setminus E$ such that $h(y) = x$. But this means $h'(y) = x = 0$, which, as we noted previously, implies $f'(x) = \infty$. But we already have shown that the set of points where $f'(x) = \infty$ has measure zero. We can then write $[0, 1]$ as a union of two sets of measure zero, a contradiction. Thus, the set E does not have measure zero. □

The proof of Theorem 3.2 relies on the fact that functions with bounded derivatives map sets of measure to sets of measure zero. We can prove a more general result using the notion of absolute continuity.

Definition 3.3. A function f on an interval I is *absolutely continuous* if, for each $\epsilon > 0$, there exists $\delta > 0$ satisfying the following condition: if the set $\{[c_i, d_i] : 1 \leq i \leq n\}$ is a finite collection of disjoint intervals in I that satisfies the inequality $\sum_{i=1}^{n}(d_i - c_i) < \delta$, then $\sum_{i=1}^{n} |f(d_i) - f(c_i)| < \epsilon$. □

It is easy to see from Definition 3.3 that absolute continuity implies uniform continuity (simply take $n = 1$ as it is used in the definition). What is less obvious is that there are uniformly continuous functions that are not absolutely continuous. The Cantor function, which we won't explore here, is a canonical example, but we will now give an example of a more elementary function which is uniformly continuous, but not absolutely continuous.

Example 3.4. Consider the function $f\colon [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x \sin \frac{\pi}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

From some elementary properties of continuous functions, it is clear that f is continuous on $(0, 1]$. By the Squeeze Theorem,

$$\lim_{x \to 0^+} x \sin \frac{\pi}{x} = 0,$$

and so f is continuous on $[0, 1]$. This also shows that f is uniformly continuous on its domain. We note that when

$$\frac{\pi}{x} = \frac{\pi}{2} + 2\pi n,$$

for $n \in \mathbb{Z}$, then

$$f(x) = x = \frac{2}{4n + 1}.$$

Similarly, if $\dfrac{\pi}{y} = -\dfrac{\pi}{2} + 2\pi n$, then $f(y) = -y = -\dfrac{2}{4n-1}$. For the same values x and y, we have

$$|f(y) - f(x)| = x + y = \frac{2(4n-1) + 2(4n+1)}{16n^2 - 1} = \frac{16n}{16n^2 - 1} > \frac{1}{n}.$$

Now let $\epsilon = 1$ and let $\delta > 0$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series, each tail $\sum_{n=N}^{\infty} \frac{1}{n}$ is also divergent. Take N such that $\frac{2}{4n-1} < \delta$. Since $\sum_{n=N}^{\infty} \frac{1}{n}$ diverges, there is some positive integer M such that $\sum_{n=N}^{M} \frac{1}{n} > 1 = \epsilon$. Consider the set of disjoint intervals $\{[x_n, y_n] : N \leq n \leq M\}$ where

$$x_n = \frac{2}{4n+1}, \qquad \text{and} \qquad y_n = \frac{2}{4n-1}.$$

Then $\sum_{n=N}^{M} (y_i - x_i) < \delta$ and $\sum_{n=N}^{M} |f(y_i) - f(x_i)| > \sum_{n=N}^{M} \frac{1}{n} > 1 = \epsilon$, so f is not absolutely continuous. $\square$

3.1 Absolute continuity and bounded derivatives

Theorem 3.1 stated that a function f which has a bounded derivative maps any set of measure zero to a set of measure zero. Here, we will prove a similar result for absolutely continuous functions. Finally, we will give an alternate proof of Theorem 3.1 by showing that if f has a bounded derivative on $[a, b]$, then f is absolutely continuous on $[a, b]$.

Theorem 3.5. *If $f\colon [a, b] \to \mathbb{R}$ is absolutely continuous and A has measure zero, then $B = f(A)$ has measure zero.*

Proof. Let $\epsilon > 0$. Since f is absolutely continuous, there is some $\delta > 0$ such that, for any sequence $\{[r_k, s_k]\}$ of disjoint intervals satisfying $\sum_{k=1}^{\infty} (s_k - r_k) < \delta$, we have $\sum_{k=1}^{\infty} |f(s_k) - f(r_k)| < \epsilon$. Since A has measure zero, there exists a sequence of open sets $\{(a_k, b_k)\}$ such that $A \subseteq \bigcup_{k=1}^{\infty}(a_k, b_k)$ and $\sum_{k=1}^{\infty}(b_k - a_k) < \delta$. Since any open set is the union of disjoint open intervals, we may assume that the (a_k, b_k)'s are disjoint. By the absolute continuity of f,

$$\sum_{k=1}^{\infty}|f(b_k) - f(a_k)| \leq \epsilon.$$

Since f is continuous, by the Extreme and Intermediate Value Theorems, we know that f maps closed intervals to closed intervals. Define the intervals $[c_k, d_k] = f([a_k, b_k])$. Again, using the continuity of f, the preimage of each $[c_k, d_k]$ is a closed interval whose endpoints must fall in $[a_k, b_k]$. This shows that $\sum_{k=1}^{\infty}|d_k - c_k| \leq \epsilon$, and since

$$B \subseteq \bigcup_{k=1}^{\infty} f([a_k, b_k]) = \bigcup_{k=1}^{\infty}[c_k, d_k],$$

we may also write

$$B \subseteq \bigcup_{k=1}^{\infty}(c_k - \epsilon/2^k, d_k + \epsilon/2^k).$$

Because $\epsilon > 0$ was arbitrary and $\sum_{k=1}^{\infty}(d_k - c_k + \epsilon/2^{k-1}) \leq \epsilon + 2\epsilon$, it follows that B has measure zero. $\square$

We have shown that if a function f has a bounded derivative, or if f is absolutely continuous, then f maps sets of measure zero to sets of measure zero. Next we will show that, if f' is bounded, then f is absolutely continuous. Since it follows from Theorem 3.5 that f maps sets of measure zero to sets of measure zero, Theorems 3.5 and 3.6 provide an alternate proof to Theorem 3.1.

Theorem 3.6. *If $f\colon [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and f' is bounded, then f is absolutely continuous on $[a, b]$.*

Proof. Let $\epsilon > 0$ and let $M > 0$ be a bound for f'. Suppose that the set $\{[c_i, d_i] : 1 \leq i \leq n\}$ is a finite collection of disjoint intervals in $[a, b]$ that satisfies the inequality $\sum_{i=1}^{n}(d_i - c_i) < \epsilon$. By the Mean Value Theorem, for each $i \in \{1, \ldots, n\}$, there is some $t_i \in (c_i, d_i)$ such that $|f(d_i) - f(c_i)| < M(d_i - c_i)$. Then

$$\sum_{i=1}^{n} |f(d_i) - f(c_i)| < \sum_{i=1}^{n} M(d_i - c_i) < \epsilon M,$$

and so f is absolutely continuous. □

3.2 Some properties of Baire class one functions

In Section 2.2, we constructed a strictly increasing, differentiable function whose derivative vanished on a dense subset of its domain. Using some ideas about F_σ sets and Baire class one functions, we showed that Pompeiu's non-integrable derivative vanishes at even more points than that dense subset. To expand on this, we will explore some more general properties of F_σ sets and Baire class one functions. The goal of this section will be to arrive at a necessary and sufficient condition for a function to be Baire class 1. Many readers will recognize the following fact, which is common in a real analysis course:

- The function $f\colon [a, b] \to \mathbb{R}$ is continuous if and only if the preimage of any open set under f is an open set.

If we think of Baire class one functions as one level of abstraction removed from continuous functions, and F_σ sets as one level removed from open sets, then the following analogy, which we will prove, is fitting:

- The function $f\colon [a,b] \to \mathbb{R}$ is a Baire class one function if and only if the preimage of any open set under f is an F_σ set.

Before we turn our attention to this main idea, we first prove another fact which alludes back to a common fact about uniform continuity. In an analysis course, one might learn that uniform convergence "preserves" continuity. That is, if $\{f_n\}$ converges uniformly to f, and each f_n is continuous, then f is continuous. Again, in a pleasing analogy, it turns out that, if each f_n is a Baire class one function, then f is a Baire class one function. To prove this, we will first consider the sequence $\{f_n\}$ in a series representation.

Lemma 3.7. *Suppose that $f\colon [a,b] \to \mathbb{R}$ is a Baire class one function. If $|f(x)| \leq M$ for all $x \in [a,b]$, then there is a sequence of continuous functions $\{h_k\}$ which converge pointwise to f on $[a,b]$ and each h_k is bounded by M.*

Proof. Since f is a Baire class one function, there is a sequence of continuous functions $\{f_n\}$ converging pointwise to f on $[a,b]$. For each $k \in \mathbb{Z}^+$, define the function h_k as follows:

$$h_k(x) = \begin{cases} M, & \text{if } f_k(x) > M, \\ f_k(x), & \text{if } |f_k(x)| \leq M, \\ -M, & \text{if } f_k(x) < -M. \end{cases}$$

Now let $c \in [a,b]$. If $|f_k(c)| < M$, then on some open interval containing c, $h_k(x)$ is equal to $f_k(x)$, and so h_k is continuous at c. Similarly, if $|f_k(c)| > M$, then h_k is constant on some neighborhood of c. If $f_k(c) = M$, then for any $\epsilon > 0$, there is $\delta > 0$ such that $|x - c| < \delta$ implies $|f_k(x) - M| < \epsilon$. Then, if $|x - c| < \delta$ and $f_k(x) > M$, we have $|h_k(x) - h_k(c)| = |M - M| < \epsilon$. If $-M < f_k(x) \leq M$, we have $|h_k(x) - h_k(c)| = |f_k(x) - M| < \epsilon$. The case where $f_k(c) = -M$ is similar. Thus h_k is continuous for each k. For each $x \in [a, b]$, $\{f_k(x)\}$ converges to $f(x)$. If $|f(x)| < M$, there is some K such that, if $k > K$, then $|f_k(x)| < M$. Thus $\{h_k(x)\}$ converges to $f(x)$. If $f(x) = M$, then for any $\epsilon > 0$, there is some K such that, if $k > K$, then $|f_k(x) - K| < \epsilon$. Then, if $f_k(x) > M$, $|h_k(x) - M| = 0$, and, if $f_k(x) < M$, then $|h_k(x) - M| = |f_k(x) - M| < \epsilon$. We have shown that $\{h_k\}$ is a sequence of continuous functions converging pointwise to f on $[a, b]$, where each h_k is bounded by M. $\square$

As we mentioned earlier, we first prove the following result for series. Afterwards, we will relate our findings back to sequences.

Theorem 3.8. *Suppose that the series* $\sum_{j=1}^{\infty} g_j$ *converges to* h *on* $[a, b]$*, and, for each* $j \geq 1$*, there is a positive constant* N_j *such that* $|g_j(x)| < N_j$ *for all* $x \in [a, b]$*, where* $\sum_{j=1}^{\infty} N_j$ *converges. If each* g_j *is a Baire class one function, then* h *is a Baire class one function.*

Proof. Suppose that $\sum_{j=1}^{\infty} g_j$ converges to h on $[a, b]$, where each g_j is a Baire class one function and bounded by $N_j > 0$, where $\sum_{j=1}^{\infty} N_j = N$. By the Weierstrass M-test, the convergence is uniform. Each g_j is bounded by $M_j = N_j + 1/2^j$, and $\sum_{j=1}^{\infty} M_j$ converges to $M = N + 1$. By Lemma 3.7, for each $j \geq 1$, there is a sequence of continuous functions $\{f_{j,i}\}_{i=1}^{\infty}$ which converge pointwise to g_j on $[a, b]$, and where each $f_{j,i}$ satisfies $|f_{j,i}(x)| < M_j$ for all $x \in [a, b]$ and for all $i \geq 1$. Note: to avoid confusion, we emphasize here that the series $\sum_{j=1}^{\infty} g_j$ converges uniformly to h, and, for each j, the sequence $\{f_{j,i}\}_{i=1}^{\infty}$ converges pointwise to g_j. We claim that the sequence

$$\{f_{1,1}, f_{1,2} + f_{2,2}, f_{1,3} + f_{2,3} + f_{3,3}, \ldots\} = \left\{\sum_{j=1}^{p} f_{j,p}\right\}_{p=1}^{\infty} \tag{9}$$

converges pointwise to h on $[a, b]$. Since each term in this sequence is a finite sum of continuous functions, this will show that h is a Baire class one function.

Fix $x \in [a, b]$ and let $\epsilon > 0$. Since the series $\sum_{j=1}^{\infty} M_j$ converges, there is a positive integer Q such that, if $n > Q$, then

$$\sum_{j=n}^{\infty} M_j < \epsilon. \tag{10}$$

For each $j \in \{1, \ldots, Q\}$, there is a positive integer P_j such that, if $p > P_j$, then

$$|f_{j,p}(x) - g_j(x)| < \epsilon/n. \tag{11}$$

Using (10) and (11), if $n > \max\{Q, P_1, \ldots, P_Q\}$,

$$\begin{aligned}
\left|\sum_{j=1}^{n} f_{j,n}(x) - h(x)\right| &= \left|\sum_{j=1}^{n} f_{j,n}(x) - \sum_{j=1}^{\infty} g_j(x)\right| \\
&\leq \left|\sum_{j=1}^{Q} f_{j,n}(x) - \sum_{j=1}^{Q} g_j(x)\right| + \left|\sum_{j=Q+1}^{\infty} g_j(x)\right| + \left|\sum_{j=Q+1}^{n} f_{j,n}(x)\right| \\
&\leq \sum_{j=1}^{n} |f_{j,n}(x) - g_j(x)| + \sum_{j=n+1}^{\infty} |g_j(x)| + \sum_{j=Q+1}^{n} |f_{j,n}(x)| \\
&< 3\epsilon.
\end{aligned}$$

This shows that the sequence in (9), when evaluated at a fixed value of x, indeed converges to $h(x)$. This suffices to show that h is a Baire class one function. □

We now prove a similar result for sequences of functions. Corollary 3.9 will be instrumental in our main result (Theorem 3.13).

Corollary 3.9. *If the sequence $\{h_k\}$ converges uniformly to h on $[a, b]$ and each h_k is a Baire class one function, then h is a Baire class one function.*

Proof. Suppose $\{h_k\}$ is a sequence of Baire class one functions converging uniformly to h on $[a, b]$. Choose a subsequence $\{h_{k_i}\}$ of $\{h_k\}$, such that, for all $i \geq 1$ and $x \in [a, b]$, we have $|h_{k_i}(x) - h(x)| < \epsilon/2^i$. Define $g_1 \colon [a, b] \to \mathbb{R}$ by $g_1(x) = h_{k_1}(x)$ and if $i > 1$, define $g_i \colon [a, b] \to \mathbb{R}$ by $g_i(x) = h_{k_i}(x) - h_{k_{i-1}}(x)$. Then we have $\sum_{i=1}^{n} g_i(x) = h_{k_i}(x)$, and for all $i > 1$ and $x \in [a, b]$,

$$\begin{aligned}
|g_i(x)| &\leq |h_{k_i}(x) - h(x)| + |h(x) - h_{k_{i-1}}(x)| \\
&< \frac{\epsilon}{2^i} + \frac{\epsilon}{2^{i-1}} \\
&< \frac{\epsilon}{2^{i-2}}.
\end{aligned}$$

Since g_1 is also bounded by some $M > 0$, it follows that h is a Baire class one function. □

We now can begin working towards the main result of this section, which gives a necessary and sufficient condition for a function to be Baire class one. Recall that Theorem 2.11 gave a necessary condition; it remains to show that, if the preimage of any open set under f is an open set, then f is a Baire class one function. To that end, we will partition the *range* of f, rather than the domain, into open intervals. The preimage of these open intervals are F_σ sets, and we will construct a sequence converging to f by using these sets. Readers familiar with concepts like Lebesgue measure, Lebesgue integration, and simple sets may find some of these concepts familiar. As our discussion of F_σ sets concerns unions of closed sets, the following canonical result will be useful, and we will not prove it here.

Theorem 3.10 (Tietze Extension Theorem)**.** *Let K be a closed set. If the function $f\colon K \to \mathbb{R}$ is continuous on K, then there is a continuous function $g\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x)$ for all $x \in K$.*

As we discussed earlier, the proof of Theorem 3.13 consists of "partitioning" the range of a function f, taking preimages of those sets and then reconstructing f. This "reconstruction" will be done using indicator functions, which we define now. Define the *indicator function* of a set S by $\mathcal{X}_S\colon S \to \{0,1\}$ by

$$\mathcal{X}_S(x) = \begin{cases} 0, & \text{if } x \notin S, \\ 1, & \text{if } x \in S. \end{cases}$$

Lemma 3.11 will streamline the proof of Theorem 3.13.

Lemma 3.11. *Suppose that $\{A_1, \ldots, A_n\}$ is a set of pairwise disjoint F_σ sets whose union contains $[a,b]$. Then the function $f\colon [a,b] \to \mathbb{R}$, defined by*

$$f(x) = \sum_{k=1}^{n} a_k \mathcal{X}_{A_k}(x),$$

is a Baire class one function, where each a_k is a real constant.

Proof. Since each A_k is an F_σ set, we can write

$$A_k = \bigcup_{i=1}^{\infty} B_i^k$$

where each B_i^k is closed. Now define the function f_j on a subset of $[a,b]$ such that $f_j(x) = a_k$ if $x \in \bigcup\limits_{i=1}^{j} B_i^k$. Note that f_j is a piecewise function with j "pieces," and it is defined on a finite union of closed sets. Then, since the domain of f_j is closed, by the Tietze Extension Theorem (Theorem 3.10), for each j, there is a continuous function $g_j\colon [a,b] \to \mathbb{R}$ such that $g_j(x) = f_j(x)$ for any x in the domain of f_j. We claim that the sequence $\{g_j\}$ converges pointwise to f. Let $x \in [a,b]$. Since $[a,b] \subseteq \bigcup_{k=1}^{n} A_k$, x must be contained in some A_k. Then there is an integer J such that $x \in \bigcup_{i=1}^{J} B_i^k$. It follows that, for any $j \geq J$, $g_j(x) = a_k$. Thus f is a Baire class one function. □

We now turn our attention to the main proof of this section. We will first give a necessary and sufficient condition for bounded functions to be Baire class one functions, and then prove a similar result for all Baire class one functions.

Theorem 3.12. *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Then f is a Baire class one function if and only if the preimage of any open set under f is an F_σ set.*

Proof. For convenience, this proof uses superscript when listing both sets and real numbers; they do not represent exponents.

Theorem 2.11 shows that, if f is a Baire class one function, then the preimage of any open set under f is an F_σ set. It remains to prove the converse.

Suppose $f\colon [a,b] \to \mathbb{R}$ is a function bounded by $M > 0$ such that, if U is open, then $f^{-1}(U)$ is an F_σ set. For positive integers n, k, where $0 \leq k \leq n$, define

$$y_k^n = -M + \frac{2M}{n}k.$$

Then, for each $k \in \{1, \ldots, n-1\}$, define

$$B_k^n = f^{-1}((y_{k-1}^n, y_{k+1}^n)) = \{x \in [a,b] : y_{k-1}^n < f(x) < y_{k+1}^n\}.$$

By our hypothesis, each B_k^n is an F_σ set. Define $A_1^n = B_1^n$ and, for each integer $k \in \{2, \ldots, n-1\}$, define $A_k^n = B_k^n \setminus \bigcup_{l=1}^{k-1} A_{l-1}^n$. Since the set difference of two F_σ sets is an F_σ set, the set $\{A_1^n, \ldots, A_{n-1}^n\}$ contains disjoint F_σ sets whose union contains $[a, b]$. For each positive integer n, define $f_n \colon [a, b] \to \mathbb{R}$ where

$$f_n(x) = \sum_{k=1}^{n-1} y_k^n \mathcal{X}_{A_k^n}(x).$$

By Lemma 3.11, each f_n is a Baire class one function. We will show that $\{f_n\}$ converges uniformly to f on $[a, b]$.

Let $\epsilon > 0$ and pick a positive integer N such that $\frac{2M}{N} < \epsilon$. Then let $x \in [a, b]$. For some $k \in \{1, \ldots, n-1\}$, $x \in A_k$. Note that $f_n(x) = y_k^n$ and $f(x) \in (y_{k-1}^n, y_{k+1}^n)$. For any $n \geq N$,

$$|f_n(x) - f(x)| = |y_k^n - f(x)| < \frac{2M}{n} \leq \frac{2M}{N} < \epsilon.$$

Thus $\{f_n\}$ converges uniformly to f on $[a, b]$. By Corollary 3.9, f is a Baire class one function. □

Theorem 3.13. *Let $f \colon [a, b] \to \mathbb{R}$ be any real-valued function. Then f is a Baire class one function if and only if the preimage of any open set under f is an F_σ set.*

Proof. If $f \colon [a, b] \to \mathbb{R}$ is a Baire class one function, then, by Theorem 2.11, the preimage of any open set under f is an F_σ set. Theorem 3.12 shows that, if f is bounded, then it is a Baire class one function; it remains to show that, if f is unbounded and the preimage of any open set is an F_σ set, then f is a Baire class one function.

Suppose that $f\colon [a,b] \to \mathbb{R}$ is an unbounded real valued function for which the preimage of any open set is an F_σ set. Define $h\colon \mathbb{R} \to (0,1)$ to be a continuous, strictly increasing function mapping the real line into $(0,1)$. Then $h \circ f\colon [a,b] \to (0,1)$. If U is any open set, then, since h is continuous, $h^{-1}(U)$ is open, and thus, by our hypothesis, $(h \circ f)^{-1}(U) = f^{-1} \circ h^{-1}(U)$ is an F_σ set. Since $h \circ f$ is bounded, by Theorem 3.12, $h \circ f$ is a Baire class one function.

Since h maps an interval to an interval and is continuous and strictly increasing, h^{-1} is continuous. Let $\{f_n\}$ be a sequence of continuous functions converging pointwise to $h \circ f$ on $[a,b]$. Then, since the composition of two continuous functions is continuous, $h^{-1} \circ f_n$ is continuous for each n. Furthermore, we claim that $\{h^{-1} \circ f_n\}$ converges pointwise to $h^{-1} \circ h \circ f = f$ on $[a,b]$. Let $c \in [a,b]$. Since h^{-1} is continuous,

$$\lim_{n\to\infty} h^{-1}(f_n(c)) = h^{-1} \lim_{n\to\infty} f_n(c) = (h^{-1} \circ h \circ f)(c) = f(c)$$

Thus, $h^{-1} \circ f_n$ is a sequence of continuous functions converging uniformly to f on $[a,b]$. We have shown that f is a Baire class one function if and only if the preimage of any open set under f is an F_σ set. $\square$

4 The Henstock Integral

As we have already seen, there are some functions which are not Riemann integrable, but perhaps feel like they "should" be integrable. The simplest example uses the so-called improper Riemann integral, such as the limit describing the area under the curve $y = x^{-2/3}$ between $x = 0$ and 1. In Section 2.2, we discovered a function on a closed interval which is bounded, has an antiderivative, and yet is not Riemann integrable. In this section, we present a new integral, consistent with the Riemann integral on all Riemann integrable functions, which allows us to integrate such derivatives.

We say that a function is *Riemann integrable* on $[a, b]$ if there is some $L \in \mathbb{R}$ satisfying the following: for each $\epsilon > 0$, we can find a real number $\delta > 0$ such that, if ${}^tP$ is any tagged partition of $[a, b]$ with norm less than δ, we have $|S(f, {}^tP) - L| < \epsilon$. We will define the Henstock integral a similar way, but replacing the real number $\delta > 0$ with a function $\delta\colon [a, b] \to (0, \infty)$. Rather than a tagged partition of $[a, b]$, with a requirement about its norm, we will use a δ-fine tagged partition, whose definition follows.

Definition 4.1. Given the closed interval $[a, b]$ and a positive valued function $\delta\colon [a, b] \to (0, \infty)$, a *$\delta$-fine tagged partition* of $[a, b]$ is a set

$${}^tP_\delta = \{([x_{i-1}, x_i], t_i) : 1 \leq i \leq n\}$$

where $x_0 = a$, $x_n = b$, $x_{i-1} < x_i$, and $t_i \in [x_{i-1}, x_i]$ for all $1 \leq i \leq n$. Furthermore, for all $1 \leq i \leq n$, $[x_{i-1}, x_i] \subseteq (t_i - \delta(t_i), t_i + \delta(t_i))$. $\square$

Now we can formally define the Henstock integral.

Definition 4.2. A function $f\colon [a, b] \to \mathbb{R}$ is *Henstock integrable* on $[a, b]$ if there is a real number L satisfying the following: for each $\epsilon > 0$, there exists a function $\delta\colon [a, b] \to (0, \infty)$, such that for any δ-fine tagged partition ${}^tP_\delta$ of $[a, b]$, we have $|S(f, {}^tP_\delta) - L| < \epsilon$. If such a number L exists, we call L the *Henstock integral* of f over $[a, b]$, denoted $L = (H)\int_a^b f$. $\square$

In this section, we will use the notation $(R)\int_a^b f$ to denote the Riemann integral of f, and, if not specified, we assume that $\int_a^b f$ denotes the Riemann integral of f.

It follows easily from the definition that Riemann integrable functions are Henstock integrable.

Theorem 4.3. *If* $f\colon [a,b] \to \mathbb{R}$ *is Riemann integrable on* $[a,b]$*, then* f *is Henstock integrable on* $[a,b]$*, and* $(R)\int_a^b f = (H)\int_a^b f$*.*

Proof. Because f is Riemann integrable, there is some L such that, for each $\epsilon > 0$, there is some $\delta_0 > 0$ such that, for any tagged partition ${}^tP$ of $[a,b]$ with norm smaller then δ_0, we have

$$|S(f,{}^tP) - L| < \epsilon.$$

Then let $\delta\colon [a,b] \to (0,\infty)$ be defined by $\delta(x) = \delta_0/2$. If ${}^tP_\delta$ is a δ-fine tagged partition of $[a,b]$, then for all $1 \le i \le n$, $[x_{i-1}, x_i] \subseteq (t_i - \delta_0/2, t_i + \delta_0/2)$, and thus $|{}^tP_n| < \delta_0$. It follows that $|S(f,{}^tP_n) - L| < \epsilon$, and thus

$$(R)\int_a^b f = L = (H)\int_a^b f,$$

which completes the proof. □

Before proceeding, we make the following observation: if, given a function f, for each $\epsilon > 0$, we could construct a function $\delta\colon [a,b] \to (0,\infty)$ for which there be no δ-fine tagged partition of $[a,b]$, then f would vacuously satisfy the criterion for Henstock integrability. As it turns out, we can always find a δ-fine tagged partition.

Theorem 4.4. *For any interval* $[a,b]$ *and function* $\delta\colon [a,b] \to (0,\infty)$*, there exists a* δ*-fine tagged partition of* $[a,b]$*.*

Proof. Let D be the set of all numbers x, where $a < x \le b$, for which there is a δ-fine tagged partition of $[a,x]$. By taking $x = a + \delta(a)/2$, we see that

$\{([a, a+\delta(a)/2], a)\}$ is a δ-fine tagged partition of $[a, a+\delta(a)/2]$. Thus, D is non-empty. Since D is bounded above by b, it has a least upper bound — call it β. We claim that $\beta \in D$ and $\beta = b$, and conclude that there is a δ-fine tagged partition of $[a, b]$.

First we must show that $\beta \in D$. Since β is the least upper bound of D, there is some $r \in (0, \delta(\beta)/2)$ such that we have a δ-fine tagged partition of $[a, \beta - r]$. If necessary, we can remove tagged intervals or part of tagged intervals from this partition to obtain a δ-fine tagged partition of $[a, \beta - \delta(\beta)/2]$. By adjoining the singleton $\{([\beta - \delta(\beta)/2, \beta], \beta)\}$, we have a δ-fine tagged partition of $[a, \beta]$.

Now suppose that $\beta < b$, and let ${}^tP_\delta = \{([x_{i-1}, x_i], t_i) : 1 \leq i \leq n\}$ be a δ-fine tagged partition of $[a, \beta]$. Then since $\delta(\beta)$ is positive, we can extend ${}^tP_\delta$ to the tagged partition ${}^tP_\delta \cup \{([\beta, \beta+\delta(\beta)/2], \beta)\}$. Since this is a δ-fine tagged partition of $[a, \beta+\delta(\beta)/2)]$, we have a contradiction (since we said β is an upper bound for D and have shown that $\beta + \delta(\beta)/2 \in D$). Thus $\beta = b$, as desired. □

Theorem 4.5. *If f is differentiable on $[a, b]$, then f' is Henstock integrable on $[a, b]$ and $(H)\int_a^b f' = f(b) - f(a)$.*

Proof. Let f be differentiable on $[a, b]$ and let $\epsilon > 0$. For each $x \in [a, b]$, define $\delta(x)$ as a positive number for which, if $0 < |y - x| < d(x)$, then

$$\left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < \epsilon. \tag{12}$$

(The existence of such a $\delta(x)$ follows from the fact that f is differentiable at

x.) Note that we may rewrite (12) as

$$|f'(x)(y-x) - (f(y) - f(x))| < \epsilon|y-x|. \tag{13}$$

Now suppose that ${}^tP_\delta = \{([x_{i-1}, x_i], t_i) : 1 \leq i \leq n\}$ is a δ-fine tagged partition of $[a, b]$ (and such a partition does exist, from Theorem 4.4). Using the inequality in (13), for each of the tagged intervals $([x_{i-1}, x_i], t_i)$, we have the following:

$$\begin{aligned}
|(x_i - x_{i-1})f'(t_i) &- (f(x_i) - f(x_{i-1}))| \\
&= |(x_i - t_i + t_i - x_{i-1})f'(t_i) - (f(x_i) - f(t_i) + f(t_i) - f(x_{i-1}))| \\
&\leq |f'(t_i)(x_i - t_i) - (f(x_i) - f(t_i))| \qquad (14) \\
&\qquad + |f'(t_i)(t_i - x_{i-1}) - (f(t_i) - f(x_{i-1}))| \\
&< \epsilon(x_i - t_i) + \epsilon(t_i - x_{i-1}) \\
&= \epsilon(x_i - x_{i-1}) \qquad (15)
\end{aligned}$$

Then, using (15), if we take the Riemann sum of f' with the partition ${}^tP_\delta$, we find

$$\begin{aligned}
\left|S(f', {}^tP_\delta) - (f(b) - f(a))\right| &= \left|\sum_{i=1}^{n} \Big((x_i - x_{i-1})f'(t_i) - (f(x_i) - f(x_{i-1}))\Big)\right| \\
&< \sum_{i=1}^{n} \epsilon(x_i - x_{i-1}) \\
&= \epsilon(b-a).
\end{aligned}$$

Thus f' is Henstock integrable on $[a, b]$ and $(H)\int_a^b f' = f(b) - f(a)$. $\square$

Corollary 4.6. *Pompeiu's derivative h', as defined in Section 2.2, is not Riemann integrable on any subinterval of $[a, b]$, yet it is Henstock integrable on* *every subinterval of* $[a,b]$*. Furthermore, for any* $[c,d] \subseteq [a,b]$*,*

$$(H)\int_c^d h' = h(d) - h(c),$$

which is a real (positive) number.

4.1 A worked example using the Henstock integral

We have successfully engineered a new integral which can integrate derivatives. However, Pompeiu's function and its derivative are difficult (if not impossible) to visualize, and the proof of Theorem 4.5 may not provide much intuition for the problem. We turn to a more elementary example of a function which is Henstock integrable but not Riemann integrable, and hopefully gain some intuition into the difference between the two integration techniques.

Example 4.7. Define the function $f\colon [-a,a] \to \mathbb{R}$ where $a > 0$ and

$$f(x) = \begin{cases} x^2 \cos \dfrac{1}{x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

We will show that f is differentiable on $[-a,a]$, f' is not Riemann integrable on $[-a,a]$, and $(H)\int_{-a}^{a} f' = 0$. It is clear that f is differentiable away from the origin. At the origin, we have

$$\lim_{h \to 0} \frac{h^2 \cos \frac{1}{h^2}}{h} = 0$$

by the Squeeze Theorem. Thus,

$$f'(x) = \begin{cases} 2x\left(\cos \dfrac{1}{x^2}\right) + \dfrac{2}{x}\left(\sin \dfrac{1}{x^2}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Note that f' is unbounded near the origin, and thus f' is not Riemann integrable on $[-a, a]$. However, since f is even, we know from Theorem 4.5 that $(H)\int_a^a f' = 0$. But how do two reasonable definitions of the integral arrive at different conclusions? Let's consider the function we are trying to integrate, shown in Figure 5.

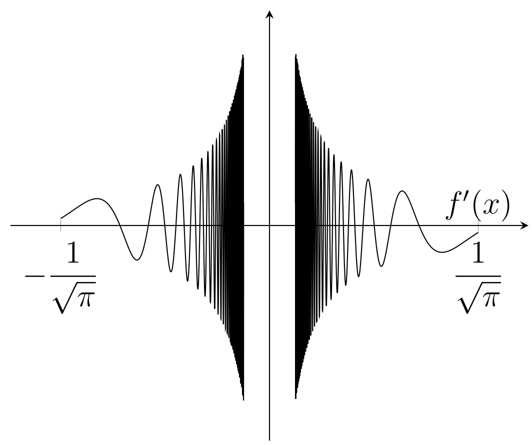

Figure 5: A partial graph of f'. Note that f' is unbounded near the origin and that f' is an odd function.

Recall that, if f' is Riemann integrable and we take any sequence of partitions of $[-a, a]$ whose norm converges to 0, then any corresponding sequence of Riemann sums of f' must converge to some number. Let's consider the sequence of partitions $\{P_n\}$. The distance between two consecutive points in P_n is greater than some $M > 0$, and so, for some neighborhood of the origin, each interval in P_n will contain a relative maximum *and* minimum. Intuitively, then, the upper and lower Riemann sums will always differ by some amount. Furthermore, since f' is unbounded in each direction near the origin, the amount by which the upper and lower sums differ does not converge to 0. This gives us some insight into why f' is not Riemann integrable. On the other hand, f' is an odd function, and, as long as we ignore a symmetric interval around the origin, one expects the integral of f' on its domain to be 0. We can ascribe meaning to this by writing

$$\lim_{r \to 0^+} \left((R)\int_{-a}^{-r} f' + (R)\int_{r}^{a} f' \right) = 0.$$

However, we still cannot write $(R)\int_{-a}^{a} f' = 0$. The Henstock integral solves this problem for us by allowing δ to vary throughout the domain. Rather than using a fixed-value $\delta > 0$, we use a positive-valued function δ to construct a δ-fine tagged partition of the domain. Intuitively, one can imagine that, as we approach the origin, the values of δ get smaller so as to closely estimate the "waves" of the curve. Then, when we take a Riemann sum using this partition, the "evenness" of the function come into play, leaving a value which converges to 0. The question remains: how does this approach solve the issue at the origin, with unbounded magnitude and oscillation? Let's consider the interval of our δ-fine tagged partition which contains the origin. Call it $([x_{i-1}, x_i], t_i)$. Then t_i cannot be nonzero, because we constructed δ in a such a way that $[x_{i-1}, x_i]$ closely estimated a part of the "wave" of the curve containing t_i (imagine that each interval is restricted to one single "hump" of the sinusoid). Thus, we must have $t_i = 0$. This gives us some clearer understanding for how the Henstock integral can solve problems that the Riemann integral cannot.$\square$

4.2 Further properties of the Henstock integral

In this paper, we developed the Henstock integral with the problem of integrating derivatives in mind. In Theorem 4.5, we showed that, with the Henstock integral, we can indeed integrate all derivatives by calculating the difference of the endpoints in the antiderivative. In Theorem 4.3, we saw that, for Riemann integrable functions, the Riemann and Henstock integrals give the same result. The Henstock integral thus solves a problem that the Riemann integral has, and aligns with our intuition *vis a vis* the Fundamental Theorem of Calculus.

In this section, we explore some further properties of the Henstock integral, focusing on classes of functions which may not be Riemann integrable but which are Henstock integrable. We already saw that a derivative is Henstock but not necessarily Riemann integrable. Next, we prove the stronger result that a continuous function may be not differentiable at infinitely many places, and its "derivative" will still be Henstock integrable. In this section, we will say that a property holds *nearly everywhere* on S if it holds everywhere but a countable subset of S.

Theorem 4.8. *If $F\colon [a,b] \to \mathbb{R}$ is continuous and differentiable nearly everywhere, we define $f\colon [a,b] \to \mathbb{R}$ as*

$$f(x) = \begin{cases} F'(x), & \text{if } F \text{ is differentiable at } x, \\ 0, & \text{if } F \text{ is not differentiable at } x. \end{cases}$$

Then

$$(H)\int_a^b f = F(b) - F(a).$$

Proof. Let $F\colon [a,b] \to \mathbb{R}$ be a continuous function which is differentiable on $[a,b] \setminus \{q_k : k \in \mathbb{Z}^+\}$, where each $q_k \in [a,b]$. Define f as in the statement of Theorem 4.8. Let $\epsilon > 0$, and define the function $\delta\colon [a,b] \to (0,\infty)$ as follows: if $x \neq q_k$ for all $k \geq 1$, $\delta(x)$ is a positive number such that, if $0 < |y-x| < \delta(x)$, then

$$\left| \frac{F(y) - F(x)}{y - x} - f(x) \right| < \epsilon;$$

if $x = q_k$, then define $\delta(x)$ as the positive number such that, if $|y - x| < \delta$, then $|F(y) - F(x)| < \epsilon/2^k$. Now let ${}^tP_\delta$ be a δ-fine tagged partition of $[a,b]$, where ${}^tP_\delta = \{([x_{i-1}, x_i], t_i) : 1 \leq i \leq n\}$. We now split ${}^tP_\delta$ into two sets, based on whether or not F is differentiable at t_i. That is, define ${}^tR_\delta$ as the set

$$ {}^tR_\delta = \{([y_{i-1}, y_i], r_i) : 1 \leq i \leq p\}, $$

where $([y_{i-1}, y_i], r_i) = ([x_{j-1}, x_j], t_j)$ for some i, j, and F is differentiable at t_i; define ${}^tS_\delta$ as the set

$$ {}^tS_\delta = \{([z_{i-1}, z_i], s_i) : 1 \leq i \leq q\}, $$

where $([z_{i-1}, z_i], s_i) = ([x_{j-1}, x_j], t_j)$ for some i, j, and F is *not* differentiable at t_i.

Recall the inequality in line (15) of the proof of Theorem 4.5, which says that, for $1 \leq i \leq p$,

$$ |(y_i - y_{i-1})f(t_i) - (F(y_i) - F(y_{i-1}))| < \epsilon(y_i - y_{i-1}). \tag{16} $$

Then note that, because of the continuity of F and the way we defined δ, together with the fact that $f(t_i) = 0$, for $1 \leq i \leq q$,

$$ \begin{aligned} |(z_i - z_{i-1})f(t_i) - (F(z_i) - F(z_{i-1}))| &= |F(z_i) - F(t_i) + F(t_i) - F(z_{i-1})| \\ &\leq |F(z_i) - F(t_i)| + |F(t_i) - F(z_{i-1})| \\ &< \epsilon/2^{k-1}. \end{aligned} \tag{17} $$

Incorporating (16) and (17),

$$\begin{aligned}
|S(f,{}^tP_\delta) - (F(b) - F(a))| &= \left|\sum_{i=1}^{n}(x_i - x_{i-1})f(t_i) - (F(x_i) - F(x_{i-1}))\right| \\
&\le \left|\sum_{i=1}^{p}(y_i - y_{i-1})f(t_i) - (F(y_i) - F(y_{i-1}))\right| \\
&\quad + \left|\sum_{i=1}^{q}(z_i - z_{i-1})f(t_i) - (F(z_i) - F(z_{i-1}))\right| \\
&\le \left|\sum_{i=1}^{p}\epsilon(y_i - y_{i-1})\right| + \left|\sum_{i=1}^{q}\frac{\epsilon}{2^{k-1}}\right| \\
&\le \epsilon(b - a + 2).
\end{aligned}$$

Therefore f is Henstock integrable on $[a,b]$, and $(H)\int_a^b f = F(b) - F(a)$. $\square$

Remark. A similar result to Theorem 4.8 is the following: if $F\colon [a,b] \to \mathbb{R}$ is differentiable on $[a,b] \setminus S$ and continuous on $[a,b]$, where S is a finite set, and we define its "derivative" $f\colon [a,b] \to \mathbb{R}$ where $f(x) = F'(x)$ for all $x \in [a,b] \setminus S$ and $f(x) = 0$ for all $x \in S$, then f is Riemann integrable, and the integral has the expected value $f(b) - f(a)$.

References

[1] A. Bruckner, J. Bruckner, T. Bruckner, B. Thomson, *Elementary Real Analysis*, Prentice Hall, 2nd Ed., 2001.

[2] H. Lebesgue, *Leçons sur L'intégration et la Recherche des Fonctions Primitives*, Chelsea Publishing Company, New York, 3rd Ed., 1973.

[3] R. Gordon, *A Bounded Derivative That Is Not Riemann Integrable*, Mathematics Magazine, Vol. 89, No. 5, 2016.

[4] R. Gordon, *Real Analysis: A First Course*, Addison-Wesley Higher Mathematics, 2nd Ed., 2002.

[5] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, New York, 3rd Ed., 1976.

[6] V. Volterra, *Sui Principii del Calcolo Integrale*, Giornale di Matematiche, Vol. XIX, Naples, 1881.