Minimizing the Calculus in Optimization Problems

Teylor Greff

Mathematics Department
Whitman College
May 2016

Abstract: Do we actually need calculus to solve maximum/minimum problems? Optimization problems are explored and solved using the AM/GM inequality and Cauchy Schwarz inequality, while simultaneously finding trends and evolutions in these optimization problems as we look at a textbooks ranging from 1902 – 2015.

Contents

1 Introduction

The focus of this paper is optimization problems in single and multi-variable calculus spanning from the years $1900 - 2016$. The main goal was to see if there was a way to solve most or all optimization problems without using any calculus, and to see if there was a relationship between this discovery and the published year of the optimization problems. The primary method used to accomplish this was the Arithmetic Mean/Geometric Mean inequality, but upon further exploration, the Cauchy Schwarz inequality and other generalized solutions proved to be incredibly helpful as well. This paper lays out how these problems are solved, gives examples of similar problems, and also explores the historical evolution of these problems.

2 The AM/GM Inequality

The Arithmetic Mean/Geometric Mean inequality was the main method when it came to solving optimization problems without differentiation. Before looking at applications of the inequality, a firm grasp on the inequality concept is key. The result states that the Arithmetic Mean is always greater than or equal to the Geometric Mean. The AM/GM inequality is stated below with three proofs. Two for intuition and one formal proof. We will begin by looking at a diagram proof and an algebraic proof for two positive numbers a and b before seeing the AM/GM applied to the nth case.

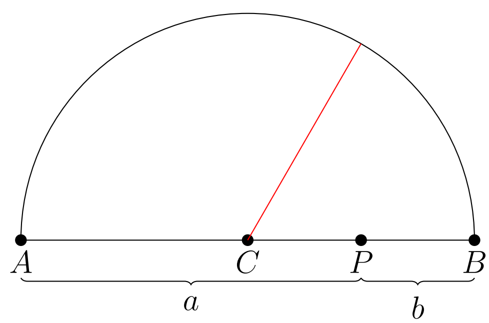
2.1 Diagram Proof

The following proof for the AM/GM inequality is useful for two numbers. First, note that the arithmetic mean and geometric mean for two numbers are as follows:

$$AM = \frac{a+b}{2}, \quad GM = \sqrt{ab}.$$

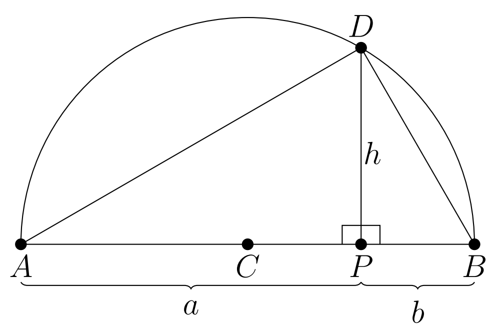
Using a semicircle where AB is the diameter representing $a + b$ and P is the point distinguishing a from b, we can illustrate the AM/GM.

Since the diameter of the circle is $a + b$, its radius is $(a + b)/2$. The red line represents the arithmetic mean of a and b.

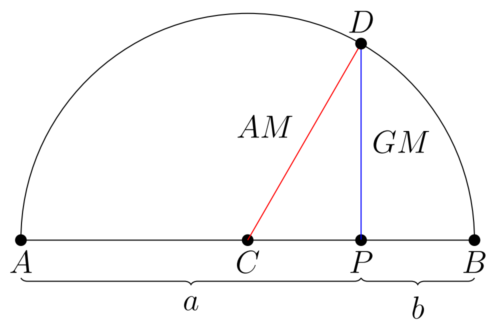


We now draw an altitude from point P to point D on the semicircle and line segments from point D to points A and B. Thus, since ΔADP is similar to ΔDBP it follows that ΔADB is a right triangle, since an angle inscribed in a semicircle is a right angle, or:

$$\frac{a}{h} = \frac{h}{b}, \quad h = \sqrt{ab}.$$



Therefore, the altitude is the geometric mean shown in blue, and is clearly less than the hypotenuse.



Therefore, no matter what the lengths of a and b, AM $\geq$ GM, and equality holds whenever $a = b$.

2.2 Algebra Proof

Another way to easily reason the inequality with two positive numbers a and b, is by first recognizing $0 \leq (\sqrt{a} - \sqrt{b})^2$, then it follows easily:

$$0 \leq a - 2\sqrt{a}\sqrt{b} + b,$$

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

2.3 Formal Proof

Now that we have proved the AM/GM inequality for two numbers, we prove the inequality for n numbers. To prove this inequality, we use the following lemma.

Lemma 1. *Let $n \geq 2$ be an integer. Suppose that $b_1, b_2, b_3, ..., b_n$ are positive real numbers that are not all equal. If $b_1 b_2 b_3 \cdots b_n = 1$, then $b_1 + b_2 + b_3 + \cdots + b_n > n$.*

Proof. We will use the Principle of Mathematical Induction. For the case $n = 2$ we know that $b_1 \neq b_2$ and $b_1 b_2 = 1$. It follows that,

$$0 < (\sqrt{b_1} - \sqrt{b_2})^2 = b_1 - 2\sqrt{b_1 b_2} + b_2 = b_1 - 2 + b_2 \quad \text{and thus} \quad b_1 + b_2 > 2,$$

showing that the result is true when $n = 2$. Now suppose the result is valid for some positive integer $p \geq 2$. Let $b_1, b_2, ..., b_p, b_{p+1}$ be positive real numbers that are not all equal and satisfy $b_1 b_2 \cdots b_p b_{p+1} = 1$. Without loss of generality, we may assume that the numbers are in increasing order, that is, $b_1 \leq b_2 \leq \cdots \leq b_p \leq b_{p+1}$. By the assumptions on these numbers, we must have $b_1 < 1 < b_{p+1}$. Since the conclusion of the lemma is assumed to be true when $n = p$, we consider the product $(b_1 b_{p+1}) b_2 \cdots b_p = 1$, which is a product of p numbers. If all of these numbers are equal (and thus all equal 1), then

$$b_2 + b_2 + \cdots + b_p = p - 1 \quad \text{and} \quad b_1 + b_{p+1} > 2$$

(the inequality follows from the first part of the proof) and it follows that

$$b_1 + b_2 + \cdots + b_p + b_{p+1} > p + 1.$$

If the numbers $b_1 b_{p+1}, b_2, ..., b_p$ are not all equal, then

$$b_1 b_{p+1} + b_2 + \cdots + b_p > p$$

by the induction hypothesis. Since the quantity $(b_{p+1} - 1)(1 - b_1)$ is positive, we find that

$$\begin{aligned} b_1 + b_2 + \cdots + b_{p+1} &= (b_1 b_{p+1} + b_2 + \cdots + b_p) + 1 + (b_{p+1} - 1)(1 - b_1) \\ &> p + 1 + (b_{p+1} - 1)(1 - b_1) \\ &> p + 1. \end{aligned}$$

This shows that the result holds when $n = p + 1$. By the Principle of Mathematical Induction, the conditional statement given in the lemma is valid for all integers $n \geq 2$. [7] $\square$

Theorem 2 (Arithmetic Mean/Geometric Mean Inequality)**.** *Let n be a positive integer. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then*

$$(a_1 a_2 \cdots a_n)^{\frac{1}{n}} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

Proof. The equality clearly holds when $a_1 = a_2 = \cdots = a_n$. If $n = 1$, then the result is trivial. Also, if any of the $a_k = 0$, the result is trivial as well.

Now, suppose for $n \geq 2$, all a_k's are positive, and that not all of a_k's are equal to each other. If we let $r = (a_1 a_2 ... a_n)^{\frac{1}{n}}$, we get the following result,

$$\frac{a_1}{r}\frac{a_2}{r}\cdots\frac{a_n}{r} = \frac{a_1 a_2 \cdots a_n}{r^n} = 1.$$

By the lemma, it follows that,

$$\frac{a_1}{r} + \frac{a_2}{r} + \cdots + \frac{a_n}{r} > n.$$

Multiplying through by r and dividing by n, we find that,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} > r = (a_1 a_2 \cdots a_n)^{\frac{1}{n}}. \quad [7]$$

This completes the proof. $\square$

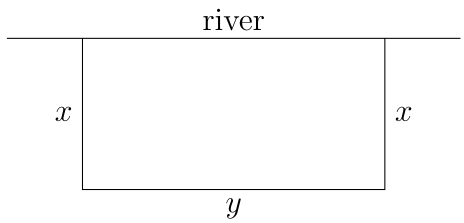
3 Optimization Problems and the AM/GM Inequality

The following problems are typical problems seen in most calculus textbooks. These can all be solved using the AM/GM inequality, and are categorized into a few different types of problems that often appear in maximum/minimum sections of calculus textbooks. Starting with a simple example, the derivative approach is used, then a solution is shown using the AM/GM inequality. The rest of the problems that follow, are solved using the AM/GM inequality.

3.1 Fence Problems

Fence problems are extremely popular in calculus textbooks, and perhaps the most basic. There are many variations of this problem, but all can be solved using the AM/GM inequality and provide an easy starting place to see an application of the inequality.

Example 3. *A landowner wishes to use 2 miles of fencing to enclose a rectangular region of maximum area. However, one side runs along a stream, so only three sides must be fenced in. Find the lengths of the sides of the rectangular region having the largest area that can be enclosed. [5]*



3.1.1 Derivative Approach

We first solve the problem using calculus. Suppose that x is the width of the rectangle, and y is its length. We want to maximize the area A, where $A = xy$. The problem of maximizing the area of the region can be reduced to the following:

$$\text{maximize} \;\; xy \;\; \text{subject to} \;\; 2x + y = 2.$$

Noticing that $y = 2 - 2x$ and thus,

$$A(x) = x(2 - 2x) = 2x - 2x^2,$$

we can take the derivative and set it equal to zero to find our maximal points:

$$0 = 2 - 4x \;\; \text{and hence} \;\; x = \frac{1}{2}.$$

We have thus shown for the fencing to have maximum area, it will have a width of 1/2 miles, and length of 1 mile.

3.1.2 AM/GM Approach

As above, we seek a solution to the following problem:

$$\text{maximize } xy \;\; \text{subject to} \;\; 2x + y = 2.$$

By the AM/GM inequality, we find that

$$2 = 2x + y \geq 2\sqrt{2xy} \;\; \text{and hence} \;\; xy \leq \frac{1}{2}.$$

The maximum value for xy is thus (1/2) and this value is attained when

$$2x = 1 = y.$$

We have thus shown that the maximum area of the landowners fencing has length of $1/2$ miles, and width of 1 mile.

Fence problems, such as this one, give a good idea of how to apply the AM/GM inequality to optimization problems. However, as we will see in the next section, some care is needed when applying it, as not all problems work out as cleanly as this one.

3.1.3 Similar Problem Examples

Example 4. *A farmer wishes to set aside one acre of his land for corn and wheat. To keep out the cows, the field is enclosed by a fence costing* 50 *per running foot. In addition, a fence running down the middle of the field is needed with a cost per foot of \$1. Given that* 1 *acre* $= 43560$ *square feet, what dimensions should the field have so as to minimize his total cost? The field is rectangular. [1]*

Example 5. *A farmer wishes to divide* 20 *acres of land along a river into* 6 *smaller plots by using a one fence parallel to the river and* 7 *fences perpendicular to the it. Show that the total amount of fencing is minimized if the sum of the lengths of the* 7 *cross fences equals the length of the one fence parallel to the river. [1]*

Example 6. *A dairy farmer plans to fence in a rectangular pasture adjacent to a river. The pasture must contain* $180,000$ *square meters in order to provide enough grass for the herd. What dimensions would require the least amount of fencing if no fencing is needed along the river? [2]*

Example 7. *A rancher has* 200 *feet of fencing with which to enclose to two adjacent rectangular corrals. What dimensions should be used so that the enclosed area will be a maximum? [2]*

Example 8. *One side of an open field is bounded by a straight river. How would you put a fence around the other three sides of a rectangular plot in order to enclose as great an area as possible with a given length of fence? [4]*

3.2 Number Problems

Number problems are extremely popular when it comes to early single variable calculus textbooks. They involve finding two numbers that satisfy certain conditions. The following example shows a problem that can occur when using the AM/GM inequality. It often can appear unhelpful if used carelessly. We will show what must occur in order for the AM/GM inequality to solve an optimization problem correctly by first showing what goes wrong if applied carelessly, and then showing how to carefully apply it.

Example 9. *Divide the number* 20 *into two parts such that the product of one part by the square of the other part shall be a maximum. [8]*

3.2.1 Incorrect Application

Suppose that x is the part being squared, and y is the part being multiplied. We want to maximize the product F, where $F = x^2y$. The problem of maximizing the product of the numbers can now be reduced to the following:

$$\text{maximize } \; x^2y \; \text{ subject to } \; x + y = 20.$$

By the AM/GM inequality, we find that

$$20 = x + y \geq 2\sqrt{xy} \; \text{ and hence } \; xy \leq \left(\frac{20}{2}\right)^2 = 100.$$

However, we notice that this states a maximum value for xy, when we are looking for a maximum value of x^2y. This is where the inequality appears unhelpful. Fortunately, there is a way to work around this as shown below.

3.2.2 Correct Application

Since we need two x values to appear, as that is what we are trying to maximize, we write $x + y$ as $\frac{1}{2}x + \frac{1}{2}x + y$. By the AM/GM inequality,

$$20 = \frac{1}{2}x + \frac{1}{2}x + y \geq 3\sqrt[3]{\frac{1}{4}x^2y} \; \text{ and hence } \; x^2y \leq 4\left(\frac{20}{3}\right)^3.$$

We notice that this states the maximum value for x^2y is 32000/27 and this value is attained when

$$\frac{1}{2}x = \frac{20}{3} = y.$$

Thus, we have shown that the maximum product of x^2y is when $y = 20/3 \approx 13.33$ and $x = 40/3 \approx 6.67$.

This same idea will appear in many optimization problems when using the AM/GM inequality. In order to solve these problems correctly, we must make sure that the AM/GM is maximizing the quantity desired. The constraint must multiply through to the get the value being maximized or vice versa. In situations where both the constraint and the max/min function contain addition, the AM/GM proves to be unhelpful. This will further be explored in the Non-AM/GM Generalized Problems section.

3.2.3 Similar Problem Examples

Example 10. *The product of two numbers is* 16*. Determine them so that the square of one plus the cube of the other is as small as possible. [1]*

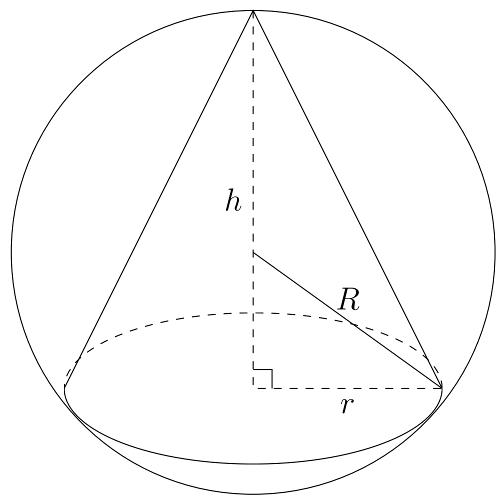
Example 11. *Divide the number* 10 *into two parts such that the square of one part multiplied by the cube of the other shall be a maximum. [12]*

3.3 Geometry Problems

In addition to number problems, geometry problems were a popular type of optimization problem used in the early 1900's. To this day almost all text books include a problem much like the one below, and most can be solved used the AM/GM as long as applied carefully as discussed above. Below is an example and solution to one of these problems.

3.3.1 Application

Example 12. *Determine the ratio of significant dimensions so as to attain the maximum volume of a cone in a sphere. [6]*



Suppose that R is the radius of the sphere. Let h and r be the height and radius of the cone, respectively. We want to maximize the volume V of the cone, where $V = \pi r^2 h/3$. Referring to the figure, we note that h will be larger than R when the volume of the cone is maximized. By the Pythagorean Theorem,

$$(h-R)^2 + r^2 = R^2 \quad \text{and thus} \quad h^2 + r^2 = 2Rh.$$

The problem of maximizing the volume of the cone can now be reduced to the following:

$$\text{maximize} \quad r^2 h \quad \text{subject to} \quad 2h + \frac{2r^2}{h} = 4R.$$

By the AM/GM inequality, we find that

$$4R = h + h + \frac{2r^2}{h} \geq 3\sqrt[3]{2r^2 h} \quad \text{and hence} \quad r^2 h \leq \frac{1}{2}\left(\frac{4R}{3}\right)^3 = \frac{32}{27}R^3.$$

The maximum value for $r^2 h$ is thus $32R^3/27$ and this value is attained when

$$h = \frac{4R}{3} = \frac{2r^2}{h}.$$

Since $h^2 = 2r^2$, the ratio of height to radius for the optimal cone is $\sqrt{2}$. We have thus shown that the maximum volume of a cone inscribed in a sphere of radius R is

$$\frac{1}{3}\pi r^2 h = \frac{\pi}{3} \cdot \frac{32}{27}R^3 = \frac{8}{27} \cdot \frac{4}{3}\pi R^3,$$

that is, the maximal cone occupies $8/27$ of the volume of the entire sphere.

3.3.2 Similar Problem Examples

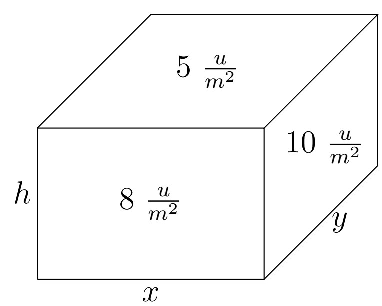
Example 13. *Find the maximum right cylinder that can be inscribed in a sphere of radius* a*. [12]*

3.4 Three Dimensional Problems

Not only can the AM/GM inequality be applied to single variable optimization problems, but also multi-variable calculus problems. These problems are typically solved using Lagrange Multipliers, but the AM/GM inequality gives a clean and straightforward result. Below is an example of one of these problems.

3.4.1 Application

Example 14. *A rectangular building is being designed to minimize heat loss. The east and the west walls lose heat at a rate of* 10 *units/*m^2 *per day, the north and south walls at a rate of* 8 *units/*m^2 *per day, the floor at a rate of* 1 *unit/*m^2*, and the roof at a rate of* 5 *units/*m^2 *per day. The volume must be exactly* 4000 m^3*. Find the dimensions that minimize heat loss. [9]*



Suppose that x is the length of the box, y is the width of the box and h is the height of the box. We want to minimize the heat loss function, $f(x)$, where $f(x) = 20hy + 16xh + 6xy$. The problem of minimizing the heat loss can be reduced to the following:

$$\text{minimize } \ 20hy + 16xh + 6xy \ \text{ subject to } \ xyh = 4000.$$

By the AM/GM inequality, we find that

$$20hy+16xh+6xy \geq 3(1920x^2y^2h^2)^{1/3} \text{ and hence } 3\sqrt[3]{1920(4000)^2} \leq 20hy+16xh+6xy.$$

Since it follows that $y = 8/3h$ from $20hy = 16xh = 6xy$, the minimum value is thus attained when

$$(20h)(8/3h) \geq \sqrt[3]{1920(4000)^2}.$$

Solving for h we find the optimal box to have $h \approx 7.6$, $y = 20.4$ and $x = 25.5$.

Another interesting thing to note is the use of economic ideas in this calculus problem. Looking through old books, economic and medical applications did not begin appearing until around the 1970's when calculus became a requirement for other disciplines besides just mathematics. Further exploration of this will be shown in the Historical Bit section of this paper.

3.4.2 Similar Problem Examples

Example 15. *An open box is to be made from a rectangular piece of material by cutting equal squares from each corner and turning up the sides. Find the dimensions of the box of maximum volume if the material has dimensions* 2 *feet by* 3 *feet. [2]*

Example 16. *An open rectangular box is to be made from a piece of cardboard* 8 *in. wide and* 15 *in. long by cutting a square from each corner and bending up the sides. Find the dimensions of the box of maximum volume. [4]*

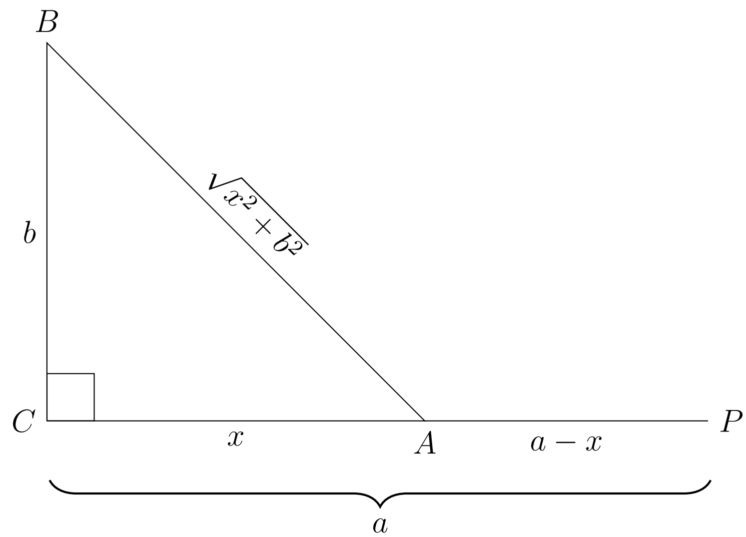
4 Non-AM/GM Generalized Problems

Many optimization problems cannot be solved using the AM/GM inequality. I was able to find that if the equation being maximized/minimized and the constraint both involved addition the AM/GM inequality proved to be unhelpful. In these instances, calculus proved to be unavoidable, but a convenient solution could be found to help with further similar problems. Below are a few examples of these types of problems.

4.1 "Triathlon" Problems

"Triathlon" Problems take the form of a person or thing that needs to travel in one direction at a certain speed, and in another direction at a different speed. While the activity or subject may change, these problems can be generalized to a simple solution. If a student was able to recognize a problem as a "Triathalon" Problem, they could plug into the generalized solution without using any calculus. One of these problems is shown below, first using calculus, then using the generalized solution on a "different" problem.

Example 17. *A man who can row at a speed of* 4 *miles per hour and run at a speed of* 6 *miles per hour wishes to reach the point* P *from a boat at point* B *as shown in the figure below in the least amount of time as possible. Find the distance* AP *that the man must run on the beach. [8]*



4.1.1 Specific Solution

Using the derivative, suppose that a is the distance from C to P, b is the distance from B to C, and x is the distance from C to A. Recognizing, distance=(rate)(time), then the problem of minimizing the time, $t(x)$ can be reduced to the following:

$$t(x) = \frac{\sqrt{10^2 + x^2}}{4} + \frac{10 - x}{6} \quad \text{where} \quad 0 \le x \le 10.$$

We can take the derivative and set it equal to zero to find our minimal points:

$$t'(x) = \frac{x}{4\sqrt{10^2 + x^2}} - \frac{1}{6} = 0,$$

$$9x^2 = 2\sqrt{10^2 + x^2},$$
$$5x^2 = 400,$$
$$x = \sqrt{80}.$$

We have shown to minimize the time to get from B to P is by traveling to the point A at $\sqrt{80}$ miles from point C. The distance from C to P is thus $10 - \sqrt{80} \approx 1.056$ miles.

While this result is correct, we can make it much simpler for later similar problems by generalizing the solution.

4.1.2 Generalized Solution

If we let the rate from point B to A be equal to r, and the rate from A to P be equal to s, the minimum time $t(x)$ is:

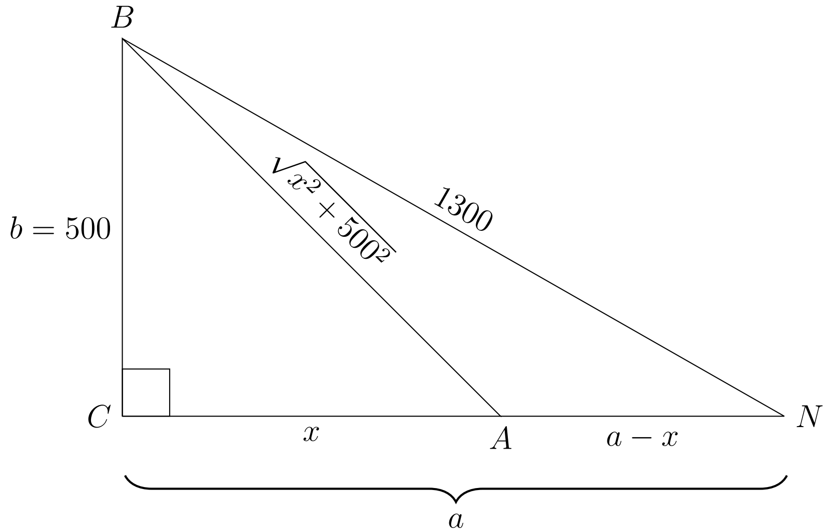
$$t(x) = r\sqrt{x^2 + b^2} + s(a - x).$$

Taking the derivative and setting equal to 0 gives,

$$\frac{x}{\sqrt{x^2 + b^2}} = \frac{s}{r},$$
$$r^2x^2 = s^2x^2 + s^2b^2,$$
$$x^2 = \frac{s^2b^2}{r^2 - s^2},$$
$$x = \frac{sb}{\sqrt{r^2 - s^2}}.$$

What is perhaps the most interesting result about this is that a does not appear in our general equation, suggesting that after a certain point, no matter how long a is, the person should still always arrive at the same point. Also note $r > s$. Further, any problem set up in a similar manner can simply applied with the above equation. An example of this is shown below.

Example 18. *It is known that homing pigeons fly faster over land than over water. Assume that they fly* 10 *meters per second over land, but only* 8 *meters per second over water. If a pigeon is located at the edge of a straight river* 500 *meters wide and must fly to its nest, located* 1300 *meters away on the opposite side of the river. What path would minimize its flying time? [5]*



Once we use the Pythagorean Theorem to find that $a = 1200$, all we have to do is plug into the general equation,

$$x = \frac{(1/10)(500)}{\sqrt{(1/8)^2 - (1/10)^2}} = \frac{2000}{3}.$$

Therefore, the bird should fly to a point A, which is $2000/3 \approx 666.67$ meters east of point C to minimize their time.

4.1.3 Similar Problem Examples

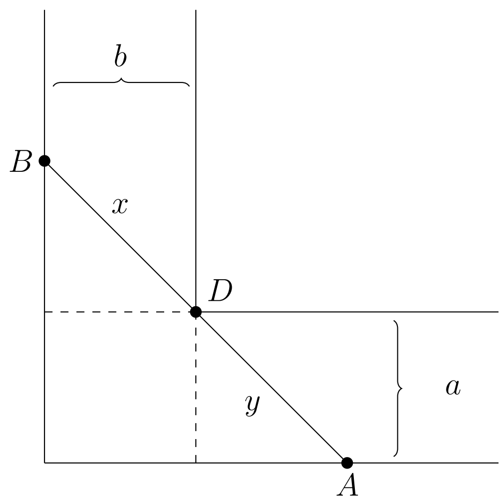
Example 19. *An oil pipeline is built with two different kinds of tubing. From a point P on one side of a river, the line must cross the river and then proceed to a point Q along the bank on the other side. The tubing used in crossing the river costs 50 percent more than the tubing which can be used on dry land. The river is $1/2$ km wide and the point Q is 5 km down river from P. To what point R across the river must the pipe be directed so as to minimize the total cost? [1]*

Example 20. *A man is in a boat 2 miles from the nearest point on the coast. He is to go to a point Q, 3 miles down the coast and 1 mile inland. If he can row at 2 miles per hour and walk at 4 miles per hour, toward what point on the coast should he row in order to reach point Q in the least time? [2]*

4.2 Pipe Corridor Problem

The Pipe Corridor problem is another frequently used problem, especially in later textbooks. Again, we can find a generalized solution so that once we know how to identify a Pipe Corridor problem, we can simply plug into our generalized result.

Example 21. *Find the length of the longest thin, rigid pipe that can be carried from one* 10 *foot wide corridor to a similar corridor at right angles to the first. Assume that the pipe has negligible diameter. [5]*



4.2.1 Generalized Solution

Let $x = \overline{BD}$ and $y = \overline{DA}$. The problem of maximizing the length of the pole, L can be reduced to the following constraint by similar triangles:

$$\frac{x}{b} = \frac{y}{\sqrt{y^2 - a^2}}.$$

Since $x = by/\sqrt{y^2 - a^2}$, we want to minimize L, but find the longest pipe,

$$L(y) = \frac{by}{\sqrt{y^2 - a^2}} + y.$$

We can take the derivative and set equal to zero to find our minimal points:

$$L'(y) = 1 - \frac{a^2 b}{(y^2 - a^2)^{(3/2)}} = 0,$$

$$(y^2 - a^2)^{(3/2)} = a^2 b,$$
$$y = \sqrt{a^{4/3} b^{2/3} + a^2}.$$

Much like the "Triathlon Problem," any problem with a similar set-up can be plugged into the above equation. We will now solve the original problem given the equation obtained.

4.2.2 Specific Solution

From the problem, we know that $a = b = 10$. Plugging into the general equation,

$$y = \sqrt{(10)^{4/3}(10)^{2/3} + 10^2},$$
$$y = 10\sqrt{2}.$$

Since x=y, we find the minimum length of the pole to be $10\sqrt{2}+10\sqrt{2} = 20\sqrt{2}$ feet.

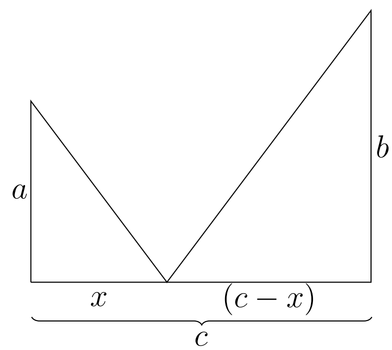
4.2.3 Similar Problem Examples

Example 22. *Find the length of the longest rod which can be carried horizontally around a corner from a corridor* 8 *m wide to one* 4 *m wide. [3]*

4.3 Post Problems

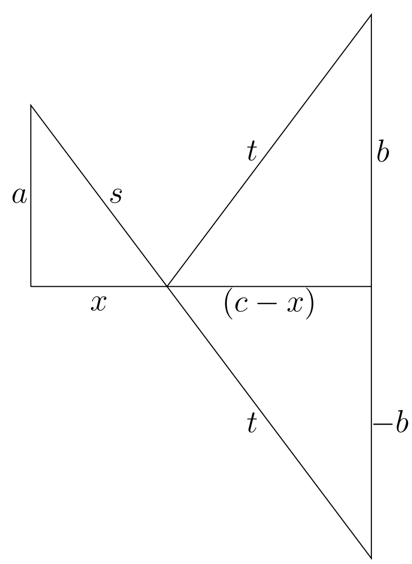
The Post Problems are another example of a type of optimization problem that cannot be solved using the AM/GM, but no calculus is involved either. After a very simple observation, all that is necessary is some basic algebra skills to find a general solution.

Example 23. *Two posts, one* 8 *feet high and the other* 12 *feet high, stand* 15 *ft apart. They are to be supported by wires attached to a single stake at ground level, the wires running to the tops of the posts. Where should the stake be placed, to use the least amount of wire? [13]*



4.3.1 Generalized Solution

Let a be the height of the first pole, b be the height of the second pole, and c be the distance between the two poles. We can recognize that the angle will remain unchanged if a straight line is drawn from $(0, a)$ to $(c, -b)$ as shown below.



We are looking for the shortest distance between these two points as we are trying to minimize the amount of wire used. Finding the slope between the two points, and plugging into point slope form gives:

$$\frac{\Delta y}{\Delta x} = \frac{a+b}{0-c} = \frac{a+b}{c},$$

$$y = -\frac{a+b}{c}x + a.$$

Finding where this line intersects the x axis gives the desired result.

$$0 = -\frac{a+b}{c} + a,$$

$$x = \frac{ac}{a+b}.$$

4.3.2 Specific Solution

From the problem, we know $a = 8$, $b = 12$ and $c = 15$. Plugging into the general equation,

$$x = \frac{(8)(15)}{8+12} = 6.$$

Thus we have found to minimize the wire used, the stake should be placed 6 feet from the 8 foot high post.

4.3.3 Similar Problem Examples

Example 24. *Two saplings,* 6 *and* 8 *feet high, are planted* 10 *feet apart. To prevent bending, poles the height of the trees are pounded in, then attached to each tree, and a rope tied to the top of each pole is then fixed to the ground (between the two trees) after being pulled taut. How close to the taller tree will the rope be fixed if the total length of the rope is to be minimized? [1]*

Example 25. *Two towns, located on the same side of a straight river, agree to construct a pumping station and filtering plant at the river's edge, to be used jointly to supply the towns with water. If the distances of the two towns from the river are a and b and the distance between them is c, show that the sum of the lengths of the pipe lines joining them to the pumping station is at least as great as $\sqrt{c^2 + 4ab}$. [4]*

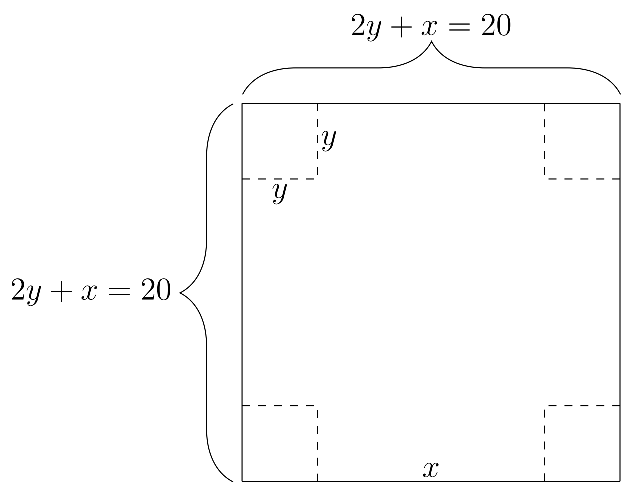
5 Problems that Have Evolved

Flipping through old and new calculus textbooks, perhaps the most interesting thing found was the evolution of some of these problems. Most early textbooks stick to number or geometrical problems, while later ones begin making more complex story problems including Triathlon, Pipe Corridor, and Post Problems. What we find is that while later books appear to include more complex examples, when reduced down to what they are maximizing/minimizing and their constraints, we see that they are in fact the same problem. Below are two examples of problems and how they evolved from their early format.

5.1 Number Problem → Box Problem

Consider the following problem:

Example 26. *From a square piece of cardboard of length* 20 *feet, we make an open box of maximum volume by cutting small squares out of the corners and turning up the sides. Find the dimensions of the box. [6]*



Suppose that y is the length of the corner square, x is the distance between the y's. The problem of maximizing the volume of the box can now be reduced to the following:

$$\text{maximize } \; x^2y \; \text{ subject to } \; 2y + x = 20.$$

But, if we recall from earlier the problem,

Example 27. *Divide the number* 20 *into two parts such that the product of one part by the square of the other part shall be a maximum. [8]*

We are trying to maximize the product considering the following:

$$\text{maximize } \ x^2 y \ \text{ subject to } \ x + y = 20.$$

While at first glance, these problems appear very different, they are reduced down to the same problem. From here, each problem can easily be solved using the AM/GM inequality. Another example of a problem we saw evolve twice from $1900 - 2016$ is shown below.

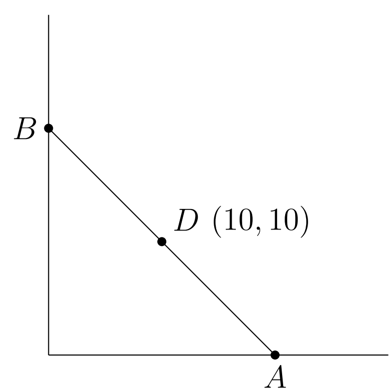
5.1.1 Similar Problem Examples

Example 28. *A box of maximum contents is to be made from a rectangular piece of tin* 30 *inches by* 14 *inches; required the side of the square to be cut out of each corner of the tin sheet. [12]*

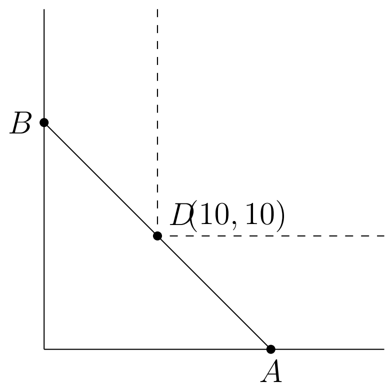
5.2 Graph Problem → Pipe Problem → Ladder Problem

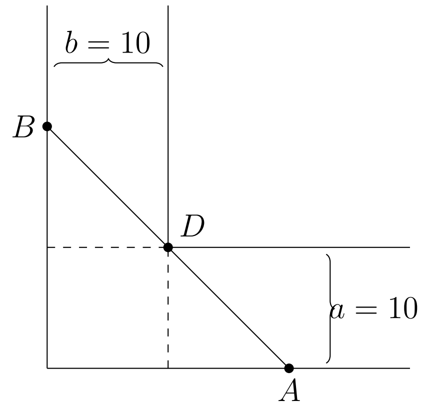
Consider the following problem:

Example 29. *A right triangle in the first quadrant has two coordinate axes as sides and the hypotenuse passes through the point* $(10, 10)$*. Find the vertices of the triangle such that the length of the hypotenuse is a minimum. [8]*



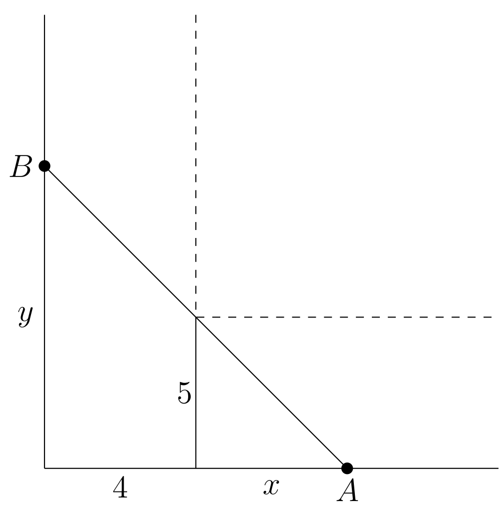
Without much effort, we can recognize this problem is the same problem as the Pipe Down the Corridor problem that we solved earlier. The following figure shows where the similarities occur.





Further, we can look at yet another problem that appears different, but is actually the same as these two problems.

Example 30. *A 5-ft fence stands 4-ft from a high wall. How long is the shortest ladder that can reach from the ground outside the fence to the wall? [10]*



5.2.1 Similar Problem Examples

Example 31. *Find the shortest length that can be drawn through a given point* (a, b) *and terminate in the rectangular axes to which the point is referred. [12]*

6 The CS Inequality

In addition the AM/GM inequality, the Cauchy Schwarz Inequality also has some useful applications when it comes to optimization problems. It can be used in some situations where the max/min and constraint involve addition, as the AM/GM inequality is not useful in problems such as these. We will begin by proving the inequality, looking at a typical application of the CS inequality, and then an example of an optimization problem that can be solved using the CS inequality.

6.1 Formal Proof - CS

The Cauchy Schwarz inequality can be written in two different forms - one in coordinate form, one in vector form. The coordinate form is as follows:

Theorem 32 (Cauchy Schwarz Inequality)**.** *Let $x_1, ..., x_n$ and $y_1, ..., y_n$ be real numbers. Then,*

$$(x_1y_1 + \cdots + x_ny_n) \leq \sqrt{(x_1^2 + \cdots + x_n^2)}\sqrt{(y_1^2 + \cdots + y_n^2)}.$$

Equality holds if and only if either there exists a real number k such that $x_1 = ky_1$, $x_2 = ky_2$,...,$x_n = ky_n$, or there exists a real number m such that $y_1 = kx_1$, $y_2 = kx_2$,...,$y_n = kx_n$.

The vector form of the theorem follows.

Theorem 33 (Cauchy Schwarz Inequality)**.** *If X and Y are any two vectors then*

$$|X \cdot Y| \leq |X||Y|,$$

with equality if and only if one of the vectors is a scalar multiple of the other.

Proof. If $Y = 0$, then both sides are 0, so the equation holds with equality. In this case $Y = 0 = X$. Now suppose that $Y \neq 0$ and t is any real number. Then

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n}(x_i - ty_i)^2 \\ &= \sum_{i=1}^{n} x_i^2 - 2t\sum_{i=1}^{n} x_iy_i + t^2\sum_{i=1}^{n} y_i^2 \end{aligned}$$

$$= |X|^2 - 2(X \cdot Y)t + t^2|Y|^2.$$

The last expression is a second-degree polynomial p in t. From the quadratic formula, the zeros of p are

$$t = \frac{(X \cdot Y) \pm \sqrt{(X \cdot Y)^2 - |X|^2|Y|^2}}{|Y|^2}.$$

Hence,

$$(X \cdot Y)^2 \leq |X|^2|Y|^2,$$

because if not, then p would have two distinct real zeros and therefore be negative between them, contradicting the inequality. Taking square roots yields the inequality if $Y \neq 0$.
If $X = tY$, then $|X \cdot Y| = |X||Y| = |t||Y|^2$, so equality holds. Conversely, if equality holds, then p has the real zero $t_0 = (X \cdot Y)/|Y|^2$, and

$$\sum_{i=1}^{n}(x_i - t_0 y_i)^2 = 0.$$

Therefore, $X = t_0 Y$. [11] □

7 Optimization Problems and the CS Inequality

Below is an example of where the CS inequality is useful in an optimization problem. First, there is an example of a typical application of the inequality, then we will move to an optimization problem that can be solved using the CS inequality, but not the AM/GM inequality.

7.1 Typical Problem

The Cauchy Schwarz inequality is typically taught in multi-variable calculus. Below is an example of the type of problem that typically shows up when learning how to apply this inequality.

Example 34. *What is the minimum value of x^2+9y^2 given that $4x+9y = 36$? [4]*

Recognizing that $4x+9y=4x+3(3y)$, and plugging into the inequality,

$$\begin{aligned} 4x+3(3y) &\leq \sqrt{4^2+3^2}\sqrt{x^2+(3y)^2}, \\ 36^2 &\leq 25(x^2+9y^2), \\ \frac{36^2}{25} &\leq x^2+9y^2. \end{aligned}$$

Therefore, the minimum value of x^2+9y^2 subject to $4x+9y=36$ is approximately 51.84. From the CS inequality, we know:

$$x=4t, \quad 3y=3t.$$

Solving for x and y gives:

$$\begin{aligned} \frac{3y}{x} &= \frac{3t}{4t}, \\ y &= \frac{1}{4}x. \end{aligned}$$

Plugging into our constraint gives,

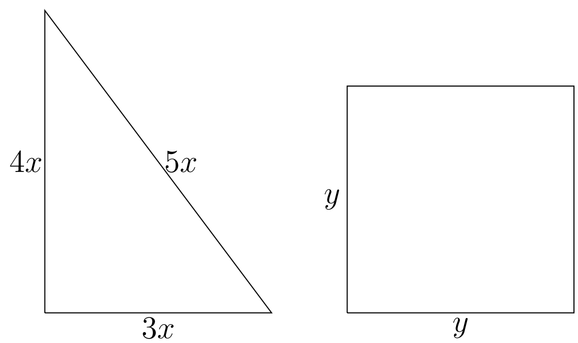
$$16y+9y=36,$$

$$y=\frac{36}{25}, \quad x=\frac{144}{25}.$$

7.2 Optimization Problem

Finding applications of the CS inequality on optimization problems prove to be much more difficult than the AM/GM. However, in certain instances, (such as the one below), we are able to see some really concise applications of the CS inequality.

Example 35. *A length of wire* 28 *feet long is cut into two pieces. One piece is bent into a* $3:4:5$ *right triangle and the other piece is bent into a square. Find the minimum combined area of the two shapes. [13]*



Let y be the side length of the square, and x be the multiple of the $3:4:5$ triangle. We know that the perimeter of the above figures add to the following:

$$12x + 4y = 28,$$

$$3x + y = 7,$$

and that the combined area of the triangle and square that we are trying to minimize is as follows:

$$\frac{(3x)(4x)}{2} + y^2,$$

$$6x^2 + y^2.$$

Recognizing,

$$3x + y = \left(\frac{\sqrt{6}}{2}\right)\left(\sqrt{6}x\right) + y,$$

then plugging into the CS inequality gives:

$$\langle\frac{\sqrt{6}}{2}, 1\rangle\langle\sqrt{6}x, y\rangle \leq \sqrt{\left(\frac{\sqrt{6}}{2}\right)^2 + 1^2}\sqrt{\left(\sqrt{6}x\right)^2 + y^2},$$

$$\langle\frac{\sqrt{6}}{2}, 1\rangle\langle\sqrt{6}x, y\rangle \leq \sqrt{\frac{5}{2}}\sqrt{6x^2 + y^2},$$

$$3x + y \leq \sqrt{\frac{5}{2}}\sqrt{6x^2 + y^2},$$

$$\frac{7}{\sqrt{5/2}} \leq \sqrt{6x^2 + y^2},$$

$$\frac{98}{5} \leq 6x^2 + y^2.$$

Therefore, the minimum value of the area of the triangle and the square is 19.6 ft ^{2}.

8 Historical Bit

Looking through a hundred years of calculus problems proved to be extremely interesting for a multitude of reasons. As was referenced in the "Problems that Have Evolved" section, we saw a few problems that in the early 1900's were given in a particular way and more recently have been worded differently. We found these problems particularly interesting, as it appears as the years go on the authors are trying to express how calculus is useful in more real world type problems. Also, as touched on earlier, the implementation of other disciplines also became very evident as the years went by in the texts. The more recent textbooks included more economic, physical, and medical problems, we suspected as calculus became a requirement for more disciplines. Listed below are a few examples of these problems. Another aspect that was especially interesting to look at was the introductions of each of these books. One of the books stated in the introduction,"In this calculus book for young men..." Clearly, in more recent calculus books this would not be an acceptable introduction, so seeing that mindset being phased out after the 1913 textbook was also a neat progression to see.

8.1 Economic/Medical/Physical Problems

Example 36. *In constructing the new Trump Colosseum, projected to occupy the entire state of Rhode Island, the builder estimates the initial costs (buying Rhode Island, etc.) as* 450 *times the cost of the first floor. The second floor is projected to cost twice as much as the first floor, the third floor three times as much as the first floor, etc. What number of floors in the building will give the cheapest average cost per floor? [13]*

Example 37. *The strength of a drug is given by $R(M)$ where M measures the dosage. The sensitivity of the patient's body to the drug is the derivative. The strength of a patients reaction to a dose of M milligrams of a certain drug is $R(M) = c_1M^2(c_2 - M)$ where c_1 and c_2 are positive constants. For what value of M is the strength a maximum? [13]*

9 Conclusion

While we got a good grasp on what the AM/GM inequality is, and when it can be used there is still more to learn. We did many optimization problems using this technique, however there are so many more problems out there. Becoming familiar with more of these problems, and learning small tricks to make the equality work would only continue to happen with exploration. Further, a look at a few more Cauchy Schwarz inequality applications would be an interesting direction for further research. Lastly, looking at how optimization problems were solved before calculus existed would bring about some unexpected results as well.

References

[1] S. Grossman *Calculus* Second Edition, New York, 1981.

[2] R. Larson, R.Hostetler, B. Edwards. *Calculus with Analytic Geometry* Fourth Edition, D.C. Heath, 1990.

[3] M. Protter, C. Morrey. *College Calculus with Analytic Geometry* Third Edition, Addison-Wesley, 1977.

[4] G. Thomas. *Calculus with Analytic Geomentry* Fourth Edition, Addison-Wesley, 1968.

[5] Robert Ellis, Denny Gulick. *Calculus One and Several Variables* University of Maryland, Maryland 1991.

[6] G.E.F. Sherwood, Angus E. Taylor *Calculus* University of California at Los Angeles, California 1943.

[7] P. Keef, D. Guichard, R. Gordon. *An Introduction to Higher Math.* Whitman College, Washington 2010.

[8] W.S. Franklin, B. MacNutt, R.L. Charles. *An Elementary Treatise* South Bethlehem, Pennsylvania, 1913.

[9] J. Stewart *Calculus: Early Transcendentals* Seventh Edition, Cengage Learning, 2012.

[10] Flanders, Prince. *something* 1978

[11] W. Trench. *Introduction to Real Analysis* Pearson Education, Trinity University, 2015.

[12] E. Nichols. *Differential and Integral Calculus with Applications* D. C. Heath & Co., Boston, 1902.

[13] R. Fraga, D. Hart, E. Herman, J. Stain, L. Welch. *Calculus Problems for a New Century* The Mathematical Association of America, Washington D.C., 1993.