The Platonic Solids

An Exploration of the Five Regular Polyhedra
and the Symmetries of Three-Dimensional Space

Abstract

The five Platonic solids (regular polyhedra) are the tetrahedron, cube, octahedron, icosahedron, and dodecahedron. The regular polyhedra are three dimensional shapes that maintain a certain level of equality; that is, congruent faces, equal length edges, and equal measure angles. In this paper we discuss some key ideas surrounding these shapes. We establish a historical context for the Platonic solids, show various properties of their features, and prove why there can be no more than five in total. We will also discuss the finite groups of symmetries on a line, in a plane, and in three dimensional space. Furthermore, we show how the Platonic solids can be used to visualize symmetries in $\mathbb{R}^3$.

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1 Introduction

The Platonic solids, or regular polyhedra, permeate many aspects of our world. They appear in crystals, in the skeletons of microscopic sea animals, in children's toys, and in art. They have been studied by many philosophers and scientists such as Plato, Euclid, and Kepler. They are of great interest in classical geometry, such as the work of Euclid which focuses on the figures themselves. They play a very interesting role in modern geometry as well, where geometry becomes an application of group theory. We will discuss the Platonic solids in both respects.

1.1 Definitions

In order to build an understanding of what a regular polyhedron is, we must first present definitions for two dimensional shapes.

Definition 1.1 (Polygon [5])**.** A polygon is a geometric object consisting of a number of points, vertices, and an equal number of line segments, sides, namely a cyclically ordered set of points in a plane, with no three successive points collinear, together with the line segments joining consecutive pairs of the points.

Definition 1.2 (Regular Polygon)**.** A regular polygon is a p-sided polygon in which the sides are all the same length and are symmetrically placed about a common center (i.e., the polygon is both equiangular and equilateral).

Note that there are a countably infinite number of regular polygons, one for each positive integer p such that $p \geq 3$. We will find that this is not the case for the regular solids. Now we will present definitions in order to prepare us to discuss three dimensional shapes.

Definition 1.3 (Polyhedron)**.** a polyhedron is a three-dimensional solid which consists of a collection of polygons joined at their edges.

Note that the plural of polyhedron is polyhedra.

Definition 1.4 (Convex Polyhedra [1])**.** A polyhedron is said to be convex if the planes that bound the solid do not enter its interior.

Definition 1.5 (Regular Polyhedra)**.** A regular polyhedron is convex, with all of its faces congruent regular polygons, and with the same number of faces at each vertex.

It is the case that there are only five solids in total that satisfy this definition. They are the five Platonic solids: tetrahedron, cube, octahedron, dodecahedron, and icosahedron. These can be seen respectively in figure 1.1.

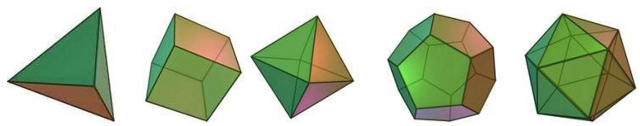
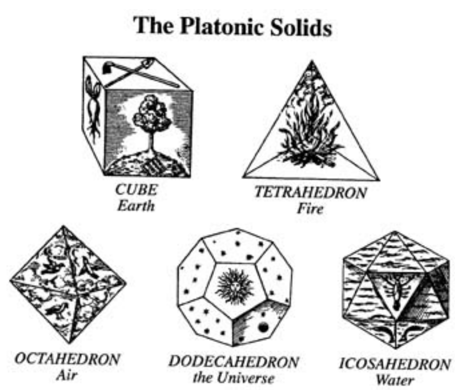


Figure 1.1: The five Platonic Solids

1.2 Historical Background

The Platonic solids have a rich history. We will briefly discuss some of the components of their history here.

The original discovery of the platonic solids is unknown. The five regular polyhedra all appear in nature whether in crystals or in living beings. They also appear all throughout history in children's toys, dice, art, and in many other areas. Around 360 B.C. the regular polyhedra are discussed in the dialogues of Plato, their namesake. The varying solids are likened to each of the elements, earth, fire, water, and air. He compared the tetrahedron to fire because the sharp stabbing points and edges reminded Plato of the stabbing heat from flames. The octahedron was associated with air because its many small smooth parts makes it seem as though it is barely there. Plato saw the icosahedron as water because it flows, like water, out of ones hands. And the cube was associated with earth because its causes dirt to crumble and fall apart, and at the same time it resembles the solidity of earth. The fifth solid, the dodecahedron, was considered the shape that encompasses the whole universe and was used for arranging the constellations in the heavens. [1]



In Euclid's famous text, *Elements* (published circa 300 B.C.), he begins with a discussion of constructing the equilateral triangle. *Elements* concludes with a detailed construction of the five regular solids. Euclid claims that these are in fact the only five with the necessary properties to be considered regular. Some people believe that Euclid created this text with the ultimate goal of establishing the five Platonic Solids [2].

Similar to Plato, the German astronomer Johannes Kepler also searched for connections between the regular polyhedra and the natural world. In 1596 he published his astronomy book, *Mysterium Cosmographicum*, translated as The Cosmographic Mystery, where he proposed his model for our solar system. At the time they knew of five planets other than earth. Kepler predicted connections between these five planets and the five Platonic solids. His model had each planet's orbit associated with a sphere and the distance between the spheres was determined by a Platonic solid, as seen in figure 1.2. The spheres of orbits circumscribed and inscribed each Platonic solid. The out-most sphere represented the orbit of Saturn. The remaining planets, moving in towards the sun, were Jupiter, Mars, Earth, Venus, and Mercury. The Platonic solids, again moving inwards to the sun, were the cube, tetrahedron, dodecahedron, icosahedron, and octahedron. Kepler ultimately determined this model to be incorrect however his astronomical research was quite fruitful. Notably, he developed Kepler's laws of planetary motion. The first of these laws asserts that a planet's orbit is in fact an ellipse.

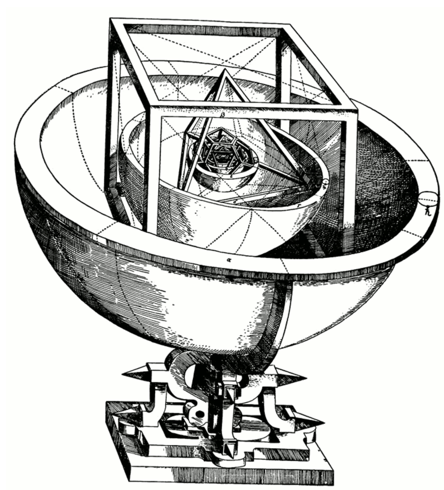


Figure 1.2: Kepler's model of the solar system using the Platonic solids

1.3 Objectives

In this paper we will cover a wide variety of topics relating to the Platonic solids. we will begin with a more geometric look at the regular polyhedra. We will do this by discussing the number of faces, edges, and vertices they have and by drawing information from these results. We will present two different proofs that explain why we can have no more than five Platonic solids. We will end the paper with an application of group theory. That is, we will use the Platonic solids to help describe the finite groups of congruences in three dimensions.

In Section 2 we begin by setting up some terminology to support our discussion of the Platonic solids. We see how we can define each of the five regular polyhedra by the ordered pair $\{p, q\}$. Then we begin our discussion on configurations. We will determine a useful equation that demonstrates an important relationship between the number of points and the number of lines in a configuration in two dimensions. Moving to three dimensions reveals analogous relations between the number of points, lines, and planes. Then we use these ideas to develop a very useful relation that helps describe the Platonic solids, $qN_0 = 2N_1 = pN_2$. Next we use this to determine the number of vertices, edges, and faces each regular polyhedron has with respect to the values p and q. Finally we conclude this section with a discussion of the Platonic duals. This idea will be quite important in our discussion of the symmetries of $\mathbb{R}^3$.

In Section 3 we present two proofs for why there are at most five Platonic solids. The first proof uses the geometry required by the definition of a regular polyhedron and the fact that the sum of the angles around a convex vertex must be less that 2π. The second proof uses two important equations: Euler's polyhedral formula $N_0 + N_2 - N_1 = 2$ and the previously derived equation, $qN_0 = 2N_1 = pN_2$. This proof relies on algebraic manipulation of these two results which lead us to the inequality $4 > (p-2)(q-2)$. This inequality has 5 ordered pair solutions that each correspond to one of our five Platonic solids.

Section 4 is dedicated to proving theorem 4.1, which describes all the possible finite groups of rotations of $\mathbb{R}^3$. We will show that every finite orientation preserving group of congruences of three dimensional space must be one of the following. It could be the trivial case where the group consists of only the identity, it could be a cyclic group, or it could be a dihedral group D_n of order $2n$. These first three possible finite groups describe all the finite symmetry groups in $\mathbb{R}^2$. If the group is none of the previously listed then it must be the group of congruences that take a Platonic solid to itself. We will see that the set of orientation-preserving congruences which are symmetries of some Platonic solid, are the same as those congruences for its dual. Hence there are exactly three additional finite groups of rotation for $\mathbb{R}^3$.

2 The Geometry of the Platonic Solids

In this section we will take a closer look at the relationships between the Platonic solids. We will see how we can use configurations to obtain an important equation relating the number of vertices, edges, and faces on each Platonic solid. We will present a table summarizing the important values associated with each regular polyhedron. Finally, we will conclude this section with a definition and description of Platonic duals.

Recall from definition 1.5 that the faces of a regular polyhedron are congruent regular polygons and that there are an equal number of such faces surrounding each vertex. Thus we will associate each Platonic solid with the notation $\{p, q\}$. The value p represents the number of edges surrounding each face. That is, the regular polyhedron defined by $\{p, q\}$ is made of p-gon faces. The value q represents the number of faces that surround each vertex.

2.1 Configurations

In this section we will introduce a configuration in two dimensions. We will then discuss how configurations function in three dimensions and how they can help us understand the regular polyhedra.

2.1.1 Configurations in Two Dimensions

Let us begin with some useful definitions.

Definition 2.1 (Binary Relation [1])**.** A binary relation on a set is a collection of ordered pairs on that set.

Definition 2.2 (Incidence Relation)**.** An incidence relation is a binary relation between two objects that touch each other.

For example when a line passes through a point we can say the line is incident with the point. Similarly, if a point lies on a line they are said to have an incidence relation. This concept retains its meaning in three dimensions as well. That is, a line contained in a plane is incident with the plane.

Definition 2.3 (Configuration in Two Dimensions [1])**.** A configuration in the plane is a set of finitely many points and lines such that each point is incident to the same number of lines and each line is incident to the same number of points

Each line in a configuration has the same number of points lying on it and each point has the same number of lines passing through it. Let N_0 be the number of points and let N_1 be the number of lines. Additionally, let us define N_{01} as the number of lines that pass through each point and N_{10} as the number of points that lie on each line. View the examples in the figures below.

Figure 2.2 and 2.3 have lines in general position.

Definition 2.4 (Lines in General Position)**.** Lines in general position are defined to be an arrangement of lines such that no three intersect at a single point, and no two lines are parallel.

Another way to view this definition is that, every point has exactly 2 lines passing through it; that is, $N_{01} = 2$. Furthermore, observe that since each line is intersected by every other line in the configuration, each line will have $N_1 - 1$ points on it; that is $N_{10} = N_1 - 1$.

As demonstrated in figure 2.2 a p-gon is a special case of a configuration with lines in general position where, $N_0 = N_1 = p$ and $N_{01} = N_{10} = 2$. Specifically, in figure 2.2 we have $p = 3$, a triangle.

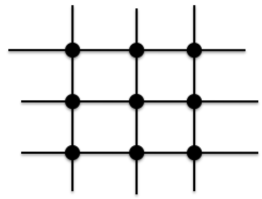


Figure 2.1: $N_0 = 9$ and $N_1 = 6$

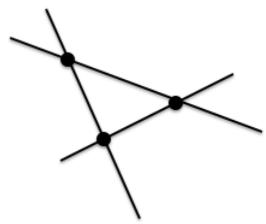


Figure 2.2: $N_0 = 3$ and $N_1 = 3$

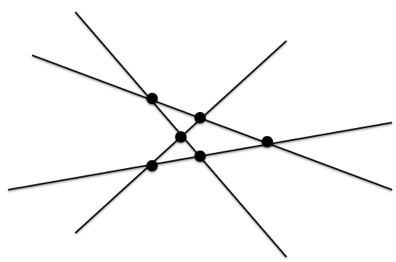


Figure 2.3: $N_0 = 6$ and $N_1 = 4$

Now we would like to derive a formula that relates these values (i.e. N_0, N_1, N_{01}, and N_{10}) for any configuration. One way to think about this is to consider each incidence relation as an ordered pair, (P_i, L_i), where P_i is the i^{th} point and L_i is the i^{th} line. There are two ways to count how many ordered pairs we have. One way is to begin with P_1 and count an ordered pair for each L_i that passes through that point; that is N_{01} ordered pairs. Then do the same for P_2. Now we have $2N_{01}$ ordered pairs. Continue in this manner until you reach P_{N_0}. Now we see that we must have N_0N_{01} incidence relations.

Another way to determine the number of incidence relations is to begin with L_1 and form an ordered pair with each point that it passes through. This will give N_{10} ordered pairs. Continuing in the same manner as we did previously, we will eventually arrive at the last line, line L_{N_1}, and determine we have $N_1 N_{10}$ incidence relations. Hence we have demonstrated two ways of arriving at the same number. This gives us the following formula,

$$N_0 N_{01} = N_1 N_{10}.$$

2.1.2 Configurations in Three Dimensions

Now that we have demonstrated what a configuration is in two dimensions and a significant equation relating points and lines, we will extend this concept into three dimensions. Furthermore, we will look at how our regular polyhedra interact with these results.

Definition 2.5 (Configuration in Three Dimensions [1])**.** A configuration in space is a finite set of points, lines, and planes, where each object is incident with the other two the same number of times.

Explicitly stated this definition says that, each point is incident to the same number of lines, each line is incident to the same number of planes, each plane is incident to the same number of points. The reciprocal holds true for each of these three statements.

Let N_0 be the number of points, N_1 be the number of lines, and N_2 be the number of planes. Each point, line, and plane in the configuration is incident with each other the same number of times. We will denote this as follows. Let $j = 0, 1, 2$ and let $k = 0, 1, 2$ but $j \neq k$. Then we see that the analogous equation from the previous section is,

$$N_j N_{jk} = N_k N_{kj}.$$

This could also be written as three separate equations,

$$N_0 N_{01} = N_1 N_{10}, \quad N_0 N_{02} = N_2 N_{20}, \quad N_1 N_{12} = N_2 N_{21}$$

which relates points and lines, points and planes, and lines and planes, respectively. It may be convenient to view these numbers in a matrix where N_{jj} is the number we have been calling N_j.

$$\begin{bmatrix} N_{00} & N_{01} & N_{02} \\ N_{10} & N_{11} & N_{12} \\ N_{20} & N_{21} & N_{22} \end{bmatrix}$$

We recall that we can denote each Platonic solid as $\{p, q\}$ where p represents the number of sides on each face and q represents the number of faces surrounding each vertex. Furthermore each edge (line) will pass through exactly two vertices (points), and exist between two faces (planes). Thus we have

$N_{10} = N_{12} = 2$. Now we see that the configuration of any Platonic solid, $\{p, q\}$ can be represented as,

$$\begin{bmatrix} N_0 & q & q \\ 2 & N_1 & 2 \\ p & p & N_2 \end{bmatrix}.$$

We would also like to note that applying the formula $N_j N_{jk} = N_k N_{kj}$ to the Platonic solid configuration gives,

$$qN_0 = 2N_1 = pN_2.$$

This formula will be significant in our topological proof of the fact that there are no more than five Platonic solids.

2.2 Derivation of the Values for N_0, N_1, and N_2

Now we will derive the formulas for the number of vertices, edges, and faces on each Platonic solid given their values for p and q. In the preceding section we established $qN_0 = 2N_1 = pN_2$. Rearranging gives,

$$N_0 = \frac{2N_1}{q}, \qquad N_2 = \frac{2N_1}{p}.$$

Now we will introduce Euler's polyhedral formula, which we will discuss in more depth in Section 3.2. Euler's formula, for any convex polyhedron with no holes, is $N_0 - N_1 + N_2 = 2$. Substituting the above results for N_0 and N_2 into Euler's formula yields,

$$\frac{2N_1}{p} + \frac{2N_1}{q} - N_1 = 2.$$

Now we can solve for N_1.

$$\begin{aligned}
\frac{2N_1}{p} + \frac{2N_1}{q} - N_1 &= 2 \\
\frac{2qN_1}{pq} + \frac{2pN_1}{pq} - \frac{pqN_1}{pq} &= \\
\frac{2qN_1 + 2pN_1 - pqN_1}{pq} &= \\
N_1 \frac{2q + 2p - pq}{pq} &= \\
\frac{4 + 2q + 2p - pq - 4}{pq} &= \frac{2}{N_1} \\
\frac{pq}{4 - (pq - 2p - 2q + 4)} &= \frac{N_1}{2} \\
\frac{2pq}{4 - (p-2)(q-2)} &= N_1
\end{aligned}$$

Observe that since both p and q are positive we have bounded their values by our denominator. That is, since $2pq > 0$, it follows that $4-(p-2)(q-2) > 0$, which yields $p > 2$ and $q > \frac{2p}{p-2}$.

Now note that, $N_1 = \frac{qN_0}{2} = \frac{pN_2}{2}$. Thus substitution and simple cancellation gives the remaining formulas. Hence we have,

$$N_0 = \frac{4p}{4-(p-2)(q-2)}, \quad N_1 = \frac{2pq}{4-(p-2)(q-2)}, \quad N_2 = \frac{4q}{4-(p-2)(q-2)}.$$

The following table lists each of the five Platonic solids with its respective values for p and q and their corresponding number of vertices, edges, and faces.

Name	p	q	Vertices, N_0	Edges, N_1	Faces, N_2
Tetrahedron	3	3	4	6	4
Cube	4	3	8	12	6
Octahedron	3	4	6	12	8
Icosahedron	3	5	12	30	20
Dodecahedron	5	3	20	30	12

2.3 Platonic Duals

The preceding table begins to bring to light some important relations between the Platonic solids. In this section we will discuss and define these relationships.

Let us begin by discussing the cube and octahedron. We first note that they both have 12 edges. We also observe that, while the cube has 8 vertices, the octahedron has 8 faces. Furthermore, while the cube has 6 faces, the octahedron has 6 vertices. There is something intriguing going on here.

Further observation reveals that we have consistency between every pair of Platonic solids $\{p_1, q_1\}$ and $\{p_2, q_2\}$ such that $p_1 = q_2$ and $q_1 = p_2$. Specifically, these pairs will have equal numbers of edges, and the number of faces and vertices they have will be switched. We call this type of pairing a dual relationship. That is, $\{p_1, q_1\}$ is the dual Platonic solid to $\{p_2, q_2\}$ and vice versa.

Definition 2.6 (Dual Polyhedra)**.** For every polyhedron, there exists a polyhedron in which faces and polyhedron vertices occupy complementary locations. This polyhedron is known as the dual.

As we see in figure 2.4, we can form a dual by taking the center of each face of the polyhedron and letting those points become the vertices of the dual. We then connect the new vertices with edges that cross perpendicular to each edge of the original solid. We note that the cube and the octahedron are duals, the icosahedron and dodecahedron are duals, and the tetrahedron is its own dual.

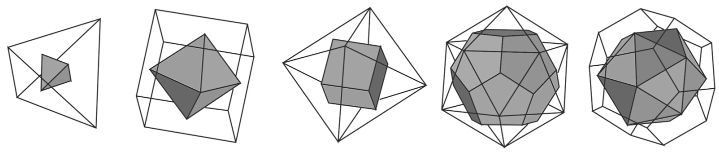


Figure 2.4: The Platonic solids and their duals [6]

Now we recall the matrix that represents a three dimensional configuration, which we can think of as a polyhedron.

$$\begin{bmatrix} N_{00} & N_{01} & N_{02} \\ N_{10} & N_{11} & N_{12} \\ N_{20} & N_{21} & N_{22} \end{bmatrix}$$

As we form a dual polyhedron, vertices and faces are interchanged, or in the language of configurations the points and planes are interchanged, while the number of edges (lines) stays consistent. In order to form a dual configuration we simply replace each 0 with a 2 and each 2 with a 0 in the matrix above. For example, N_{02} becomes N_{20}, N_{10} becomes N_{12}, etc. A less explanatory but more rigorous way to state the process of finding the dual configuration is, replace each N_{jk} with $N_{j'k'}$, where $j + j' = k + k' = 2$ [1].

Also recall that the configuration of any Platonic solid $\{p, q\}$, can be represented as,

$$\begin{bmatrix} N_0 & q & q \\ 2 & N_1 & 2 \\ p & p & N_2 \end{bmatrix}.$$

This representation demonstrates that the regular polyhedra have the property that every dual of a Platonic solid $\{p, q\}$ is a Platonic solid $\{q, p\}$.

3 Proving There Are Only Five

In this section we will present two different proofs that show why we can have at most five Platonic solids. It is significant to note that the following proofs guarantee that no more than five regular polyhedra exist, however it does not ensure each of their existence. That can only be shown through careful and explicit construction.

Note that definitions for a convex and regular polyhedron (definition 1.4 and definition 1.5) will be necessary in our proofs. We will show that there are no more than five, three dimensional shapes that satisfy the conditions outlined in these definitions.

3.1 Geometric Proof

In this section we will present a proof of why there are at most five Platonic solids that relies on the geometry of the polyhedra. A version of this proof appeared in Euclid's book, *Elements*.

An important result required to complete our geometric proof is:

Lemma 3.1. *The sum of the angles of the faces that surround each vertex on a convex polyhedron must total less than* 2π*.*

Note that if they equal 2π, then the surrounding faces create a flat surface. A flat surface cannot be a vertex of a polyhedron. If the sum of angles is greater than 2π, some bounding plane will necessarily enter the interior of the solid, and therefore it will not be a convex polyhedron. Now we are ready to present our proof.

Theorem 3.1. *There exist no more than five regular polyhedra.*

Proof. We will prove this result in several parts. From definition 1.5 we know that the faces on a given regular polyhedron must be congruent regular polygons. Furthermore, we know that each vertex is surrounded by the same number of faces. We will show that there are exactly five cases in which congruent regular polygons placed around a given point (the vertex) have angles that add up to less than 2π. Our first case will be equilateral triangles. Then we will look at squares and eventually work our way through all the sets of possible polygon faces.

3.1.1 Regular Triangles

Now we consider the case in which the faces of a regular polyhedron are regular triangles, the simplest regular polygon.

A regular triangle has three sides and angles of $\frac{\pi}{3}$ radians. We need at least three faces to define a vertex on a polyhedra. Three triangular faces add up to π radians as seen in figure 3.1. Four triangular faces add up to $\frac{4\pi}{3}$ as seen in figure 3.2. Five triangular faces add up to $\frac{5\pi}{3}$ as seen in figure 3.3. For each of these cases the sum of the angles is less than 2π. With six regular triangular faces we have exactly 2π radians and thus a flat surface as seen in figure 3.4. With six or more triangular faces we will not be able to form a vertex on a convex polyhedra. Thus, there are exactly three possible cases of a regular polyhedra with triangular faces.

The figures that follow are the shapes we would obtain if we isolated the polygons surrounding a single vertex, cut along one of the edges separating two of these faces, and laid the polygon faces flat. As long as there is some gap between one set of edges we are able to fold the faces up to form a convex vertex.

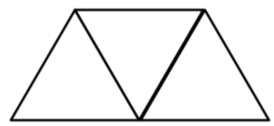


Figure 3.1: $\{3, 3\}$, the sum of the angles is π.

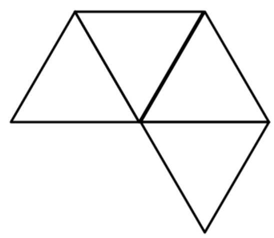


Figure 3.2: $\{3, 4\}$, the sum of the angles is $\frac{4\pi}{3}$.

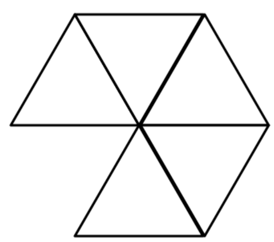


Figure 3.3: $\{3, 5\}$, the sum of the angles is $\frac{5\pi}{3}$.

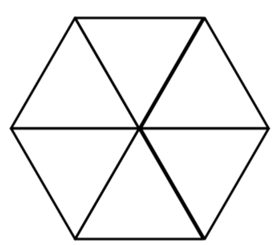


Figure 3.4: Six triangles leave no gap to form a convex vertex

3.1.2 Squares

Now we consider the case in which the faces of a regular polyhedron are squares.

The square has interior angles of $\frac{\pi}{2}$ radians. Therefore three squares surrounding a vertex give a sum of $\frac{3\pi}{2}$ as seen in figure 3.5. Four squares give exactly 2π, as seen in figure 3.6. Thus we cannot form a convex vertex with four squares surrounding a single point. Hence we have found the only possibility of a regular polyhedron with square faces.

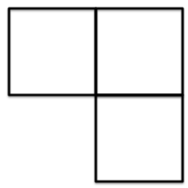


Figure 3.5: $\{4, 3\}$, the sum of the angles is $\frac{3\pi}{2}$.

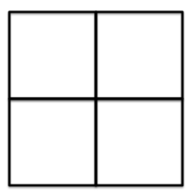


Figure 3.6: Four squares leave no gap to form a convex vertex

3.1.3 Regular Pentagons and Beyond

Now we consider the case in which the faces of a regular polyhedron are regular pentagons.

The regular pentagon's interior angles are $\frac{3\pi}{5}$ radians. Therefore three regular pentagons have a sum of $\frac{9\pi}{5}$ radians which is still less than 2π as seen in figure 3.7. Clearly, adding a fourth pentagon would add up to greater than 2π. Thus this is the only possibility for a regular polyhedron with pentagon faces.

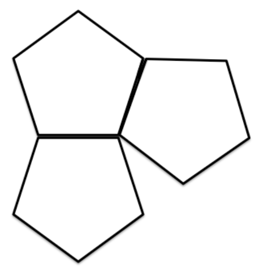


Figure 3.7: $\{5,3\}$, the sum of the angles is $\frac{9\pi}{5}$.

Now we consider the case in which the faces of a regular polyhedron are regular hexagons which have six sides. The regular hexagon's interior angles are $\frac{2\pi}{3}$ radians. Three hexagons surrounding a point form a flat surface with 2π radians, as seen in figure 3.8. Thus we cannot make a regular polyhedron with pentagons for faces. Note that each successive regular polygon has angles of greater measure. Therefore there are no other possibilities for the type of face and the number of faces surrounding each vertex than the cases outlined here.

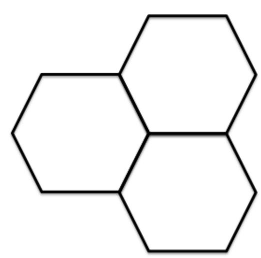


Figure 3.8: Three hexagons leave no gap to form a convex vertex

Thus the fives cases that remain as possible regular polyhedra are 3, 4, or 5 triangles around a vertex, 3 squares around a vertex, and 3 pentagons around a vertex, as shown in figure 3.9. Hence there are at most five regular solids. $\square$

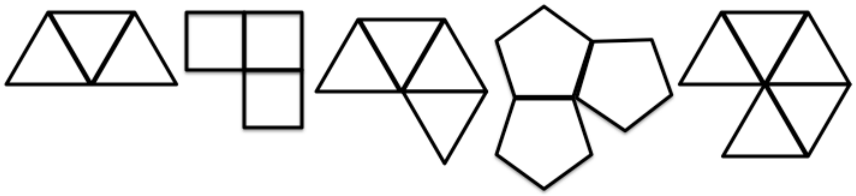


Figure 3.9: The Five Platonic Solids

As we know, each of these five possibilities correspond with a Platonic solid. The triangular faces correspond to the tetrahedron, the octahedron, and the icosahedron. The square faces corresponds to the cube. The pentagon faces corresponds to the dodecahedron.

3.2 Topological Proof

Now we will present another proof. This version relies on Euler's Polyhedral Formula.

Definition 3.1 (Simply-Connected [1])**.** A polyhedron is simply-connected if every simple closed curve drawn on the surface can be shrunk to a point.

The following formula relates the number of N_0 vertices, N_1 edges, and N_2 faces of a simply-connected convex polyhedron.

Theorem 3.2 (Euler's Polyhedral Formula)**.** *The polyhedral formula states*

$$N_0 + N_2 - N_1 = 2.$$

The other formula required for this proof is $qN_0 = 2N_1 = pN_2$. This was first suggested in our section on configurations. We will now verify it considering our regular polyhedra directly.

Let us recall a bit of notation. We say that q is the number of faces surrounding each vertex, and p is the number of edges surrounding each face. We seek to establish a relationship between the number of vertices, edges, and faces, on a regular polyhedra. Let A represent the number of instances where a vertex touches an edge on a Platonic solid. As we have defined q, each vertex is in contact with q edges. Hence,

$$\begin{aligned} A &= q + q + \cdots + q \\ &= \sum_{n=1}^{N_0} q \\ &= qN_0. \end{aligned}$$

That is, on any given Platonic solid there are qN_0 instances where an edge comes into contact with a vertex. Furthermore, each edge begins and ends at a vertex. Therefore, each edge touches 2 vertices. Thus we also have that,

$$\begin{aligned} A &= 2+2+\cdots+2 \\ &= \sum_{n=1}^{N_1} 2 \\ &= 2N_1. \end{aligned}$$

We have thus found a different way of counting the number of vertex-edge intersections on a regular polyhedra. Putting these results together gives $qN_0 = 2N_1$.

We will now, similarly, show two different ways of counting the number of face-edge intersections. Let B represent the number of instances where a face touches an edge on a Platonic solid. Each face comes into contact with p edges. Hence,

$$\begin{aligned} B &= p+p+\cdots+p \\ &= \sum_{n=1}^{N_2} p \\ &= pN_2. \end{aligned}$$

That is, on any given Platonic solid there are pN_2 instances where a face touches an edge. Since every edge necessarily exists between exactly two faces we know that each edge touches two faces. Thus we also have,

$$\begin{aligned} B &= 2+2+\cdots+2 \\ &= \sum_{n=1}^{N_1} 2 \\ &= 2N_1. \end{aligned}$$

Putting these results together gives $pN_2 = 2N_1$. Now we see that we can combine our two equalities to obtain,

$$qN_0 = 2N_1 = pN_2,$$

as desired. Now we will use the relations $N_0 - N_1 + N_2 = 2$ and $qN_0 = 2N_1 = pN_2$ to obtain an important new result. Through simple algebraic manipulation and substitution we have,

$$\begin{aligned} N_0 - N_1 + N_2 &= 2 \\ \frac{N_0 - N_1 + N_2}{2N_1} &= \frac{2}{2N_1} \\ \frac{N_0}{2N_1} - \frac{N_1}{2N_1} + \frac{N_2}{2N_1} &= \frac{1}{N_1} \\ \frac{N_0}{qN_0} - \frac{1}{2} + \frac{N_2}{pN_2} &= \\ \frac{1}{q} - \frac{1}{2} + \frac{1}{p} &= \quad . \end{aligned}$$

This gives us,

$$\frac{1}{N_1} = \frac{1}{p} + \frac{1}{q} - \frac{1}{2},$$

which expresses N_1 in terms of p and q. Further algebraic manipulation gives,

$$\begin{aligned}
\frac{1}{N_1} &= \frac{1}{p} + \frac{1}{q} - \frac{1}{2} \\
\frac{1}{N_1} + \frac{1}{2} &= \frac{1}{p} + \frac{1}{q} \\
\frac{1}{2} &< \frac{1}{p} + \frac{1}{q} \\
&< \frac{q}{pq} + \frac{p}{pq} \\
&< \frac{q+p}{pq} \\
2 &> \frac{pq}{q+p} \\
2(q+p) &> pq \\
2q + 2p + 4 &> pq + 4 \\
4 &> 4 - 2q - 2p + pq \\
4 &> (p-2)(q-2).
\end{aligned}$$

This inequality matches the bounding of p and q that we observed in the previous section. Recall that the denominator of the equations for N_0, N_1 and N_2 is $4-(p-2)(q-2)$, which must be strictly greater than 0. The only solutions to this inequality for the regular polyhedron $\{p, q\}$, are

$$\{3,3\},\ \{3,4\},\ \{4,3\},\ \{3,5\},\ \{5,3\}.$$

Each one of these solutions corresponds to one of the five Platonic solids, that is, the tetrahedron, octahedron, cube, icosahedron, and dodecahedron.

4 Describing the Finite Groups of Congruences

One of our main objectives is to discuss the symmetry groups of the Platonic solids and how they relate to the symmetries of three dimensional space. Over the course of this section we will be striving to provide both intuition and rigorous proof for the following theorem.

Theorem 4.1. *Suppose G is a finite group consisting of rotations of $\mathbb{R}^3$. Then G is isomorphic to one of the following:*

(a) The group consisting of only the identity;

(b) A cyclic group of order n;

(c) *A dihedral group of order* $2n$*;*

(d) *The group of orientation-preserving congruences which are symmetries of a tetrahedron;*

(e) *The group of orientation-preserving congruences which are symmetries of a cube (or an octahedron);*

(f) *The group of orientation-preserving congruences which are symmetries of an icosahedron (or a dodecahedron).*

We will begin building up to this result with the simplest case. We will start with congruences in one dimension and work up to three dimensional space.

We will define a congruence now, and provide an alternative definition once we begin our discussion of congruences in two dimensions. Note that we will use the terms symmetry and congruence interchangeably.

The first way to define the congruence f is as a distance preserving function. Let A and B be points in $\mathbb{R}^k$, where $k = 1, 2, 3$. We will denote the distance between points A and B by $d(A, B)$.

Definition 4.1 (Congruence)**.** A function $f : \mathbb{R}^k \to \mathbb{R}^k$ is a congruence if for all $A, B \in \mathbb{R}^k$, we have

$$d(A, B) = d(f(A), f(B)).$$

A function that preserves distances will preserve angles as well. We are aiming to prove results about finite groups of congruences. Thus we will now define a group and related relevant terminology.

Definition 4.2 (Binary Operation [4])**.** Let G be a set. A binary operation on G is a function that assigns each ordered pair of elements of G an element of G.

Definition 4.3 (Group [4])**.** Let G be a set together with a binary operation that assigns to each ordered pair (a, b) of elements of G an element in G denoted by ab. We say G is a group under this operation if the following properties are satisfied.

1. Associativity. The operation is associative; that is, $(ab)c = a(bc)$ for all a, b, and c in G.

2. Identity. There is an element e (called the identity) in G such that $ae = ea = a$ for all a in G.

3. Inverses. For each element a in G, there is an element b in G (called an inverse of a) such that $ab = ba = e$.

Definition 4.4 (Order of a Group [4])**.** The number of elements of a group (finite or infinite) is called its order.

Definition 4.5 (Order of an Element [4])**.** The order of an element g in a group G is the smallest positive integer n such that $g^n = e$. If no such integer exists, we say that g had infinite order.

4.1 The Symmetries of $\mathbb{R}^1$

In this section we will prove the following theorem:

Theorem 4.2 (Symmetries of the Real Number Line)**.** *There are two finite groups of symmetries of the real number line. They are,*
(a) The group containing a single element, the identity;
(b) The group containing two elements, the identity and a reflection.

The proof of this result is broken up into the following steps. First, we show that the only possible congruences in one dimension, aside from the identity, are translations and reflections. Then, we will show that a translation cannot be an element of a finite group. Finally, we will conclude that these groups listed are the only two possible finite groups of congruences. That is, we show that any two element group of congruences (such that it contains the identity and a reflection) is isomorphic to $\mathbb{Z}_2$.

Now let us define all of the distance preserving functions in one dimension. These congruences are the identity, reflections, and translations.

Definition 4.6 (Identity)**.** The identity is the transformation that takes every point to itself.

Definition 4.7 (Reflection about a point)**.** A reflection about a point P is a congruence such that P is taken to P and for all other points, A is taken to A' where P is the midpoint of $\overline{AA'}$.

Definition 4.8 (Translation)**.** A translation is a congruence that moves every point the same distance and in the same direction.

Proof. First we will show that the only symmetries of a line are translations and reflections.

Suppose a and b are numbers on the real number line with a distance between them of d. Suppose f is a congruence and f takes a to c, as seen in figure 4.1. We know that f must preserve distances. Therefore, it follows that f must map b to either $c - d$ or $c + d$.

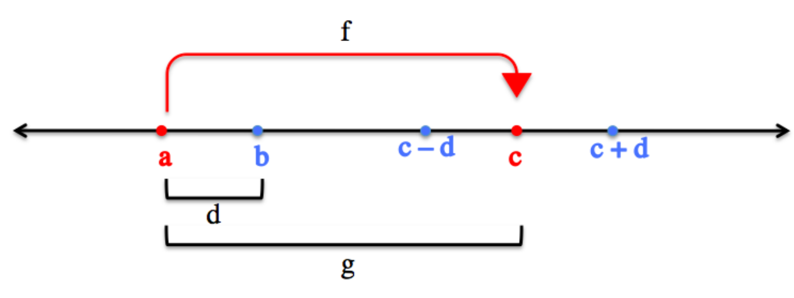


Figure 4.1: A congruence f in one dimension

Let us consider how f must be defined in each of these cases. First suppose that $f(a) = c$ and $f(b) = c + d$. If the distance between a and c is g, that is $|c - a| = g$, then it follows that f must be defined as,

$$f(x) = x + g.$$

This is a translation that shifts the real number line by g. Observe that,

$$f(a) = g + a = c$$

and

$$f(b) = g + b = (c - a) + b = c + (b - a) = c + d$$

as desired. Therefore, when $f(a) = c$ and $f(b) = c + d$ we have that f is a translation.

The second case is when $f(a) = c$ and $f(b) = c - d$. We want to show that in this case f is a reflection on the point $\frac{e}{2}$ which lies at the midpoint of line segment $\overline{ac}$; that is $e = a+c$ (see figure 4.2). In order to achieve this congruence we see that f is defined as,

$$f(x) = -x + e.$$

This is the desired reflection. Let us check how it transforms our points a and b. Observe that,

$$f(a) = -a + e = -a + (a + c) = c$$

and

$$f(b) = -b + e = -b + (c + a) = c - (b - a) = c - d$$

as desired. Therefore when $f(a) = c$ and $f(b) = c - d$ we have that f is a reflection.

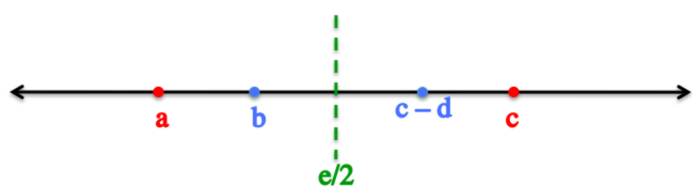


Figure 4.2: A reflection about the point $\frac{e}{2}$

As seen in figure 4.1, these are the only two distance preserving transformations possible when a is mapped to c, and therefore the only two symmetries possible in one dimension.

Now we are set up to prove our final result. Observe that by performing a given reflection twice we arrive at our original point. Hence a reflection has order 2. Now consider a translation. A translation moves a point a given distance in a given direction. Thus no matter how many times we apply the translation we will never return to our original number, as demonstrated in figure 4.3. Therefore a translation has infinite order.

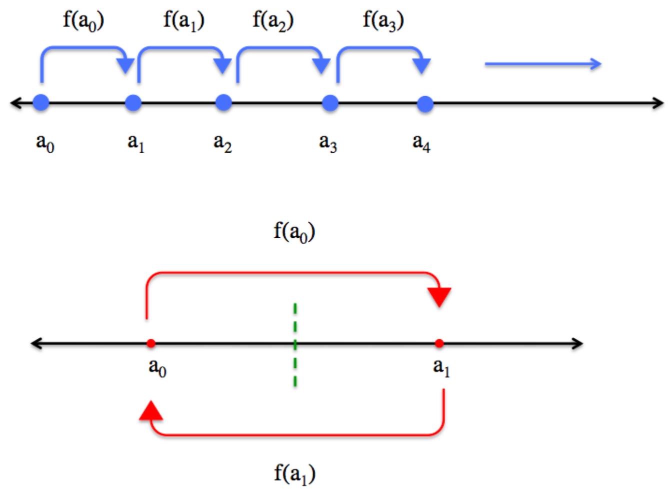


Figure 4.3: Translations have infinite order, while reflections have order 2.

We know that any group that contains an element of infinite order must be a group of infinite order. Hence when forming the finite groups of symmetries of the real number line we must exclude translations.

From our definition of a group we know that every group must have an identity element. Consider the group consisting of the single element, the identity. This is the trivial group and is the first case in theorem 4.2.

The next possibility is the group consisting of the identity and a single reflection. Since a reflection has order two, it is its own inverse. Thus the only property left to verify is associativity, that is $(ab)c = a(bc)$. We have associativity because the group operation in the finite group of congruences is composition of functions, which we know to be associative. Therefore we have found two finite groups of symmetries in one dimension.

Lastly, we need to confirm that no other finite groups exist. The only other possibilities are groups with the identity and multiple reflections. Suppose we had a group with the identity and two reflections about different points, say r_1 and r_2.

Suppose first we perform r_1, a reflection about point B, such that B is unchanged and C is taken to C_0. Then suppose we take a second reflection, r_2, about point D, which is half the distance between B and B_1. This will take B to B_1 and C_0 to C_1 as seen in figure 4.4. Now we see that the composition of these two reflections produces a translation.

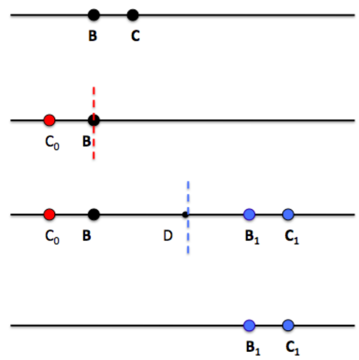


Figure 4.4: Reflecting about points B and D

Since B and C are arbitrary points we see that we can always create a translation by performing two reflections. This leads us to our desired result that, the composition of any two reflections about two points in $\mathbb{R}^1$ is a translation.

Thus, the composition of r_1 and r_2 is necessarily a translation. By the definition of a group this translation must also be in the group. However, translations have infinite order, therefore our group must be infinite. This applies to a group with any number of reflections greater than one. Hence, we have shown that there are no other possible finite groups of symmetries in one dimension. □

4.2 The Symmetries of $\mathbb{R}^2$

Now we are prepared to begin our discussion of the finite group of congruences in two dimension. This will provide us with the first three cases in our ultimate result, theorem 4.1. We will prove the following theorem.

Theorem 4.3. *Suppose G is a finite group consisting of symmetries of $\mathbb{R}^2$. Then G is isomorphic to one of the following*

(a) The group consisting of only the identity;

(b) A cyclic group of order n;

(c) A dihedral group of order $2n$.

Now we would like to present a second definition for a congruence. First however, we must define an orthogonal matrix.

Definition 4.9 (Orthogonal Matrix)**.** An $n \times n$ matrix A is an orthogonal matrix if $AA^T = I$, where A^T is the transpose of A and I is the identity matrix. In particular, an orthogonal matrix is always invertible, and $A^{-1} = A^T$.

Our new definition for a congruence is as follows,

Definition 4.10 (Congruence)**.** A function $f : \mathbb{R}^k \to \mathbb{R}^k$ is a congruence if there is an orthogonal matrix M and a fixed element $\mathbf{v}$ such that for all $\mathbf{X} \in \mathbb{R}^k$,

$$f(\mathbf{X}) = \mathbf{v} + M\mathbf{X}.$$

In this section we will be working in $\mathbb{R}^2$; thus we are assuming $\mathbf{X}$ is a two-entry column vector.

4.2.1 Inverse Congruences

In this section we will prove that congruences have inverses that are also congruences. We need this result in order to create groups of congruences, by definition 4.3.

Theorem 4.4. *If f is a congruence then the inverse function, f^{-1}, is given by $f^{-1}(\boldsymbol{X}) = -M^T\boldsymbol{v} + M^T\boldsymbol{X}$ and is also a congruence.*

Proof. We know that f has an inverse if and only if it is bijective. A congruence is clearly bijective because it takes every point in $\mathbb{R}^k$ to exactly one point in $\mathbb{R}^k$. We seek to find the inverse of $f(\mathbf{X}) = \mathbf{v} + M\mathbf{X}$. Let $f(Y) = \mathbf{X}$ such that $f^{-1}(\mathbf{X}) = Y$

Also recall that orthogonal matrices have the property that $M^{-1} = M^T$. It follows that the inverse of an orthogonal matrix is orthogonal.

Observe,

$$\begin{aligned}
\mathbf{X} &= \mathbf{v} + MY \\
\mathbf{X} - \mathbf{v} &= MY \\
M^{-1}(\mathbf{X} - \mathbf{v}) &= Y \\
M^{-1}\mathbf{X} - M^{-1}\mathbf{v} &= \\
M^T\mathbf{X} - M^T\mathbf{v} &=
\end{aligned}$$

Hence,

$$f^{-1}(\mathbf{X}) = -M^T\mathbf{v} + M^T\mathbf{X}.$$

Now that we have found the inverse function, let us show that it is a congruence as well. Observe that $-M^T\mathbf{v}$ is a fixed element. Furthermore note that M^T is an orthogonal matrix whenever M is orthogonal. Therefore f^{-1} satisfies the form described in our second congruence definition. $\square$

Another way to confirm this result is to show that $f(f^{-1}(\mathbf{X})) = \mathbf{X}$. Observe that,

$$\begin{aligned}
f(f^{-1}(\mathbf{X})) &= f(-M^T\mathbf{v} + M^T\mathbf{X}) \\
&= \mathbf{v} + M(-M^T\mathbf{v} + M^T\mathbf{X}) \\
&= \mathbf{v} + M(-M^{-1}\mathbf{v} + M^{-1}\mathbf{X}) \\
&= \mathbf{v} + -\mathbf{v} + \mathbf{X} \\
&= \mathbf{X},
\end{aligned}$$

as desired.

4.2.2 Composition of Congruences

In this section we will prove the following result about the composition of congruences.

Theorem 4.5. *The composition of two congruences is a congruence.*

Proof. Suppose that $\mathbf{v}_1, \mathbf{v}_2$ are fixed elements and that M_1, M_2 are orthogonal matrices, such that the functions f and g, defined as

$$f(\mathbf{X}) = \mathbf{v}_1 + M_1\mathbf{X} \qquad \text{and} \qquad g(\mathbf{X}) = \mathbf{v}_2 + M_2\mathbf{X},$$

are both congruences.

Now observe,

$$\begin{aligned} f(g(\mathbf{X})) &= f(\mathbf{v}_2 + M_2\mathbf{X}) \\ &= \mathbf{v}_1 + M_1(\mathbf{v}_2 + M_2\mathbf{X}) \\ &= \mathbf{v}_1 + M_1\mathbf{v}_2 + M_1M_2\mathbf{X}. \end{aligned}$$

Note that $\mathbf{v}_1 + M_1\mathbf{v}_2$ is a fixed element. Furthermore, recall that the product of two orthogonal matrices is orthogonal, thus M_1M_2 is an orthogonal matrix. Hence $f(g(\mathbf{X}))$ is of the form $\mathbf{v} + M\mathbf{X}$, described in definition 4.10. Therefore the composition of two congruences is a congruence. □

Note that we have found inverses and closure under composition of congruences. Hence we may form a group whose elements are congruences. We also note that these results hold for congruences $f : \mathbb{R}^k \to \mathbb{R}^k$ where $k = 1, 2, 3$.

4.2.3 Determinants and Orientation

In this section we will take a look at the determinant of an orthogonal matrix for any given congruence.

Recall the following property of orthogonal matrices from Linear Algebra: The determinant of an orthogonal matrix is equal to 1 or -1. If $f(\mathbf{X}) = \mathbf{v} + M\mathbf{X}$ is a congruence, let $\delta(f) = \det(M) = \pm 1$.

Let us define a homomorphism.

Definition 4.11 (Homomorphism)**.** A group homomorphism is a map

$$f : G \to H$$

between two groups such that the group operation is preserved. That is,

$$f(g_1g_2) = f(g_1)f(g_2)$$

for all $g_1, g_2 \in G$, where the product on the left-hand side is in G and on the right-hand side in H.

Now we would like to show the following result.

Theorem 4.6. *If $f(\boldsymbol{X}) = \boldsymbol{v} + M\boldsymbol{X}$ is a congruence and $\delta(f) = \det(M)$, then the function δ is a homomorphism from the group of all congruences, under composition, to the set $\{-1, 1\}$, under multiplication.*

Proof. Suppose G is the group of congruences. The function δ is a map $\delta : G \to \{-1, 1\}$. We want to show that operations are preserved under δ. That is, $\delta(fg) = \delta(f)\delta(g)$ for all $f, g \in G$ where the product on the left-hand side is function composition and the product on the right is multiplication.

Let f and g be defined as they are above, $f(\mathbf{X}) = \mathbf{v}_1 + M_1\mathbf{X}$ and $g(\mathbf{X}) = \mathbf{v}_2 + M_2\mathbf{X}$. Recall that, for any $A, B \in M_{m\times n}$, $\det(AB) = \det(A)\det(B)$. Observe that,

$$\begin{aligned} \delta(fg) &= \delta(f(g(\mathbf{X}))) \\ &= \delta(\mathbf{v}_1 + M_1\mathbf{v}_2 + M_1M_2\mathbf{X}) \\ &= \det(M_1M_2) \\ &= \det(M_1)\det(M_2) \\ &= \delta(f)\delta(g). \end{aligned}$$

Thus we have $\delta(fg) = \delta(f)\delta(g)$, as desired. □

Now let us define orientation.

Definition 4.12 (Orientation)**.** An orientation preserving linear transformation takes counterclockwise angles to counterclockwise angles, and it takes clockwise angles to clockwise angles. An orientation reversing linear transformation takes counterclockwise angles to clockwise angles, and it takes clockwise angles to counterclockwise angles.

Equivalently, the linear transformation $f(\mathbf{X}) = M\mathbf{X}$ is orientation preserving if the determinant of M is 1 and orientation reversing if the determinant is -1.

If $\delta(f) = 1$ we say f is orientation preserving. If $\delta(f) = -1$ we say it is orientation reversing. From theorem 4.6 we now know that composing two orientation preserving congruences (or two orientation reversing congruences) will result in an orientation preserving congruence. Furthermore, composing an orientation preserving and an orientation reversing congruence (in either order) will result in an orientation reversing congruence.

4.2.4 Properties of Congruences

In this section we explore some important properties of congruences. We will begin by proving the following result.

Theorem 4.7. *Suppose $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{R}^k$ and $\alpha_1, \ldots, \alpha_n$ are scalars such that*

$$\alpha_1 + \cdots + \alpha_n = 1.$$

If $f : \mathbb{R}^k \to \mathbb{R}^k$ is a congruence, then

$$f(\alpha_1\boldsymbol{v}_1 + \cdots + \alpha_n\boldsymbol{v}_n) = \alpha_1 f(\boldsymbol{v}_1) + \cdots + \alpha_n f(\boldsymbol{v}_n).$$

Proof. Suppose $f : \mathbb{R}^k \to \mathbb{R}^k$ is a congruence, defined as $f(\mathbf{X}) = \mathbf{v} + M\mathbf{X}$. It follows that,

$$\begin{aligned}
\alpha_1 f(\mathbf{v}_1) + \cdots + \alpha_n f(\mathbf{v}_n) &= \alpha_1(\mathbf{v} + M\mathbf{v}_1) + \cdots + \alpha_n(\mathbf{v} + M\mathbf{v}_n) \\
&= \alpha_1\mathbf{v} + \alpha_1 M\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v} + \alpha_n M\mathbf{v}_n \\
&= (\alpha_1\mathbf{v} + \cdots + \alpha_n\mathbf{v}) + (\alpha_1 M\mathbf{v}_1 + \cdots + \alpha_n M\mathbf{v}_n) \\
&= \mathbf{v}(\alpha_1 + \cdots + \alpha_n) + (M\alpha_1\mathbf{v}_1 + \cdots + M\alpha_n\mathbf{v}_n) \\
&= \mathbf{v} + M(\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n) \\
&= f(\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n).
\end{aligned}$$

□

Now we see that a general congruence is not exactly a linear transformation, but it has a similar property. For example, a translation is not a linear transformation. For any arbitrary congruence of the form $f(\mathbf{X}) = \mathbf{v} + M\mathbf{X}$, we have the additional condition that the scalars must add up to 1. Observe in the above algebra that the term $\mathbf{v}(\alpha_1 + \cdots + \alpha_n)$, would not simplify to $\mathbf{v}$ without this condition that $\alpha_1 + \cdots + \alpha_n = 1$. Now we will discuss an important consequence of this result which holds for $k = 1, 2, 3$, but we will discuss $\mathbb{R}^2$ here.

Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^2$ are not collinear. Then $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$ will be linearly independent; that is, they are not all in the same line as seen in figure 4.5.

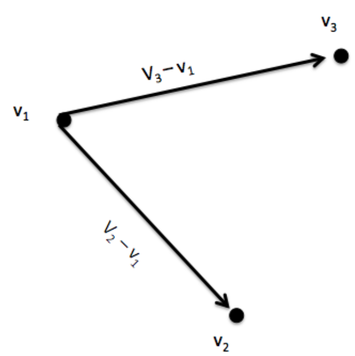


Figure 4.5: $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$ are linearly independent

Since $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$ are linearly independent they form a basis, and thus span all of $\mathbb{R}^2$. Hence any vector in $\mathbb{R}^2$ is an linear combination of $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$.

That is, we know that for any vector $\mathbf{X} \in \mathbb{R}^2$, there are unique scalars α_2, α_3 such that

$$\begin{aligned}
\mathbf{X} - \mathbf{v}_1 &= \alpha_2(\mathbf{v}_2 - \mathbf{v}_1) + \alpha_3(\mathbf{v}_3 - \mathbf{v}_1) \\
&= \alpha_2\mathbf{v}_2 - \alpha_2\mathbf{v}_1 + \alpha_3\mathbf{v}_3 - \alpha_3\mathbf{v}_1 \\
&= -\alpha_2\mathbf{v}_1 - \alpha_3\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 \\
&= (-\alpha_2 - \alpha_3)\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3.
\end{aligned}$$

Let $\alpha_1 = 1 - \alpha_2 - \alpha_3$. It follows that

$$\begin{aligned}
\mathbf{X} - \mathbf{v}_1 &= (\alpha_1 - 1)\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 \\
&= \alpha_1\mathbf{v}_1 - \mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 \\
\mathbf{X} &= \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3, \qquad \text{where} \qquad \alpha_1 + \alpha_2 + \alpha_3 = 1.
\end{aligned}$$

Thus the vector $\mathbf{X}$ in $\mathbb{R}^2$ is an affine combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. We define an affine combination as follows.

Definition 4.13 (Affine Combinations)**.** Given a set of points $x_1, x_2, \ldots, x_n$ in an affine space, an affine combination is defined to be the point

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

where the α_i are scalars and

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1.$$

We are now prepared to conclude the following result.

Theorem 4.8. *Suppose $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ are non-collinear sets in $\mathbb{R}^2$ and for all $i, j \leq 3$, $d(\boldsymbol{v}_i, \boldsymbol{v}_j) = d(\boldsymbol{w}_i, \boldsymbol{w}_j)$, then the assignments $f(\boldsymbol{v}_i) = \boldsymbol{w}_i$ for $i \leq 3$ leads to a unique congruence $f : \mathbb{R}^2 \to \mathbb{R}^2$.*

In other words, if we know where $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ map to, we know where every element in $\mathbb{R}^2$ maps. Note that this makes intuitive sense as well. Defining two points dictates what we do with position and distance, and the third determines orientation.

4.2.5 Introducing a Fixed Point

Our second definition for a congruence, definition 4.10, is useful because it depends on a particular coordinate system. In placing our transformations within a coordinate system we can choose to make our vector $\mathbf{v}$ or matrix M as simple as possible. In this section we will discuss the case where $\mathbf{v} = \mathbf{O}$, the zero vector.

Suppose that $\mathbf{O} \in \mathbb{R}^k$ is a fixed point for f, that is $f(\mathbf{O}) = \mathbf{O}$. Now we can set our coordinate system so that $\mathbf{O}$ is its origin. Now we have that for all $\mathbf{X} \in \mathbb{R}^k$,

$$f(\mathbf{X}) = \mathbf{O} + M\mathbf{X} = M\mathbf{X}.$$

Hence f is a linear transformation.

Note that we can also choose the identity matrix for M; that is, let $M = I$. It follows that $f(\mathbf{X}) = \mathbf{v} + \mathbf{X}$. Congruences of this form are called translations.

Moving forward we will focus our attention on the first case, such that the congruence has a fixed point. Let us prove the following result.

Theorem 4.9. *If G is a set of congruences that form a finite group under composition then there exists a $\boldsymbol{O} \in \mathbb{R}^k$ such that $f(\boldsymbol{O}) = \boldsymbol{O}$ for all $f \in G$.*

Proof. We want to show $f(\mathbf{O}) = \mathbf{O}$ for all $f \in G$.

Suppose $G = \{\tau_1, \dots, \tau_n\}$. Let

$$\mathbf{O} = \frac{\tau_1(\mathbf{v}) + \cdots + \tau_k(\mathbf{v})}{n}$$

when $\mathbf{v}$ is any element of $\mathbb{R}^k$.

Now recall that if $f \in G$, then $\{f\tau_1, \dots, f\tau_n\}$ is another listing of the elements of G. Therefore,

$$f\tau_1 + \cdots + f\tau_n = \tau_1 + \cdots + \tau_n.$$

From this fact, and theorem 4.7, it follows

$$\begin{aligned}
f(\mathbf{O}) &= f\left(\frac{\tau_1(\mathbf{v}) + \cdots + \tau_k(\mathbf{v})}{n}\right) \\
&= f\left(\frac{\tau_1(\mathbf{v})}{n} + \cdots + \frac{\tau_n(\mathbf{v})}{n}\right) \\
&= \frac{1}{n}[f(\tau_1(\mathbf{v})) + \cdots + f(\tau_n(\mathbf{v}))] \\
&= \frac{\tau_1(\mathbf{v}) + \cdots + \tau_n(\mathbf{v})}{n} \\
&= \mathbf{O}.
\end{aligned}$$

Thus, there is a $\mathbf{O} \in \mathbb{R}^k$ such that $f(\mathbf{O}) = \mathbf{O}$ for all $f \in G$, as desired. $\square$

Now let us apply this result in an example. Let G be a finite group of congruences with fixed point $\mathbf{O}$. Our coordinate system is constructed such that $\mathbf{O}$ is the origin. Thus each element of G is multiplication by an orthogonal matrix; that is, each element is a linear transformation. Let $k = 2$. Suppose $f \in G$ and $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Note that $f(\mathbf{i})$ must have length 1, so it will be somewhere on the unit circle and thus of the form $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ for some angle θ (see figure 4.6).

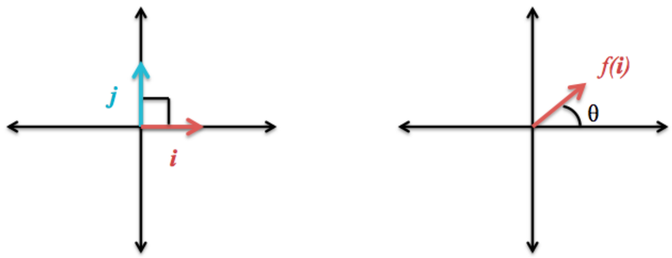


Figure 4.6: $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $f(\mathbf{i}) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

This leaves only two choices for $f(\mathbf{j})$. This is because $f(\mathbf{j})$ must be the same distance way from $f(\mathbf{i})$ as $\mathbf{j}$ is from $\mathbf{i}$, and there are only two orthogonal matrices that preserve this distance. These matrices are,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Hence $f(\mathbf{j})$ is either $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$, in which case f preserves orientation or it is $\begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$, in which case f reverses orientation as seen in figure 4.7. The dashed portion represents an orientation reversing congruence.

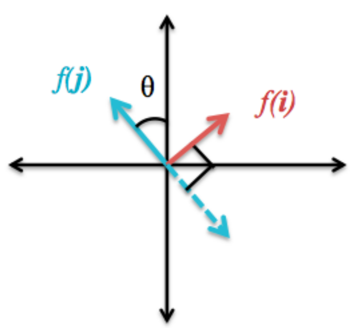


Figure 4.7: $f(\mathbf{j}) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ or $\begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$

4.2.6 Rotations

We will begin this section with the definition of a cyclic group

Definition 4.14 (Cyclic Group)**.** A cyclic group is a group that can be generated by a single element X (the group generator).

A cyclic group with finite order n and generator X satisfies $X^n = I$, where I is the identity element.

Now we will show that a finite group of orientation preserving congruences (a finite group of rotations) is a cyclic group of order n. Consider the following theorem.

Theorem 4.10. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a congruence such that f^n is the identity, where $n \in \mathbb{N}$. If f preserves orientation, then it is the rotation about some point $\boldsymbol{O}$ through an angle of $2\pi m/n$ for some $m \in \mathbb{N}$.*

Proof. Since f preserves orientation f must be a rotation through some angle θ. We will denote this as, $f = R_\theta$. Note that composing a rotation through θ with a rotation through ϕ is equivalent to rotation through $\theta + \phi$; that is,

$$R_\theta R_\phi = R_{\theta+\phi}.$$

It follows then that,

$$f^n = R_{n\theta} = R_0$$

since f^n is the identity. Therefore $n\theta$ must be equivalent to 2π or some multiple of 2π. That is,

$$\begin{aligned} n\theta &= 2\pi m, \quad m \in \mathbb{N} \\ \theta &= \frac{2\pi m}{n}, \end{aligned}$$

as desired. □

Theorem 4.11. *If G is a finite group of orientation preserving congruences and n is the order of G, then G is the group generated by a rotation about some point $\boldsymbol{O}$ through an angle of $2\pi/n$.*

Proof. We have that G is a finite group of order preserving congruences. Let $G = \{f_1, \ldots, f_n\}$. We want to show that G is the group generated by a rotation about some point **O** through an angle of $2\pi/n$, which is the cyclic group generated by $R_{2\pi/n}$. We will denote this latter group as $\langle R_{2\pi/n}\rangle$.

Theorem 4.10 tells us that an orientation preserving congruence, which returns to the identity after n iterations, is a rotation through some angle $2\pi j/k$. Therefore each element of G is a rotation of this form. Hence G must be a subgroup of $\langle R_{2\pi/n}\rangle$. Therefore,

$$\{f_1, \ldots, f_n\} \subseteq \langle R_{2\pi/n}\rangle.$$

However, from theorem 4.10 we know that f^n is the identity of $\langle R_{2\pi/n}\rangle$ and therefore it has order n. Thus G and $\langle R_{2\pi/n}\rangle$ have the same order, so they must be equal, as desired.

□

4.2.7 Reflections in Two Dimensions

In this section we will discuss the case where f reverses orientation and show that f must be a reflection. Let our coordinate system be set up around some fixed point $\mathbf{O}$. Then we have that f is multiplication by the matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$. We seek to show the following result:

Theorem 4.12. *If the linear transformation f reverses orientation, then f is the reflection across the line through $\boldsymbol{O}$ at an angle of $\theta/2$ with respect to the x-axis.*

Let us begin by building an intuition for why this is true through some figures. We will then explore some results about groups that this yields.

Observe, in figure 4.8, that f rotates $\mathbf{i}$ by some angle θ. We could also see this as a reflection across the line through $\mathbf{O}$ at an angle of $\theta/2$ with respect to the x-axis, as seen in figure 4.9. In defining f this way we see that there is only one choice for f such that f sends $\mathbf{j}$ to a location that preserves its distance with $\mathbf{i}$ and reverses orientation. When f is defined as a rotation through θ we have the orientation preserving case discussed previously.

In figure 4.9 we see that we move in the counter-clockwise direction to go from $\mathbf{i}$ to $\mathbf{j}$ but we move in the clockwise direction to go from $f(\mathbf{i})$ to $f(\mathbf{j})$. Hence we see that f is orientation reversing.

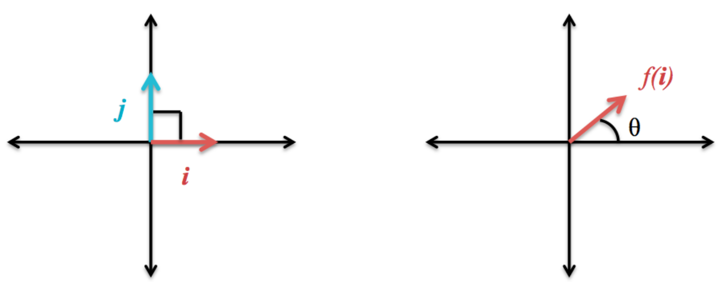


Figure 4.8: $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad f(\mathbf{i}) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

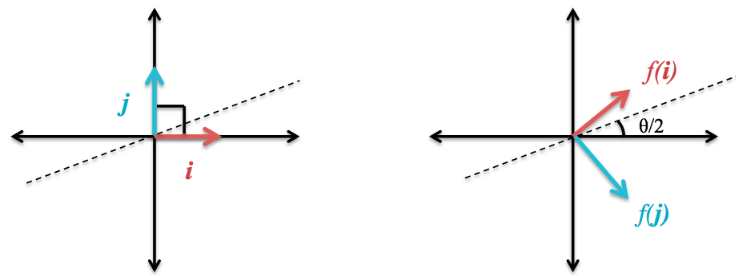


Figure 4.9: reflection across the line through **O** at an angle of $\theta/2$ with respect to the x-axis

4.2.8 The Dihedral Groups

We begin with a definition of a dihedral group.

Definition 4.15 (Dihedral Group)**.** The dihedral group D_n is the symmetry group of an n-sided regular polygon for $n > 1$. The order of D_n is $2n$.

The $2n$ motions, made up of n rotations and n reflections, together with the group operation of composition, form the dihedral group of order $2n$, denoted D_n. We think of these rotations and reflections as done on a regular n-gon. Thus D_n is generated by two elements, a rotation of $\frac{2\pi}{n}$ about the n-gon's center and a flip across any line of symmetry.

Let G be a finite group of symmetries of a plane with both orientation preserving and orientation reversing congruences. We have shown that a finite group of orientation preserving congruences is generated by a rotation about some point **O** through an angle $2\pi/n$. Furthermore, an orientation reversing function f can be expressed as the reflection across a line through the center at a certain angle, as shown above. Thus G contains both rotations and flips. It follows that $G \supseteq D_n$.

Now note that we have a rotation through an angle of $2\pi/n$, therefore G must contain n rotations. The composition of our reflection with each of these n rotations yields a total of n reflections which must also be in G. We now see that G has order $2n$. The dihedral group D_n also has order $2n$. It follows that these two groups are equivalent.

4.2.9 Finite Symmetry Groups in the Plane

Here we will put together the results from the previous sections to describe all possible finite groups, G, of symmetries of a plane.

In the previous section we have characterized one possibility. When G contains elements that are both orientation preserving and orientation reversing, G is isomorphic to D_n with order 2n.

We have also shown that if G is a finite group of solely orientation preserving congruences with order n, then G is the rotation group generated by rotations about some point $\mathbf{O}$ through an angle of $2\pi/n$. Thus, by definition, G is the cyclic group generated by some rotation with order n. Notice that the cyclic group generated by some rotation $2\pi/n$ is a subgroup of D_n.

Now let us consider Lagrange's theorem. Note that $|G|$ denotes the order of the group G.

Theorem 4.13 (Lagrange's Theorem [4])**.** *If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$. Moreover, the number of distinct left (right) cosets of H in G is $|G|/|H|$.*

By this theorem we have that if G is a finite group and H is a subgroup of G then, the order of H divides the order of G. Note that n divides $2n$, as expected. Furthermore, note that no number between $2n$ and n divides $2n$, hence there are no subgroups of D_n larger than the rotation group.

Another choice for G is the special case of a cyclic group, where not all elements preserve orientation. That is where G consists of only the identity and a reflection about some line, thus G has order two. A reflection about some line is its own inverse and thus no further element is needed to form a group.

Lastly, the trivial case is where G is only the identity, with order one. Now let us summarize the above results in the following theorem.

Theorem 4.14. *Suppose G is a finite group consisting of symmetries of $\mathbb{R}^2$. Then G is isomorphic to one of the following*

(a) The group consisting of only the identity;

(b) A cyclic group of order n;

(c) A dihedral group of order $2n$.

4.3 The Symmetries of $\mathbb{R}^3$

Our next goal is to produce a complete list of the finite rotation groups for $\mathbb{R}^3$. In the finite group of rotations there is an invariant point $\mathbf{O}$ at the origin. We will regard the elements of the group as operating on a sphere with invariant point $\mathbf{O}$ and radius 1. We we will call this sphere S. This gives us a convenient way of expressing rotations in $\mathbb{R}^3$. Note that when we refer to rotations about a point P on the surface of S, we mean the rotation around the line that passes through P and the center $\mathbf{O}$. Now we will review some definitions to help aid our discussion of rotations on a sphere.

Definition 4.16 (Great Circle)**.** A great circle is a circle on the surface of a sphere that lies in a plane passing through the sphere's center and thus divides the sphere into two equal hemispheres.

Definition 4.17 (Spherical Triangle)**.** A spherical triangle is a figure formed on the surface of a sphere by three great circular arcs intersecting pairwise in three vertices. The spherical triangle is the spherical analog of the planar triangle.

The following lemma will aid our discussion of rotations on a sphere.

Lemma 4.1. *Suppose P, Q, and R are points on S giving the spherical triangle ΔPQR with spherical angles at each vertex $\angle P$, $\angle Q$, $\angle R$. Suppose P, Q, and R are oriented clockwise around the spherical triangle. If f, g, h are functions that rotate the sphere around P, Q, R in the counter-clockwise direction through the angles $2\angle P$, $2\angle Q$, $2\angle R$, respectively, then $h \circ g \circ f$ is the identity. [1]*

Before proving this lemma we will discuss and clarify its meaning. We have our sphere S with spherical triangle ΔPQR sitting on the surface (see figure 4.10). Say we rotate the sphere counter-clockwise about P by twice the angle $\angle P$. The triangle will remain unmoved on an unattached outer surface while the sphere rotates underneath. Then we rotate counter-clockwise around Q by twice the angle $\angle Q$. Finally we rotate counter-clockwise around R by twice the angle $\angle R$. The composition of all three rotations will take the sphere back to its original positioning.

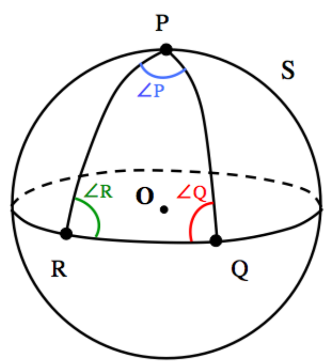


Figure 4.10: Spherical Triangle ΔPQR

Proof. To prove this result, we begin by noting that every rotation is the product of two reflections. In the sphere we reflect across great circles. We denote XY here to be the great circle which runs through points X and Y. First we will show that the counter-clockwise rotation about P through an angle of $2\angle P$ is the same as the product of the reflections across the great circles RP and PQ. Consider the spherical triangle ΔPRQ (see figure 4.11). Note that our figures appear two-dimensional for simplicity; however, these points and curves lie on the surface of a sphere.

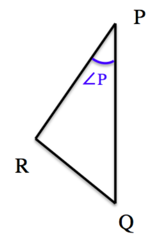


Figure 4.11: ΔPQR

First we will reflect across the great circle RP. This leaves the curve between P and R invariant and moves point Q to Q' as shown in figure 4.12. Note that $\angle Q'PR = \angle P$ and $\angle Q'PQ = 2\angle P$. Next we reflect across the great circle PQ (see figure 4.13). This leaves the curve PQ invariant and sends Q' and R to Q'' and R'' respectively. Now we are left with triangles ΔPQR and $\Delta PQ''R''$ as shown in figure 4.14. Both Q and R have rotated counter-clockwise through an angle of $2\angle P$. Thus $\Delta PQ''R''$ is ΔPQR rotated counter-clockwise by an angle of $2\angle P$.

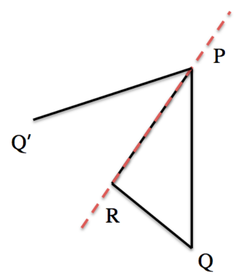


Figure 4.12: Reflection across the great circle RP

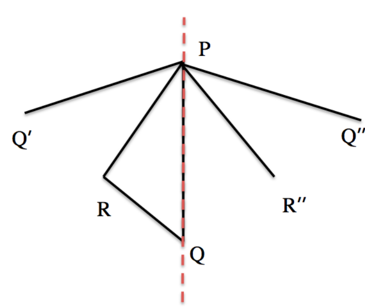


Figure 4.13: Reflection across the great circle PQ

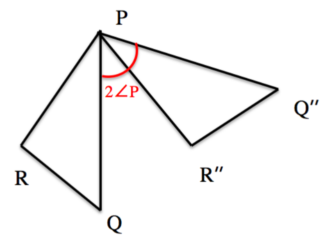


Figure 4.14: $\Delta R''PQ''$ is ΔRPQ rotated counter-clockwise by $2\angle P$

Hence we can represent this rotation by the product of reflections about RP and PQ. Let us denote the product of these reflections as $(RP)(PQ)$. By the same explanation, we have that the rotation through angle $2\angle Q$ is the product $(PQ)(QR)$ and the rotation through $2\angle R$ is $(QR)(RP)$. Recall that a reflection is its own inverse. In other words, if r is the reflection across a great circle in S, then $(r)(r) = 1_S$, where 1_S is the identity.

Definition 4.18 (Rotation about P, P_θ)**.** A rotation about point P, denoted P_θ, is a rotation of the sphere S about the line that goes through P and its antipodal point (and therefore through the center point $\mathbf{O}$) by some angle θ.

It follows,

$$\begin{aligned} h \circ g \circ f &= P_{2\angle P} R_{2\angle Q} Q_{2\angle R} \\ &= (RP)(PQ)(PQ)(QR)(QR)(RP) \\ &= (RP)(1_S)(1_S)(RP) \\ &= 1_S. \end{aligned}$$

□

Now, consider point P and Q on the surface of our sphere S as seen in figure 4.15. A line connects Q to its antipodal point Q' through the center $\mathbf{O}$. There is also a line that goes through P and $\mathbf{O}$. Note that the sum of $\angle P\mathbf{O}Q$ and $\angle P\mathbf{O}Q'$ must equal π. Hence one angle must always be less than or equal to $\frac{\pi}{2}$ radians and the other angle must be greater than or equal to $\frac{\pi}{2}$ radians. It follows that we can always label our angles such that $\angle P\mathbf{O}Q$ is at most $\pi/2$. This result will be necessary in later proofs.

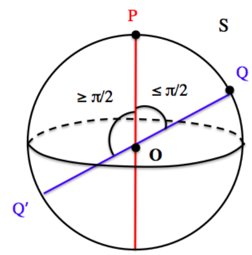


Figure 4.15: $\angle P\mathbf{O}Q$ is at most $\pi/2$

Now let us develop some terminology necessary for our discussion. Recall that G is a finite group consisting of rotations of $\mathbb{R}^3$. Note that every element of G is either a rotation about some point through some angle, as defined above in definition 4.18, or the identity element. Now we will define a subgroup of G.

Definition 4.19 (G_P)**.** The rotational subgroup G_P, is the subgroup of G consisting of all the elements of G that hold P invariant; that is, all rotations $P_\theta \in G$, along with the identity element, 1_s.

Definition 4.20 (Order of P)**.** The order of point P is defined as the order of the rotational subgroup G_P, denoted p. In other words, the order of P is the number of elements in G that are associated with point P, plus one for the identity. Thus $p \geq 2$.

Definition 4.21 (Multiple-Rotation Point)**.** If the order of P is 3 or greater then we call P a multiple-rotation point.

Note that if P is a multiple-rotation point there are multiple elements in G that are rotations around P. Moreover, it implies that there exists a rotation around P through an angle of less than π radians.

Now let us consider the following theorem.

Theorem 4.15. *Suppose P_π and Q_π are distinct elements of G. If $\angle P\boldsymbol{O}Q < \pi/2$, then there is a multiple-rotation point for G, say point R. That is, there exists a rotation in G through an angle less than π.*

In order to show that this theorem holds true we must first prove the following important consequence of lemma 4.1.

Lemma 4.2. *Consider the spherical triangle ΔPQR. If $P_{2\angle P}$ and $Q_{2\angle Q}$ are in G, then $R_{2\angle R}$ is in G.*

Proof. From lemma 4.1 we have that $P_{2\angle P} \circ Q_{2\angle Q} \circ R_{2\angle R} = 1_S$. It follows that $P_{2\angle P} \circ Q_{2\angle Q} = R^{-1}_{2\angle R}$. We know that a group is closed under its operation. The operation of our group G is composition of functions. Therefore $R^{-1}_{2\angle R}$ must be in G. From the definition of a group we know that every element has an inverse also in the group. We can then conclude that $R_{2\angle R} \in G$, and our proof is complete. $\square$

Now we are ready to show the existence of a multiple-rotation point under the conditions of theorem 4.15. Since the two rotations are distinct the points of rotation must be distinct, that is $P \neq Q$. Now consider the great circle on S that passes through P and Q. Let R be the point at the pole of one of the hemispheres created by the great circle PQ such that $\angle R = \angle P\mathbf{O}Q < \pi/2$, as seen in figure 4.16. Since R is at the pole it follows that $\angle P = \angle Q = \pi/2$. Therefore $P_{2\angle P} = P_\pi \in G$ and $Q_{2\angle Q} = Q_\pi \in G$. By lemma 4.2 it follows that $R_{2\angle R} \in G$. And since $\angle R < \pi/2$ we know that $2\angle R < \pi$. So by definition R is a multiple-rotation point for G. Thus we have shown theorem 4.15 to be true.

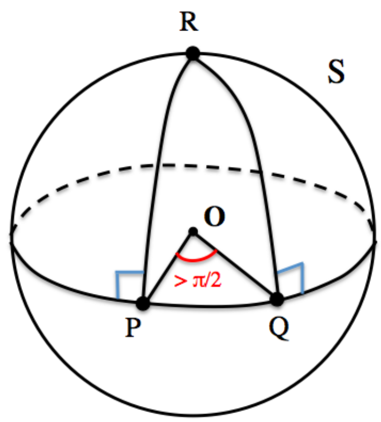


Figure 4.16: Sphere S with $\angle P\mathbf{O}Q < \pi/2$, and with R at the pole

4.3.1 Symmetries Analogous to Two-dimensions

Recall that our goal is to describe all the rotation groups of $\mathbb{R}^3$. We will now break the possible rotation groups into 3 cases determined by the number of multiple-rotation points on S for G. In case I there are no multiple-rotation points for G. In case II there is exactly one multiple-rotation point for G. In case III there are at least two multiple-rotation points for G. We begin with case I.

If G has no multiple-rotation points then the elements of G consist of the identity and rotations through π radians about certain points. It follows that the elements of G have order 1 or 2. Now we will determine all the possible formations of this G.

We will say an element g in G is associated with a point P, or P is a point for G if g is a rotation around P. The first possibility is that G contains no elements associated with points at all. This leaves only the identity. That is $G = \{1_S\}$.

The second possibility is that G contains a rotation about a single point P. By our conditions in case I (G has no multiple-rotation points), the order of P must be 2. In other words, there can only be one element associated with P which is P_π. Therefore G has order 2 and is $G = \{P_\pi, 1_S\}$. By definition of a cyclic group G must be isomorphic to the cyclic group of order 2, $\mathbb{Z}_2$.

The next possibility is that there are two points associated with rotations in G, say P and Q, both of which have order 2. We showed that $\angle P\mathbf{O}Q$ is always less than or equal to $\pi/2$. We also know, from theorem 4.15, that if $\angle P\mathbf{O}Q < \pi/2$ it follows that there exists a multiple-rotation point. Since in case I we have no multiple-rotation points we can conclude that $\angle P\mathbf{O}Q = \pi/2$. Now if we let R be the point on a pole of a hemisphere created by the great circle PQ, it follows that $\angle R = \pi/2$. We now have the spherical triangle ΔPQR where each interior angle is $\pi/2$ radians. By lemma 4.2 we know that since P and Q are points for G, R must also be a point for G. Therefore $P_\pi, Q_\pi, R_\pi \in G$.

In three dimensions we can have no more than 3 elements that are mutually perpendicular, as seen in figure 4.17. That is, we can have no more than 3 rotations around 3 distinct perpendicular lines. Therefore there are no more possible points for G under these conditions. This gives, $G = \{P_\pi, Q_\pi, R_\pi, 1_S\}$. Another clarifying way to view this group is to the think of each element as multiplication by a matrix in the collection of the following orthogonal matrices:

$$G = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

That is, a flip across the x-axis, a flip across the y-axis and a flip across z-axis. Once we have have included 2 flips, the third is necessarily included as well because the product of any two flips is the third. Also note that this set is closed under multiplication and every element in the set has an inverse in the set, as is required by the definition of a group.

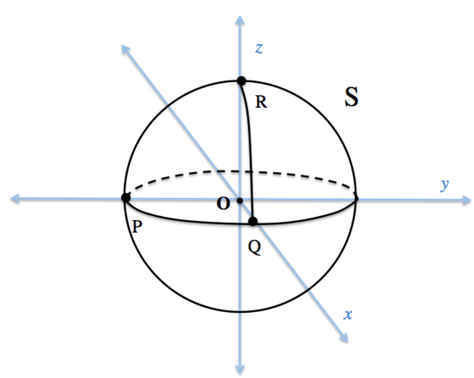


Figure 4.17: Collection of mutually perpendicular points on S

We can see that, here, G has 4 elements. We know that every group of order

4 is either isomorphic to $\mathbb{Z}_4$ (the cyclic group of order 4) or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (the external direct product of $\mathbb{Z}_2$ and itself). Let A and B be two points about which G has rotations. Recall that a and b denote the orders of A and B respectively, thus $a = b = ab = 2$. Therefore we have no element of order 4, and thus no generator of the cyclic group of order 4. Hence G is not cyclic. Therefore $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, or as it is sometimes referred to, $G = D_2$, the dihedral group of order 4.

We observed that three was the maximum number of points we could have associated with elements of G while still having no multiple-rotation points. Therefore the only possible rotation groups in $\mathbb{R}^3$ with no multiple-rotation points are the identity, cyclic group $\mathbb{Z}_2$, and dihedral group D_2.

Now we are ready to consider case II. In this case we will show that all of the finite groups of symmetries in two dimensions correspond to a rotation group in three dimensions. Suppose we have exactly 1 multiple-rotation point. Let us call it point P. All other points for G have order 2.

Lemma 4.3. *Suppose P is a multiple-rotation point for G. If Q is a point of order 2 for G and $\angle P\boldsymbol{O}Q < \pi/2$, then there is another multiple-rotation point R for G.*

Proof. Recall that $d(A, B)$ represents the distance between points A and B. Thus, since $\angle P\mathbf{O}Q < \pi/2$ we know $d(P, Q) < \pi/2$. This follows from our assumption that the sphere, S, has radius 1. Since Q has order two let us construct an arc perpendicular to PQ that goes through Q and R, such that $\angle PQR = \pi/2$. Now constructing an arc from P to R we see that $\angle R$ cannot be a right angle and P is not at a pole of the hemisphere created by the great circle RQ. Since $\angle R \neq \pi/2$, it follows that $2\angle R \neq \pi$ and therefore R is another multiple rotation point. $\square$

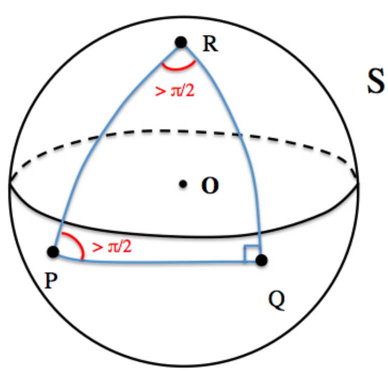


Figure 4.18: Spherical triangle on S with one right angle

Therefore, under our conditions of only a single multiple-rotation point, $\angle P\mathbf{O}Q = \pi/2$. Now let $\mathcal{M}$ be the plane through $\mathbf{O}$ that is perpendicular to the line through P and $\mathbf{O}$, as seen in figure 4.19. Let the intersection of the plane $\mathcal{M}$ and the sphere S be called the circle C. It follows that all points for G other than P lie on the circle C.

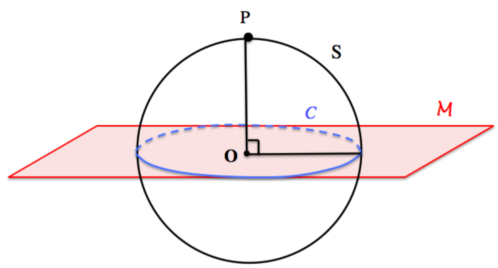


Figure 4.19: Plane $\mathcal{M}$ is perpendicular to the line through P and $\mathbf{O}$

Therefore every element of G either rotates S about point P, or rotates S about some point on C through π radians and thus takes P to its antipodal point. It follows that if $g \in G$ then every element of G takes the circle C to itself. In the case where we are rotating about P through various angles, the points on C are getting mapped around the circumference of C. All other points of rotation for G are on the circle C itself and they all rotate through an angle of π, because we can only have one multiple-rotation point. Each of these rotations is a reflection of C about a line on $\mathcal{M}$ and thus perpendicular to the line through P and $\mathbf{O}$. This action will also map all points on C to points on C.

Now, we will define the function $\phi : G \to G'$. If $g \in G$, let us define $\phi(g)$ as the resulting effect of g on the circle C. We want to show while g is a congruence that takes $\mathbb{R}^3$ to $\mathbb{R}^3$, $\phi(g)$ is the analogous congruence in two dimensions that takes $\mathbb{R}^2$ to $\mathbb{R}^2$. First we will show that $\phi(g)$ is a congruence of C. This follows from the fact that $\phi(g)$ will map C to itself and preserve distances, as explained above. It will not always be orientation preserving on C even though g is always orientation preserving on S. Specifically, when g is associated with a point on C, $\phi(g)$ will result in an orientation reversing flip across the line going through the point and the origin.

Now note that that if $f, g \in G$ then $\phi(f \circ g) = \phi(f) \circ \phi(g)$. This property of congruences in two dimensions was shown previously. Let $G' = \{\phi(g) : g \in G\}$. Note that the elements of G' are rotations and flips, therefore it is necessarily made up of congruences of C. Furthermore G' must form a group because the elements are defined by the elements of the group G. Hence inverses will carry over to G'. Previously, we showed that a finite group of order preserving congruences in $\mathbb{R}^2$ is the cyclic group generated by a rotation about a point

O through an angle of $2\pi/n$. We also have shown that if the group has order reversing elements then it is a dihedral group D_n. Therefore we can conclude these same results about G' as a group of congruences on C.

Now we will show that if $\phi(g)$ is the identity map on C then g has to be the identity map on S. Furthermore, we will show that this means ϕ is injective. If $\phi(g)$ is the identity map on C, then all points along C remain invariant. Therefore the points are not rotating along the circumference of the circle, so g is not a rotation about P. Furthermore, no points have been flipped across a line in $\mathcal{M}$, so g is not a rotation about a point on C. This leaves only the identity map as a possible choice for g. Therefore ϕ is injective. That is, there is a one-to-one and onto correspondence between the elements in G and the elements given by $\phi(g)$. We have that the identity in S maps to the identity in C. We have that rotations about P through some angle θ in S map to rotations about **O** through θ in C. Lastly, rotations about points Q_i on C in S map to flips across the line through Q_i and the center **O** in C. We also have that ϕ is onto because ϕ maps to every element of G' by definition of G'.

Since we have established a one-to-one and onto mapping between G and G', it follows that G is isomorphic to G'. Therefore we can conclude that in case II, G is isomorphic to a collection of congruence of the two dimensional plane. That is, G is either a cyclic group or a dihedral group.

4.3.2 Symmetries of the Platonic Solids

In this section we will explore case III and find the remaining three finite rotation groups for $\mathbb{R}^3$. We will show that they are isomorphic to the orientation-preserving congruences which are symmetries of the Platonic Solids. First we must do some preliminary work to prepare for this discussion.

We are once again conceptualizing our group as a collection of rotations on the sphere S with radius 1. Note that this allows us to regard distance between two points on the sphere, P and Q as equivalent to the angle $\angle P\mathbf{O}Q$, where **O** is the origin and center of our sphere S.

Now let us review some properties about spherical triangles.

Theorem 4.16. *Suppose ΔPQR is a spherical triangle on S. Then the following properties hold true:*

(a) $\angle P + \angle Q + \angle R > \pi$ *radians;*

(b) If ΔABC is another spherical triangle of S such that $\angle P = \angle A$, $\angle Q = \angle B$, and $\angle R = \angle C$, then $\Delta PQR \simeq \Delta ABC$;

(c) If $\angle P, \angle Q \leq \pi/2$, $\angle R = \pi/2$, and X, Y are two points on ΔPQR other than P, Q then $d(X,Y) < d(P,Q)$.

Part (a) of this theorem tells us that the interior angles of any spherical triangle will always sum to greater than π radians. This comes from the fact that great circles always make greater angles than their flat corresponding line segments lying below the curved surface of a sphere. We know that 2 dimensional triangles' interior angles always sum to exactly 180 degrees, thus a spherical triangle must have a greater sum of angles.

Part (b) says that if two spherical triangles have equal angles then they are congruent. Note that this differs from planar triangles. In two dimensions equal angles only implies similarity. This result comes from the fact that the deficiency of a spherical triangle is directly proportional to its area. Note that the deficiency of a spherical triangle is how far the sum of the angles differs from that of a traditional triangle (how much it exceeds 180 degrees).

Part (c) tells us that if our spherical triangle has 1 right angle and 2 angles less than $\pi/2$ at points P and Q, then P and Q have the greatest distance out of any two points on or inside the triangle.

In this section we will present and prove a series of lemmas that will support our over arching goal of proving the theorem which describes all the finite rotation groups of $\mathbb{R}^3$. First we will prove the following lemma.

Lemma 4.4. *Suppose $h \in G$, X is a point on S and $h(X) = Y$. Then X and Y have equal order.*

Proof. We will prove this lemma by showing that G_X and G_Y are conjugate subgroups. That is, that $G_Y = hG_Xh^{-1}$. Recall that G_X is the subgroup of G consisting of all the elements of G that hold X invariant and G_Y is the analogous subgroup for point Y. First we will let g' be an arbitrary element of hG_Xh^{-1} and show that it is also in G_Y. Then we will let g' be an arbitrary element of G_Y and show that it is also in hG_Xh^{-1}.

Suppose $g \in G_X$, thus $g(X) = X$. Let $g' = h \circ g \circ h^{-1}$, such that g' is an arbitrary element of hG_Xh^{-1}. Note that $h(X) = Y$ implies $h^{-1}(Y) = X$, since every function in G is injective. It then follows that,

$$\begin{aligned} g'(Y) &= h(g(h^{-1}(Y))) \\ &= h(g(X)) \\ &= h(X) \\ &= Y \end{aligned}$$

Since $g'(Y) = Y$ we have $g' \in G_Y$. Since an arbitrary element of hG_Xh^{-1} is in G_Y we have $hG_Xh^{-1} \leq G_Y$.

Now let g' be an arbitrary element in G_Y. Let $g = h^{-1} \circ g' \circ h$. We will verify that $g \in G_X$.

$$\begin{aligned} g(X) &= h^{-1}(g'(h(X))) \\ &= h^{-1}(g'(Y)) \\ &= h^{-1}(Y) \\ &= X \end{aligned}$$

Therefore $g = h^{-1} \circ g' \circ h \in G_X$ such that $g' \in hG_Xh^{-1}$. It follows that $G_Y \leq hG_Xh^{-1}$.

Putting these two results together gives, $G_Y = hG_Xh^{-1}$. Therefore G_X and G_Y are conjugate subgroups. Hence X and Y have the same order.

□

Now let us consider Case III, where we have at least two distinct, non-antipodal multiple-rotation points for our group G. We will show that G must be a group of orientation-preserving congruences which are symmetries of a Platonic solid.

Let P and Q be two distinct multiple-rotation points. Furthermore let $d(P, Q)$ be the minimum distance between any two multiple-rotation points. Therefore we must have $d(P, Q) \leq \pi/2$. Recall that p denotes the order of G_P; thus $p, q \geq 3$. Let $h = P_{2\pi/p}$, that is, h is the rotation about P through an angle of $2\pi/p$. Therefore G_P is the cyclic group generated by h, $\langle h \rangle$.

Let $h(Q) = Q'$. Thus Q' is also a multiple-rotation point such that $q = q'$, by lemma 4.4. Furthermore we have that $\angle QPQ' = 2\pi/p$. Let T be the midpoint on the arc between Q and Q'. Now see that since,

$$\begin{aligned} p &> 2 \\ \frac{\pi}{p} &< \frac{\pi}{2} \end{aligned}$$

we have $\angle QPT = \pi/p < \pi/2$ and $\angle QTP = \pi/2$.

Now we will construct a circular arc from Q to the arc PT (represented by the green dashed line in figure 4.20) so that $\angle PQR = \pi/q$. Let the point of intersection with PT be point R. That is, R is defined to be the point of intersection that allows $\angle PQR = \pi/q$. We seek to prove that R must equal T.

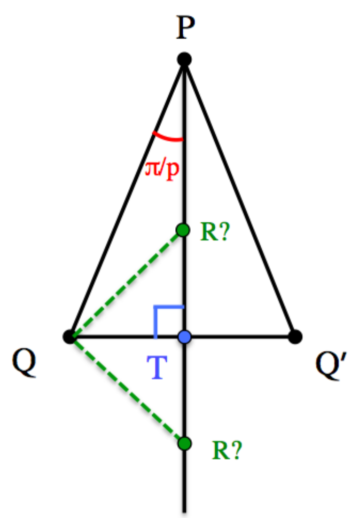


Figure 4.20: Determining a location for R such that $\angle Q = \pi/q$

Consider the spherical triangle ΔPQR. We already know that $P_{2\pi/p} \in G$ and $Q_{2\pi/q} \in G$. Therefore by lemma 4.2 a rotation around R through an angle $2\angle R$ will also be in G.

We will show $R = T$ by ruling out the other possibilities. These are, R is between P and T, or T is between P and R. First, let us suppose that R is strictly between P and T. Then $\angle R \neq \pi/2$ and $2\angle R \neq \pi$. Hence R is a multiple-rotation point for G. Now we observe that $d(P, R) < d(P, Q)$. However, no two multiple-rotation points can be closer to each other than P and Q. Thus we have a contradiction. This has shown that R cannot be strictly between T and P.

Now we know that either $T = R$ or T is strictly between P and R. By the side-angle-side theorem we can conclude that $\Delta QPR \simeq \Delta Q'PR$. Therefore, $d(Q, R) = d(Q', R)$ and $\angle QRQ' = 2\angle R$. Since $R_{2\angle R}$ is in G it must have finite order. Thus $2\angle R = 2\pi(n/r)$ where $n, r \in \mathbb{N}$ and are in lowest terms with $n < r$.

Now we will show that $n = 1$ by contradiction. Suppose $n \neq 1$. If $R = T$ then $2\angle R = 2\pi(1/2)$. By assuming $n \neq 1$ we have that

$$2\angle R = 2\pi\left(\frac{n}{r}\right) \neq 2\pi\left(\frac{1}{2}\right) = 2\angle T$$

so $\angle R \neq \angle T$ which implies $R \neq T$. Now we have T is strictly between P and R. If g is the clockwise rotation around R through the angle $2\pi/r$ then $g \in G_R$. We know $2\angle R = 2\pi(n/r) > 2\pi/r$ when $n > 1$. From lemma 4.4 we know that $g(Q)$ must have order q and therefore must be a multiple-rotation point for G. Since $2\pi/r < 2\angle R$, it follows that $g(Q)$ must be somewhere inside ΔPQT or $\Delta PQ'T$. Thus we see that $d(P, g(Q)) < d(P, Q)$. This contradicts the condition that P and Q are the nearest multiple-rotation points. Thus our beginning assumption was false and we can conclude that $n = 1$.

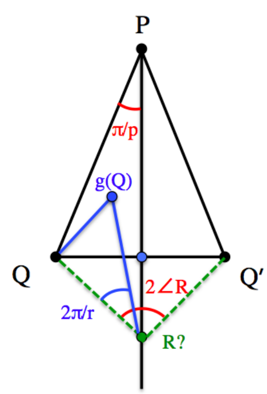


Figure 4.21: The multiple-rotation point $g(Q)$ is closer to P than the multiple-rotation point Q

We now have $2\angle R = 2\pi(1/r)$; that is, $\angle R = \pi/r$. Note that the interior angles of our triangle ΔPQR are π/p, π/q, and π/r. From property (a) in theorem 4.16 we get $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} > \pi$. Divsion by π gives,

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

Now, since P and Q are multiple-rotation points we know $p, q \geq 3$. We also know that G contains at least one rotation around R therefore $r \geq 2$. Note that $\frac{1}{p} \leq \frac{1}{3}$, $\frac{1}{q} \leq \frac{1}{3}$, and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$. So

$$\begin{aligned} \frac{1}{3} + \frac{1}{3} + \frac{1}{r} &> 1, \\ \frac{2}{3} + \frac{1}{r} &> 1, \\ \frac{1}{r} &> \frac{1}{3}, \\ r &< 3. \end{aligned}$$

Since r has to be an integer greater than or equal to 2 it follows that $r = 2$. If we chose larger values for p and q, r would need to be less than or equal to 2 to ensure the inequality holds. Therefore 2 is the only possible value for r. Note that since $r = 2$ we have $\angle R = \pi/2$, such that $R = T$, as desired.

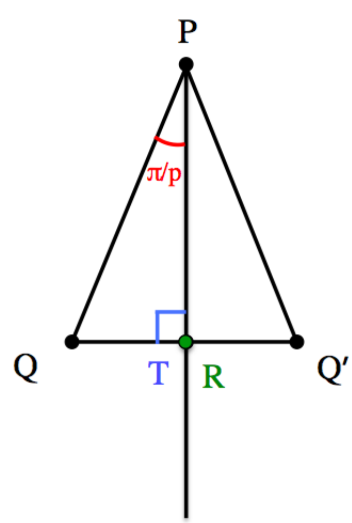


Figure 4.22: $R = T$ and $\angle R = \pi/2$

Now we plug in $r = 2$ which gives,

$$
\begin{aligned}
\frac{1}{p}+\frac{1}{q}+\frac{1}{r} &> 1\\
\frac{1}{p}+\frac{1}{q}+\frac{1}{2} &> 1\\
\frac{1}{p}+\frac{1}{q} &> \frac{1}{2}\\
(p-2)(q-2) &< 4.
\end{aligned}
$$

This last step is shown in Section 4.2. Now we see that the only possible solutions for the set $\{p, q\}$ are $\{3,3\}$, $\{3,4\}$, $\{4,3\}$, $\{3,5\}$, or $\{5,3\}$. These solutions correspond to the orders of the points P and Q on our spherical triangle ΔPQR.

This inequality and these solutions should look familiar for they are exactly the same as the solutions we derived in our topological proof of why there are no more than five Platonic solids. Let $\mathcal{P}$ be the Platonic solid corresponding to the given values of p, q. Let us inscribe our Platonic solid inside that S such that each vertex lies on the surface of S. Furthermore, let us normalize all points on $\mathcal{P}$ such that all edges of the solid become arcs along the surface of S.

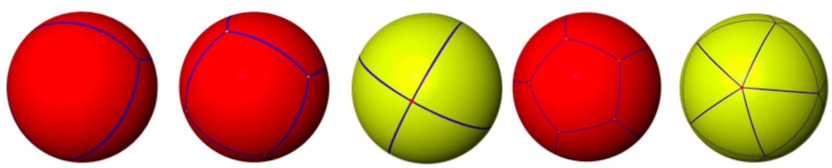


Figure 4.23: The Platonic solids normalized onto the surface of S

Let A be the center of a face of $\mathcal{P}$, BB' be one of the edges of the polygon around A, and C be the midpoint of BB'. Consider triangle ΔABC. It follows that $\angle A = \pi/p$, $\angle B = \pi/q$, and $\angle C = \pi/2$. Therefore, from property (b) of Theorem 4.16 we have $\Delta ABC \simeq \Delta PQR$. Hence, moving forward we will assume that Platonic solid $\mathcal{P}$ has P at the center of one of its faces, Q at a vertex, R at a mid point of a corresponding edge. Now we see that $\mathcal{P}$ determines the decomposition of S into spherical triangles of the form $\Delta P_k Q_k R_k$ where P_k is the center of one of the faces, and R_k is a midpoint of an edge $Q_k Q'$ of the polygon surrounding P_k.

Let H be a group of rotations that take $\mathcal{P}$ to itself. We will show that $G = H$. Recall that G is a finite group of rotations (and since we are in case III) with at least two multiple-rotation points. As shown previously our multiple-rotation points are P and Q. Thus, every multiple-rotation point is either a center of a face or a vertex. Note that every P_k and Q_k is antipodal to either a mid-face or a vertex.

Now we note that the rotational subgroups of G and H are equal. Specifically, $G_P = H_p$. Recall that this group is the collection of rotations about P through multiples of $2\pi/p$ with order p. Similarly, $G_Q = H_Q$, which is the collection of rotations about Q through multiples of $2\pi/q$ with order q. Lastly, $G_R = H_R$, which is the group consisting of the rotation around R through π and the identity, and hence has order 2. Let $\mathcal{X}$ be the set of all $X \in S$ such that $G_X = H_X$. By way of our construction $P, Q, R \in \mathcal{X}$.

Lemma 4.5. *Suppose $X, Y \in \mathcal{X}, h \in G_X = H_X$ and $Y' = h(Y)$. Then $Y' \in \mathcal{X}$.*

Proof. The proof of this comes directly from lemma 4.4. It follows from the lemma that

$$G_{Y'} = G_Y = H_Y = H_{Y'}.$$

Since $G_{Y'} = H_{Y'}$ we have $Y' \in \mathcal{X}$. □

Now consider points P, Q, and R. We can repeat the idea from Lemma 4.5 and conclude that every mid-face, vertex, and mid-edge point of $\mathcal{P}$ is in $\mathcal{X}$.

Lemma 4.6. *If $\Delta P_k Q_k R_k$ is one of the mid-face, vertex, mid-edge triangles in the decomposition of S determined by $\mathcal{P}$ then, P_k and Q_k are the only multiple-rotation points for G in the triangle.*

Proof. We begin the proof by reviewing how P_k and Q_k are defined. We have chosen P_k and Q_k to be distinct multiple-rotation points such that the distance between them is the minimum of any two multiple-rotation points on the triangle.

Now recall property (c) from Theorem 4.16. It states, if $\angle P, \angle Q \leq \pi/2$, $\angle R = \pi/2$, and X, Y are two points on ΔPQR other than P, Q then $d(X, Y) < d(P, Q)$.

Since P_k and Q_k are both multiple-rotation points their angles must each be less than $\pi/2$. Furthermore, we have previously shown that $\angle R = \pi/2$. Therefore by property (c) the distance between any two other points on the triangle must be less than the distance between P_k and Q_k. Therefore, by definition of P_k and Q_k, no other points on the triangle are multiple rotation points. □

The following lemma tells us that any rotation of order 3 or more in the group G is necessarily also a rotation in the group H and vice versa.

Lemma 4.7. *If g is some rotation about $X \in S$ and the order of g is greater than or equal 3, then $g \in G$ if and only if $g \in H$.*

Proof. If $g \in H$, then X is a multiple-rotation point for H. Therefore X must be a mid-face or a vertex for $\mathcal{P}$. It follows that $g \in H_X = G_X \leq G$. Thus $g \in G$.

Now for the other direction, assume $g \in G$. Thus X is a multiple-rotation point for G. Let S be naturally divided into mid-face, vertex, mid-edge triangles of the form $\Delta P_k Q_k R_k$. We may assume that X is in $\Delta P_k Q_k R_k$. We know that P_k and Q_k are the only multiple-rotation points in the triangle. Thus we must have $X = P_k$ or $X = Q_k$. It follows that $g \in G_X = H_X \leq H$. Hence $g \in H$. □

Now note that the identity is necessarily in both groups G and H. Thus the only element that remains is an element of order 2. The following lemma will tell us that any rotation of order 2 in the group G is necessarily also a rotation in the group H and vice versa.

Lemma 4.8. *If h is some rotation about $Y \in S$ and the order of h is 2 (that is, a rotation around Y through π), then $h \in G$ if and only if $h \in H$.*

Proof. Suppose $\Delta P_k Q_k R_k$ is a mid-face, vertex, mid-edge triangle of $\mathcal{P}$ containing Y. Now we have two cases.

First suppose that $Y = R_k$. Then it follows that $G_Y = H_Y$. Hence, $h \in H$ if and only if $h \in H$.

Now let us suppose that $Y \neq R_k$. From property (c) in theorem 4.16 we know that $d(Y, R_k) < d(P_k, Q_k) \leq \pi/2$. Let Z be a pole of a hemisphere defined by the great circle containing Y and R_k. Consider the triangle $\Delta R_k Y Z$. It follows that $\angle R_k = \angle Y = \pi/2$. Note that if f is the rotation about R_k through the angle $2\angle R_k = \pi$, then f will be in both G and H. Finally, let g be the rotation about Z through $2\angle Z$. Since $2\angle Z = 2d(Y, R_k) < \pi$, g cannot be an element of order 2. $\square$

It now follows from lemmas 4.7 and 4.8 that,

$$h \in G \text{ iff } g \in G \text{ iff } g \in H \text{ iff } h \in H.$$

We have now concluded our proof of the theorem describing the finite rotations of $\mathbb{R}^3$.

Theorem 4.17. *Suppose G is a finite group consisting of rotations of $\mathbb{R}^3$. Then G is isomorphic to one of the following:*

(a) The group consisting of only the identity;

(b) A cyclic group of order n;

(c) A dihedral group of order $2n$;

(d) The group of orientation-preserving congruences which are symmetries of a tetrahedron;

(e) The group of orientation-preserving congruences which are symmetries of a cube (or an octahedron);

(f) The group of orientation-preserving congruences which are symmetries of an icosahedron (or a dodecahedron).

5 Conclusion & Further Inquiry

In this paper we have been able to collect a fair amount of information surrounding the Platonic solids. We have a brief historical background, varying proofs of why there exist no more than five, and an exploration of different ways of viewing these intriguing solids. Most notably, we have provided a fairly robust description of how the symmetries of the Platonic solids help determine the symmetries of $\mathbb{R}^3$. However the discussion does not end here. The Platonic solids are such a rich topic and there are many further directions one could take this study. Some of the possibilities are as follows.

- A discussion of the orders of the polyhedral symmetry groups. What is the order of each group? How can we prove this result rigorously and how can we see it intuitively? Can we prove it in multiple ways? How does the order of each group correlate with the geometry of its respective solid?
- We have only so far discussed orientation preserving congruences of $\mathbb{R}^3$. What changes when we allow for orientation reversing congruences? What finite groups are added to our list of potential isomorphisms for G? What does an orientation reversing congruence look like, physically, on a Platonic solid?
- A discussion on higher dimensions. What are the analogous regular polytopes in 4 dimensions? How about 5 dimensions and beyond? How many of them are there in each dimension and why? Can we still use the higher dimensional regular polytopes to describe symmetries of $\mathbb{R}^k$. Can we build any intuition for what it means to have a four (or higher) dimensional regular figure?

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