Generalized Interval System and Its Applications

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Abstract

Transformational theory is a modern branch of music theory developed by David Lewin. This theory focuses on the transformation of musical objects rather than the objects themselves to find meaningful patterns in both tonal and atonal music. A generalized interval system is an integral part of transformational theory. It takes the concept of an interval, most commonly used with pitches, and through the application of group theory, generalizes beyond pitches. In this paper we examine generalized interval systems, beginning with the definition, then exploring the ways they can be transformed, and finally explaining commonly used musical transformation techniques with ideas from group theory. We then apply the the tools given to both tonal and atonal music. A basic understanding of group theory and post tonal music theory will be useful in fully understanding this paper.

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1 Introduction

David Lewin(1933 - 2003) is an American music theorist. He studied mathematics at Harvard and composition and music theory at Princeton. [1] His background in both mathematics and music enabled him to be able to bring mathematical rigor to music theory, which resulted in the book *Generalized Musical Intervals and Transformations*. In this book, he was especially fond of studying the concept of intervals and transformations and generalizing them beyond pitches in western scales to many other musical objects, including rhythms, timbres, and non western music scales. In this paper, we examine a portion of the text *Generalized Musical Intervals and Transformations*, focusing on the generalized interval system, an integral component of his transformational theory. Later in this paper, we take three pieces, one tonal and two atonal, and apply the theories regarding the generalized interval system to see how it actually works with music. This paper is mainly based on first four chapters of the text *Generalized Musical Intervals and Transformations*, unless stated otherwise. [2]

2 A Crash Course in Music Theory

A *pitch* is a fixed frequency of a vibrating object that is distinguishable from noise. A *note* is a symbol that indicates a pitch. Western theorists have arbitrarily decided on the frequencies for the pitches and assigned names with a combination of an element from each set $\{A, B, C, D, E, F, G\}$, $\{\sharp, \flat\}$(reads sharp and flat, respectively), and $\{1, 2, 3, 4, 5, 6, 7\}$. For example, we assign the name A_4 to the frequency of 440 Hz, meaning that this pitch is the fourth A from the left on a piano keyboard, which has seven complete octaves. The notes with the same letter, for example $A_3 = 220$ Hz and $A_4 = 440$ Hz, can be reached by multiplying the frequency by a power of 2. Figure 1 shows how the notes are arranged on a keyboard.

The distance from one note to another is called an *interval*. The shortest distance between
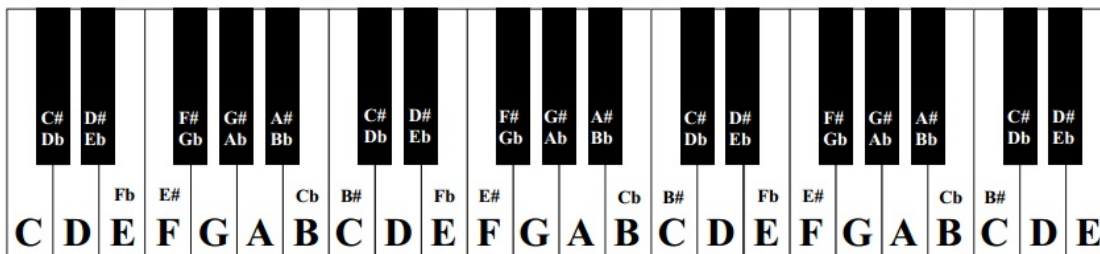
Figure 1: A diagram of a keyboard. Each key is associated with repeating sequence of notes. http://www.onlinepianocoach.com/support-files/printable-piano-keyboard-diagram2.pdf

two notes with identical names but different numbers is an *octave*. For example, C_3 and C_4 are an octave apart. *Just intonation* is a method of tuning that is more accurate to the pure sonority since it uses ratios derived from natural harmonics. However, it requires the intervals to be individually tuned, which is not possible with keyboard instruments. This limitation in just intonation led to the development in *equal temperament*, as a way to find a happy medium for instruments that cannot be readily adjusted. Equal temperament places the pitches to be the equal distance apart, such that the half step measures to be the twelfth root of 2. With a pitch with the known frequency, usually $A_4 = 440$ Hz, we can find the frequency of all the other notes by the following formula: for a note that is n half steps up from A_4, the frequency of the note is $440 \cdot 2^{\frac{n}{12}}$. [3] These two different types of tuning methods are discussed in more detail in Appendix A. Certain note names, for example, $C\sharp$ and $D\flat$, have the same frequency. These notes are called *enharmonically equivalent* but they are treated differently for analyzing music composed during the common practice period, from the seventeenth to the nineteenth century. *Octave equivalence* considers notes with the same name that are differentiated by one or more octaves to be the same note. Octave equivalence can be thought of as an equivalence relation, where two notes n and m are equivalent when there exists an integer a such that the frequency of n, $f(n)$ equals the frequency of m, $f(m)$, times 2^a, $f(n) = 2^a \cdot f(m)$.

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Figure 2: Treble, alto, tenor, and bass clef all notating C_4, the middle C. `http://www.lilypond.org/doc/v2.16/Documentation/cd/lily-850322b3.png`

Figure 3: The C Major scale notated in the treble clef. `http://upload.wikimedia.org/wikipedia/commons/a/a9/C_major_scale_letter_notation.png`

A *staff* is composed of five lines and four spaces and is where the pitches are notated. A *clef* is a symbol that determines which note indicates a certain pitch, and always accompanies a staff. There are four clefs that are commonly used today, which are treble, bass, alto, and tenor clef. Figure 2 shows the staff with treble, alto, tenor, and bass clef all notating C_4, also called *middle C*. The *range* of an instrument is a collection of pitches that a particular player on an instrument is capable of playing. When we refer to a range of a particular instrument, it is a range of notes usually playable by experienced players. Ranges of some instruments require placing notes constantly outside of the staff with a certain clef. For example, if we tried to notate a piece for a trombone on a treble clef, we would not be able to place any notes inside the staff. When a note does not fit inside the staff, we use ledger lines to create more lines and space above and below the staff. However, using ledger lines hinders readability and different clefs exist in order to accommodate for the different ranges of instruments and to avoid excessive use of ledger lines.

A *half step* is the distance between two adjacent keys on the keyboard. Attaching ♯ after

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any note raises the note by a half step, and putting ♭ after any note lowers the note by a half step. A *whole step* is composed of two half steps. A double flat, ♭♭, or a double sharp, ×, indicates to lower or raise a note by a whole step respectively. This set of symbols are called *accidentals*. They only apply to the note in the measure, a space created between two bar lines. When there is a ♮ sign after a note, the sign cancels any accidentals that were applied to a note beforehand. A *scale* is an ordered set of pitches arranged in such a way that any two adjacent pitches are either a half or whole step apart, with an exception of the harmonic minor scale. In western music, there are 4 commonly used scales, major, natural minor, harmonic minor and melodic minor. There are a total of 12 major scales on and 3 corresponding minor scales for each major scale, totaling 36 minor scales. Another commonly used scale is the *chromatic scale*, a collection of all 12 pitches in an octave. Figure 3 shows an octave of the C Major scale in treble clef. These steps and scales play an integral role in naming intervals. In Table 1 all possible interval names are shown. However, augmented and diminished intervals are not used as often as major and minor intervals are used. Moreover, augmented fourth and diminished fifth are commonly called *tritone*. Also many of these intervals sound the same; for example major second and diminished third will sound exactly the same on a piano, the difference is how they function in harmony. There are intervals that are more than an octave apart; for example, the interval from C_3 to D_4 is major ninth. Those intervals are called *compound intervals*.

Harmony is the simultaneous sounding of two or more pitches. It is constructed by stacking two or more pitches. Traditionally the pitches that are the third apart were used in constructing harmony and the chord constructed by stacking three notes in thirds is called a *triad*. Just like intervals, triads have qualities as well; major, minor, augmented, and diminished. Each note in a triad also has a name. The bottom most note when arranged in third is called the *root*. The middle note is called *the third* and the top note is called *the fifth*. Moreover, these triads come with names depending on what scale degree they are built

Full name	abbreviation	letters between notes	total half steps	example
Unison	P1	1 letter	0 half step	C to C
diminished second	d2	2 letters	0 half step	C to $D\flat\flat$
minor second	m2	2 letters	1 half step	C to $D\flat$
Major second	M2	2 letters	2 half steps	C to D
Augmented second	A2	2 letters	3 half steps	C to $D\sharp$
diminished third	d3	3 letters	2 half steps	C to $E\flat\flat$
minor third	m3	3 letters	3 half steps	C to $E\flat$
Major third	M3	3 letters	4 half steps	C to E
Augmented third	A3	3 letters	5 half steps	C to $E\sharp$
diminished fourth	d4	4 letters	4 half steps	C to $F\flat$
Perfect fourth	P4	4 letters	5 half steps	C to F
Augmented fourth	A4	4 letters	6 half steps	C to $F\sharp$
diminished fifth	d5	5 letters	6 half steps	C to $G\flat$
Perfect fifth	P5	5 letters	7 half steps	C to G
Augmented fifth	A5	5 letters	8 half steps	C to $G\sharp$
diminished sixth	d6	6 letters	7 half steps	C to $A\flat\flat$
minor sixth	m6	6 letters	8 half steps	C to $A\flat$
Major sixth	M6	6 letters	9 half steps	C to A
Augmented sixth	A6	6 letters	10 half steps	C to $A\sharp$
diminished seventh	d7	7 letters	9 half steps	C to $B\flat\flat$
minor seventh	m7	7 letters	10 half steps	C to $B\flat$
Major seventh	M7	7 letters	11 half steps	C to B
Augmented seventh	A7	7 letters	12 half steps	C to $B\sharp$
Octave	P8	1 letter	12 half steps	C to C

Table 1: The interval names are determined by the number of letters between two notes.

Scale Degree	Name of the triad
First	Tonic
Second	Supertonic
Third	Mediant
Fourth	Subdominant
Fifth	Dominant
Sixth	Submediant
Seventh	Leading Tone or Subtonic

Table 2: The names of triads built on each scale degree. The name of triad built on 7th degree depends on the quality of the triad.

on. For example, 5th scale degree in C major is G. We can build a triad using G as a root in C major, which is G, B, and D. This triad is called a *dominant* triad in C major. Table 2 shows the names for triads built on each scale degree of a western scale.

Rhythm is a way to denote the duration of notes. It is composed of a *time signature* or *meter* and different note shapes that indicates different durations namely *whole note, half note, quarter note,* and *eighth note*. A *measure* or a *bar* is the space in between two bar lines. In a time signature, the numerator indicates how many beats are in a bar and the denominator indicates what kind of notes get a beat. For example, 4/4 time means that there are four beats in a bar and each quarter note gets a beat. In the same manner, 3/4 means there are 3 beats where quarter notes get a beat and 6/8 means there are 6 beats and each eighth note gets a beat. A whole note lasts the entire measure in 4/4 time. A half note lasts half of a whole note, a quarter note is one fourth of a whole note, and an eighth note lasts an eighth of a whole note. Figure 4 shows different note figures used with time signature. Figure 5 is an excerpt from a keyboard sonata by G.F. Handel, demonstrating how the aforementioned concepts and notations are used.

In the twentieth century, composers begin to significantly bend and break the rules that were used during the common practice period, from the 17th to 19th century. A new concept called atonal music, music without a tonal center, has been created and explored.
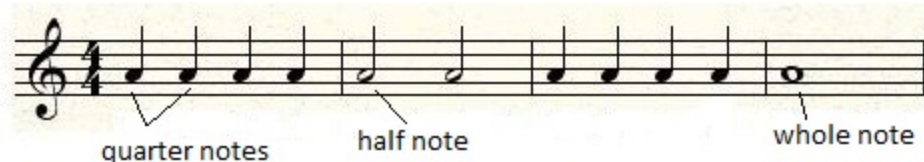
Figure 4: A sample rhythm.

Figure 5: A sample from G.F. Handel's keyboard sonata.

Compositions employed the concept of enharmonic and octave equivalence more frequently. To make the enharmonic equivalence truly enharmonic, theorists assigned numbers to the note names, beginning with C at 0 mod 12 and ending with B at 11 mod 12. In this system, $C\sharp$ and $D\flat$ are both 1 and are considered the same. This numbering system is most popularly used for analysis of post tonal music.

3   Introduction to the Generalized Interval System

In chapter 2 of his book *Generalized Musical Intervals and Transformations*, David Lewin introduces the concept of the Generalized Interval System, or the GIS for short.[2] It is an ordered triple that consists of a set called a musical space, a group of intervals, and an interval function. There are many ways to construct a musical space and its interval function. A few of the music spaces with their interval functions will be presented before we formally define the Generalized Interval System.

A music space is a set of musical objects. Contrary to the common practice of constructing one with only pitches or harmonies, it can be easily generalized to less common objects, such as beats, rhythm, and timbres.

Example 3.0.1. Some examples of the music spaces are: the set of all pitches, the set of all white key notes on the piano in an octave, the set of beats or pulses in a bar in waltz, or the set of the frequencies of the pitches in just intonation.

Many of these spaces can be transformed by using an equivalence relation. For example, the set of all pitches could become the set of all pitches in an octave if we say the two notes are equivalent if they have the same note name regardless of which octave they are in. On the other hand, the set of all white key notes on a piano in an octave can be expanded to the set of all notes that does not have ♭ or ♯.

Now we define the interval function, int.

Definition 3.0.2. Let S be a music space and G be a group under operation ∘. An *interval function* is a mapping int : $S \times S \to G$ that satisfies the following properties:

- For all $r, s, t \in S$, $\text{int}(r, s) \circ \text{int}(s, t) = \text{int}(r, t)$

- For every $s \in S$ and $g \in im(\text{int})$, there exists a unique $t \in S$ such that $\text{int}(s, t) = g$.

Example 3.0.3. One example of a musical space with an interval function is a set of pitches in a scale, arranged in scalar order. This is perhaps one of the most common spaces and the easiest one to visualize. Given pitches s and t in the set S, we define the function int : $S \times S \to \mathbb{Z}$ such that $\text{int}(s, t)$ equals the number of scale steps one must move upwards to go from the pitch s to t. For example in the C major scale, $\text{int}(C_4, C_5) = 7$, $\text{int}(C_3, A_3) = 5$, $\text{int}(C_3, C_3) = 0$, $\text{int}(C_3, D_3) = 1$, and $\text{int}(C_4, D_3) = -6$ (going -6 notes "upward"). In fact, this set of intervals forms a group under addition. It's not hard to see that the set has an identity element $\text{int}(a, a)$, inverse element $\text{int}(a, b) + \text{int}(b, a) = \text{int}(a, a)$, and is associative, $(\text{int}(a, b) + \text{int}(b, c)) + \text{int}(c, d) = \text{int}(a, b) + (\text{int}(b, c) + \text{int}(c, d))$ for all $a, b, c, d \in S$.

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Example 3.0.4. Another example of a music space is a set S containing chromatic pitches under twelve tone equal temperament. We define the function int : $S \times S \to \mathbb{Z}$ such that $\text{int}(s,t)$ is the number of half steps in upward motion to get from the pitch s to the pitch t, for $s,t \in S$. For example, $\text{int}(C_4, E_4) = 4$, $\text{int}(C_3, A_2) = -3$, and $\text{int}(C_4, C_5) = 12$.

Example 3.0.5. An interval function is not unique to the set S. Let us take the same set S from Example 3.0.4 and let G be the set of integers mod 12. We can define another function int : $S \times S \to G$ such that $\text{int}(s,t)$ is the number of half steps in upward motion mod 12 to get from the pitch s to the pitch t. Then $\text{int}(C_4, E_4) = 4$, $\text{int}(C_3, A_2) = -3 = 9$, and $\text{int}(C_4, C_5) = 12 = 0$. This function incorporates the concept of octave equivalence; for example, $\text{int}(C_4, C_5) = \text{int}(C_3, C_3) = \text{int}(C_0, C_5) = 0$.

Example 3.0.6. On a similar note, take the set S from Example 3.0.3 and let G be the set of integers mod 7. Define int : $S \times S \to G$ such that $\text{int}(s,t)$ is the number of scale steps we take from the pitch s to the pitch t mod 7. This function also brings in the octave equivalence for the set S.

These four examples can be further expanded to other scales to which octave equivalence can be applied. Not only can we apply the interval functions of Example 3.0.3 and Example 3.0.6 to other western music scales, these functions can easily be applied to more exotic scales.

Example 3.0.7. We can also look at the harmony using just intonation. Let S be a set of frequencies of the pitches. Define int : $S \times S \to \mathbb{Q}$, where $\text{int}(s,t) = s/t$ for $s,t \in S$. This is not so easy and fast to find what the $\text{int}(s,t)$ would be compared to other music spaces because not many people go around memorizing the frequencies for the pitches. An alternate way to find the $\text{int}(s,t)$ is building to t from s by using its natural harmonic series. For example, $\text{int}(C_4, F\sharp_4)$ can be calculated by building to $F\sharp$ from C in this way: Start on C_4, a perfect fifth up to G_4, an octave down to G_3, a perfect fifth up to D_4, then a major

third up to $F\sharp_4$. Since we already know the ratio of perfect fifths, octaves, and major thirds, this simply leads to multiplying the ratios together which is $3/2 \cdot 2^{-1} \cdot 3/2 \cdot 5/4 = 45/32$.

Theorem 3.0.8. *In any GIS,* $\mathrm{int}(s, s) = e$ *and* $\mathrm{int}(t, s) = \mathrm{int}(s, t)^{-1}$ *for every* s *and* $t \in S$.

Proof. Let int $: S \times S \to G$ be the interval function for a GIS. We know that $\mathrm{int}(s, s) \circ \mathrm{int}(s, s) = \mathrm{int}(s, s)$ by the transitive property of the interval functions. Since $\mathrm{int}(s, s)$ is in G, there exists an inverse element of $\mathrm{int}(s, s)$, $\mathrm{int}(s, s)^{-1}$. Then $\mathrm{int}(s, s)^{-1} \circ \mathrm{int}(s, s) \circ \mathrm{int}(s, s) = \mathrm{int}(s, s)^{-1} \circ \mathrm{int}(s, s) = e$. Consider $\mathrm{int}(s, t) \in G$. Then $\mathrm{int}(s, t) \circ \mathrm{int}(t, s) = \mathrm{int}(s, s) = e = \mathrm{int}(t, t) = \mathrm{int}(t, s) \circ \mathrm{int}(s, t)$. Thus $\mathrm{int}(s, t)$ and $\mathrm{int}(t, s)$ for any $t, s \in S$ are inverses of each other. $\square$

Now we can formally define a generalized interval system.

Definition 3.0.9. A *generalized interval system* is an ordered triple $(S, (G, \circ), \mathrm{int})$ such that S is a music space, G is a group under the operation $\circ$, and int is an interval function mapping from $S \times S$ to G.

4 Transforming GISs

A GIS also can be transformed just like music spaces, however using cosets and quotient groups instead of equivalence relations. This and the following sections, section 4 and 5, will be based on chapter 3 of *Generalized Musical Intervals and Transformations.* [2]

Proposition 4.0.10. *Let* $(S, (G, \circ), \mathrm{int})$ *be a GIS and* H *be a normal subgroup of* G. *Define* $\sim$ *such that* $a \sim b$ *when* $\mathrm{int}(a, b) \in H$ *for* $a, b \in S$. $\sim$ *is an equivalence relation.*

Proof. $\sim$ is reflexive since $a \sim a = \mathrm{int}(a, a) = e \in H$. Assume $a \sim b$. Then $\mathrm{int}(a, b) \in H$. Since H is a group, $\mathrm{int}(a, b)$ has the inverse element in H, $\mathrm{int}(b, a)$. Since $\mathrm{int}(b, a)$ implies $b \sim a$, $\sim$ is symmetric. Assume $a \sim b$ and $b \sim c$. Then both $\mathrm{int}(a, b)$ and $\mathrm{int}(b, c)$ are in

H. We know $\text{int}(a, b) \circ \text{int}(b, c) \in H$, since H is closed. Thus $\text{int}(a, c) \in H$ which implies $a \sim c$. $\qquad \square$

This equivalence relation is called the *induced equivalence relation on S*.

Lemma 4.0.11. *Let* $(S, (G, \circ), \text{int})$ *be a GIS and* H *be a normal subgroup of* G. *If* $a \sim a'$ *and* $b \sim b'$, *then* $\text{int}(a, b)$ *and* $\text{int}(a', b')$ *are in the same coset.*

Proof. Suppose that there exist elements $x, y \in G$ such that $\text{int}(a, b) \in xH$ and $\text{int}(a', b') \in yH$. Suppose by contradiction that $xH \neq yH$. Notice that $\text{int}(a, b) = \text{int}(a, a') \circ \text{int}(a', b') \circ \text{int}(b', b)$. Since $a \sim a'$ and $b \sim b'$, both $\text{int}(a, a')$ and $\text{int}(b, b')$ are in H. $\text{int}(a, a') \circ \text{int}(a', b') \circ \text{int}(b', b) \in H \cdot yH \cdot H = H \cdot yH = yH$. We assumed $xH \neq yH$, however $\text{int}(a, b) = \text{int}(a, a') \circ \text{int}(a', b') \circ \text{int}(b', b)$ which is a contradiction. Thus $xH = yH$. $\qquad \square$

Using above propositions and the lemma, we can transform one GIS to another GIS.

Theorem 4.0.12. *Let* $(S, (G, \circ), \text{int}_1)$ *be a GIS,* H *be a normal subgroup of* G, *and* $\sim$ *be an induced equivalence relation on* S. *Then* $(S/\sim, G/H, \text{int}_2)$ *is also a GIS, where* int_2 *is an interval function mapping from the Cartesian product of* $S/\sim$ *to the quotient group,* G/H. *Moreover,* int_2 *is defined by the following method: given equivalence classes* p *and* q *in* $S/\sim$, $\text{int}_2(p, q)$ *equals the coset where* $\text{int}_1(s, t)$ *belongs, where* s *and* t *are members of* p *and* q *respectively.*

Example 4.0.13. Let set S be a set of all notes and let GIS be $(S, \mathbb{Z}, int_1)$, where $\text{int}_1(s, t)$ is the number of half steps ascending from the note s to the note t. This GIS can be easily modified using the theorem above. Let H be the set of multiples of 12, $\{\cdots -12, 0, 12, 24, \dots\}$. Then we have the induced equivalence relation, where notes s and t are equivalent when $\text{int}_1(s, t)$ is a multiple of 12. This essentially is the concept of octave equivalence, since notes with the same name are considered equivalent under this definition. Thus our new set, S_2 is the set of all notes in an octave, and $\text{int}_2(s, t)$ is the number of ascending half steps mod 12,

which we already know that it is an interval function from Example 3.0.5. Our new GIS is now $(S_2, \mathbb{Z}_1 2, int_2)$.

Not only can we resize them, we can also apply other group theory concepts to them, such as doing direct product with them.

Definition 4.0.14. Let $(G, \cdot)$ and $(H, \circ)$ be groups. Define $G \times H$ as *the Cartesian product of G and H*, such that $(g_1, h_1)(g_2, h_2) = (g_1 \cdot g_2, h_1 \circ h_2)$ for $g_1, g_2 \in G$ and $h_1, h_2 \in H$. $G \times H$ is a group, and it is called *the external direct product* of G and H.

Now we apply this notion to define the external direct product of GISs.

Definition 4.0.15. Let A and B be GISs, where $A = (S, G, int_1)$ and $B = (R, H, int_2)$. Then $A \times B$ is also a GIS, where $A \times B = (S \times R, G \times H, int_3)$. The new interval function, int_3 is also a Cartesian product of int_1 and int_2.

Example 4.0.16. Perhaps the easiest one to visualize is the direct product between the notes and the time units. Let $A = (S, \mathbb{Z}, int_1)$ where S is the set of all notes and int_1 is the ascending half steps from first note to the second one. Also let $B = (R, \mathbb{Z}, int_2)$, where R is the set of time points and $int_2(s, t) = t - s$ for $s, t \in R$. Then $A \times B$ is the direct product of the notes and time points, which we can use to find different patterns in music that involves both time and intervallic differences.

5   Developmental Techniques in GIS

Composers use a variety of methods to develop, or manipulate, a given theme, melodic, rhythmic motives in their composition. Some examples of them include *transposing, inverting,* or *retrograding.* Transposition is writing a melodic object in a different key. For example, a melodic motive $C - E - G - B$ in figure 6 is in C major, and it can be transposed to A major, which would be $A - C\sharp - E - G\sharp$. Inversion is reversing the direction
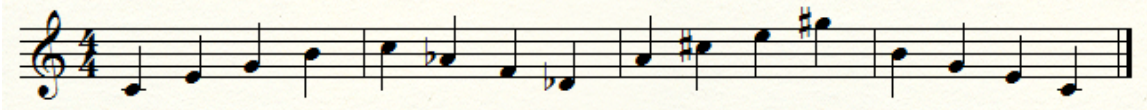
Figure 6: The original tetrachord, then inversion starting on C_5, transposing up major 6th, and retrograde.

of all the intervals between notes. The same tetrachord $C - E - G - B$ is now inverted to $C - A\flat - F - D\flat$. Retrograding is starting at the end of the musical object and reading it backwards. Thus the retrograde of $C - E - G - B$ would be $B - G - E - C$. There is no specified starting note regarding both inversion and transposition, unlike retrograde where it is required to start at the last note and read backwards.

Both transpositions and inversions alter the objects yet they preserve the intervalic relationship between each note. Thus we can conceive them as mappings from one set to itself, essentially as permutations. In following sub subsections, we will define transposition, interval preserving mappings, and inversion and take a look at properties they possess.

5.1   Transpositions

We first define the transposition mapping.

Definition 5.1.1. Given a GIS (S, G, int), let $i \in G$. Then T_i is an endomorphism mapping from S to S such that for each $s \in S$,

$$\text{int}(s, T_i(s)) = i.$$

In Definition 5.1.1, we defined T_i as an endomorphism. Moreover, we can show that T_i is an automorphism, or permutation.

Proposition 5.1.2. *Given a GIS (S, G, int), $T_i : S \to S$ is an automorphism. Moreover, the set of T_i's form a group under the function composition $T_i(T_j(s)) = T_{ij}(s)$.*

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Proof. In order to show that T_i is an automorphism, we need to show that T_i is onto and one to one regardless of which i we pick in the group G. For each $s \in S$, we want to show that there exists an inverse mapping for $T_i(s)$ such that $T_i(T_i^{-1}(s)) = s$. Since i is a group element in G, there exists an element $j \in G$ such that $ij = e$. Then

$$T_i(T_j(s)) = T_{ij}(s) = T_e(s) = s.$$

Thus the transposition mapping is onto.

To show that T_i is one to one, assume that $T_i(s) = T_i(t)$ for some $s, t \in S$. Then by the definition of transposition function, $\text{int}(s, T_i(s)) = i$ and $\text{int}(t, T_i(t)) = i$. Then

$$\text{int}(s, T_i(s)) = \text{int}(t, T_i(t))$$
$$= \text{int}(t, T_i(s))$$
$$\text{int}(s, T_i(s)) \, \text{int}(T_i(s), s) = \text{int}(t, T_i(s)) \, \text{int}(T_i(s), s)$$
$$\text{int}(s, s) = \text{int}(t, s)$$
$$e = \text{int}(t, s)$$
$$t = s.$$

Thus T_i is one to one. Therefore, the transposition mapping is indeed an automorphism.

Now we show that the set of transposition forms a group under the function composition $T_i(T_j(s)) = T_{ij}(s)$. Let j be the inverse element of i in the group G. Then

$$T_i(T_j(s)) = T_{ij}(s) = T_e.$$

$T_e(s)$ means that $\text{int}(s, T_e(s)) = e$, so it follows that $s = T_e(s)$. Hence T_e is the iden-

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tity element. Since each element $i \in G$ has the inverse element, each T_i has the inverse mapping, namely T_i^{-1}. To show associativeness, consider $(T_i(T_j(s))T_k(s) = T_k(T_i(T_j(s))) = T_k(T_{ij}(s)) = T_{ijk}(s) = T_i(Tjk(s)) = T_i(T_j(T_k(s)))$. Since G is group and its elements are associative, the above equation holds true. Thus the set of transposition functions form a group. $\qquad \square$

5.2   Interval Preserving Functions

We can further generalize the interval preserving property beyond the transposition function. However, before we do so, we define the label function, which eliminates the need of having to take in two arguments with the interval function.

The label function is the extension of the integer notation on pitches. Traditionally, C is 0 and $C\sharp$ is 1, D is 2, and so on, ending with B as 11, regardless of which octave they lie in. However, setting the starting point, labeling C as 0, is completely arbitrary and there is no particular reason other than the convenience to label such way. Therefore there is no reason not to move the reference point around for convenience, which is the point of the label function.

Definition 5.2.1. Given a GIS (S, G, int) and a referential point $r \in S$ and an element $s \in S$, define the label function $L : S \to S$ such that

$$L(s) = \mathrm{int}(r, s).$$

There exist some interesting relations between label functions and transposition functions, which we will explore next.

Proposition 5.2.2. *Given a GIS, $i \in G$, $s \in S$, and a referential point $r \in S$,*

$$L(T_i(s)) = L(s) \cdot i.$$

Proof.

$$L(T_i(s)) = \text{int}(r, T_i(s))$$

$$= \text{int}(r, s) \cdot \text{int}(s, T_i(s))$$

$$= \text{int}(r, s) \cdot i$$

$$= L(s) \cdot i$$

$\square$

In most cases, transposition function works in commutative GIS, such as collection of pitches, which the function "preserves intervals" as in it keeps the intervalic relationship between each note even after transposing. We formally define this concept of "interval preserving" function.

Definition 5.2.3. Let (S, G, int) be a GIS and $f : S \to S$ a function. f is *interval preserving* if for all $s, t \in S$,

$$\text{int}(f(s), f(t)) = \text{int}(s, t).$$

We say a GIS is commutative if G is an abelian group. Hence GISs are not inherently commutative, which means that left and right multiplication is not necessarily equal. Interval Preserving function, P_i, is precisely the function that is created by left multiplying the interval i with the label function..

Definition 5.2.4. For a referential element $r \in S$ and $i \in G$, define an endomorphism $P_i : S \to S$ such that

$$L(P_i(s)) = i \cdot L(s),$$

which is

$$\text{int}(r, P_i(s)) = i \cdot \text{int}(r, s).$$

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Theorem 5.2.5. *Let (S, G, int) be a GIS. Then the mapping P_i preserves interval. Moreover, any interval preserving endomorphism on S is P_i for some $i \in G$ any referential element $r \in S$.*

Theorem 5.2.6. *Let (S, G, int) be a GIS and i an element in G. T_i preserves intervals if and only if $T_i = P_i$ for all $i \in G$ and G is an abelian group.*

From Theorem 5.2.6, it follows that in a commutative GIS, any transposition function is interval preserving. Similarly, in a non-commutative GIS, there exists transposition functions that do not preserve intervals and interval preserving functions that are not transpositions. Moreover, they commute with each other even in non-commutative GISs.

Proposition 5.2.7. *Any transposition function commutes with any interval preserving function.*

Proof. For a GIS, a referential element $r \in S$, an element $s \in S$, and $i, j \in G$, consider T_i and P_j. Then,

$$L(P_j(T_i(s))) = j \cdot L(T_i(s))$$
$$= j \cdot L(s) \cdot i$$
$$= L(P_j(s)) \cdot i$$
$$= L(T_i(P_j(s)))$$

$\square$

5.3   Inversion Functions

Next we examine inversion. Recall from Figure 6 that inversion is taking the intervals and reversing the direction of them. Furthermore, there is no specified starting note with

inversions. Therefore we specify the ending note and the starting note of inversion by calling it u/v *inversion* where u is the ending note in original interval and v is the starting note in the inversion. For example, inverting the interval C to E in Figure 6 to C to $A\flat$ would be e/c inversion. If we had e/e inversion, the inversion would be E to C. For e/g inversion, it would be G to $D\sharp$. The following definition formally defines inversion as a function.

Definition 5.3.1. For a GIS and $u, v, s \in S$, $I_u^v : S \to S$ is an automorphism that satisfies the following equation:

$$\text{int}(v, I_u^v(s)) = \text{int}(s, u).$$

In twelve tone pitch class, same inversion effect can be achieved with many different name. As seen in earlier example, I_e^c has the same effect as I_e^e or I_e^g and so forth. However the twelve tone GIS is one of the nicer GIS to work in as it is commutative. We further investigate what kind of condition is required for two differently named inversions to have the same effect.

Lemma 5.3.2. *Let $r \in S$ be a referential element. Then*

$$L(I_u^v(s)) = L(v)L(s)^{-1}L(u).$$

Theorem 5.3.3. *Given a GIS (S, G, int), inversion functions $I_u^v = I_x^w$ if and only if $w = I_u^v(x)$ and the interval $\text{int}(x, u)$ is in $Z(G)$, center of the group G.*

Proof. Let $r \in S$ be a referential element in the set S. Then

$$L(I_u^v(s)) = L(v)L(s)^{-1}L(u)$$

and

$$L(I_x^w(s)) = L(w)L(s)^{-1}L(x)$$

for all $s \in S$ by Lemma 5.3.2. We assume $L(I_u^v(s)) = L(I_x^w(s))$, which is

$$L(v)L(s)^{-1}L(u) = L(w)L(s)^{-1}L(x)$$

$$L(w)^{-1}L(v)L(s)^{-1} = L(s)^{-1}L(x)L(u)^{-1}$$

It follows that $L(w)^{-1}L(v) = L(x)L(u)^{-1}$ which we will call $g \in G$. Note that g commutes with $L(s)^{-1}$ for all $s \in S$ Thus g is in the center of the group G.

Now consider $L(x)L(u)^{-1} = g$.

$$L(x)L(u)^{-1} = g$$

$$L(x) = gL(u)$$

$$L(x) = L(u)g$$

$$L(u)^{-1}L(x) = g$$

Thus $L(u)^{-1}L(x)$ is also in the center of G. Note that $L(x) = \text{int}(r, x)$ and $L(u) = \text{int}(r, u)$ thus $L(u)^{-1}L(x) = \text{int}(x, u)$. Since $g = L(w)^{-1}L(v) = L(u)^{-1}L(x)$, it follows that $\text{int}(v, w) = \text{int}(x, u)$. Thus $w = I_u^v(x)$ From Definition 5.3.1.

For the converse, we assume that $w = I_u^v(x)$ and $\text{int}(x, u)$ commutes with all $g \in G$. We claim that with aforementioned assumptions, $I_u^w = I_x^w$.

$$w = I_u^v(x)$$

$$\text{int}(v, w) = \text{int}(x, u)$$

$$L(w)^{-1}L(v) = L(u)^{-1}L(x)$$

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We already know that $\text{int}(x, u)$ is in the center. Thus $L(x)L(u)^{-1}$ also is in the center as well, using the same logic from the first part of the proof. Then

$$L(x)L(u)^{-1} = L(v)^{-1}L(w)$$

$$L(u)L(x)^{-1} = L(w)^{-1}L(v)$$

$$L(u)L(x)^{-1}L(s)^{-1} = L(s)^{-1}L(w)^{-1}L(v)$$

$$I_u^v = I_x^w$$

$\square$

Note that since the referential element $r \in S$ is chosen arbitrarily, and the choice of r does not change the proof.

Corollary 5.3.4. *Given a GIS (S, G, int), $I_u^v = I_v^u$ if and only if $\text{int}(v, u)$ is in the center of the group G.*

Now we examine how inversion functions interact with the transposition and interval preserving functions.

Theorem 5.3.5. *For any transposition Tn and any inversion I_u^v, the following are equivalent:*

(a) $T_n I_u^v = I_x^v$ where $x = T_n(u)$.

(b) $I_u^v T_n = I_u^w$ where $w = T_n^{-1}(v)$.

(c) T_n commutes with I_n^v if and only if n is central and $nn = e$.

Part (C) of Theorem 5.3.5 states that given an interval n in a GIS, either T_n commutes with every inversion function or T_n commutes with no inversion function. For example, consider the GIS of the twelve chromatic pitch classes, where S is the set of twelve chromatic

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pitch classes, G is $(\mathbb{Z}_{12}, +)$, and int is the number of half steps in clockwise direction. Clearly, the trivial transposition function, T_0, commutes with any inversion functions. Moreover, T_6 commutes with every inversion function as well. Inverting after transposing by a tritone or transposing by a tritone then inverting have the same net result. However, there are no other transposition functions that commute with any other inversion functions. We know there are only T_0 and T_6 that commutes with everything else because no other element in G satisfies the condition $nn = e$ in (C).

Theorem 5.3.6. *For any interval preserving function P and any inversion function I_u^v, the following are equivalent:*

(a) $P\,I_u^v = I_u^w$ where $w = P(v)$.

(b) $I_u^v\,P = I_x^v$ where $x = P^{-1}(u)$.

(c) P commutes with I_u^v if and only if $P = T_e$ for some transposition T_e such that c is in $Z(G)$ and $cc = e$.

The last two theorems show how inversion functions interact with the transposition and interval preserving functions. Earlier in Section 5, we examined how transpositions and interval preserving functions interacted with themselves and discovered that both transposition and interval preserving functions form groups among themselves. Now we investigate the ways inverse functions interact with themselves.

Theorem 5.3.7. *For a GIS , let $r \in S$ be a referential element, v, u, w, x be elements in S. Then*

$$I_u^v\,I_x^w = P_{L(v)L(x)}^{-1} T_{L(w)}^{-1L(u)}.$$

This theorem can be proven by comparing the label functions for the each side of the equation.

Corollary 5.3.8. I_v^u *is the inverse function of* I_u^v.

Corollary 5.3.9. *For a commutative GIS, let T and I be a transposition and an inversion function, respectively. Then*

$$I^{-1} = I$$

$$IT = T^{-1}I.$$

We now look globally at how inversion, transposition, and interval preserving functions play with each other.

Definition 5.3.10. Given a GIS, $PETEY$ is the set of all mappings on the set S that is equivalent to a mapping of form PT, where P is some interval-preserving function and T is some transposition function. $PETINV$ is the union of $PETEY$ and a set of all inversion mappings on S.

Proposition 5.3.11. $PETEY$ *and* $PETINV$ *each form a group under function composition.*

5.4 Interval Reversing Functions

Previously in Section 5.1, we noted that transposition mappings relate naturally to interval preserving functions. Now we define a function that is naturally related to the inversion function.

Definition 5.4.1. A function Y on the set S of a GIS is *interval reversing* if

$$\mathrm{int}(Y(s), Y(t)) = \mathrm{int}(t, s)$$

for all $s, t \in S$.

Interval reversing functions possess an interesting property. If the GIS is commutative, then the inversions are precisely the interval-reversing functions on S. On the other hand, if the GIS is non commutative, then interval reversing transformations do not exist on that particular GIS. Following lemma and theorem prove this property.

Lemma 5.4.2. *Let Y be an interval reversing mapping on S. Then for a referential point $r \in S$, there exists an interval i such that*

$$L(Y(t)) = i \cdot L(t)^{-1}$$

for all $t \in S$.

Theorem 5.4.3. *If the group of inversion functions is Abelian, then the inversion functions reverse intervals, and every interval reversing transformation is some inversion function.*

Theorem 5.4.4. *If the group of inversion functions is not Abelian, then there is no interval reversing mapping on S.*

6   Rhythmic GIS

So far, we have mostly spent time looking at GISs with sets containing pitches. In this section, we follow chapter 4 of *Generalized Musical Intervals and Transformations* for rhythmic GISs. We begin by looking at more inclusive definition of the rhythm and time point and expand to building more musically significant and concrete example of rhythmic GIS. [2]

Definition 6.0.5. A *time span* is an ordered pair (a, x), where $a \in \mathbb{R}$ and $x \in \mathbb{R}^+$. In this ordered pair, the first element points at the location of the musical event and the second element denote the length of the event. In other words, the pair tells us that a musical event "begins at time a" and "extends x units of time".

Recall that we have already seen an example of rhythmic GIS earlier in Example 4.0.16. However, that GIS was a little restricting, as we restricted the values of a to integers and x to certain proportions. Now using the previously defined concept of direct product between GISs and a more rigorous definition of time span, we can create a more inclusive rhythmic GIS.

Example 6.0.6. Let a set S be the set of time spans and G be $(\mathbb{R}, +) \times (\mathbb{R}^+, \cdot)$. The interval function is $\mathrm{int} : S \times S \to G$ such that $\mathrm{int}((a, x), (b, y)) = (b - a, y/x)$. Then (S, G, int) is a commutative GIS. Note that the interval function measures the distance and proportional length of second musical event by saying that the second event starts $b - a$ units later than the first event and it lasts y/x times longer compared to the length of the first musical event.

Example 6.0.6 is far more general than Example 4.0.16. And in next paragraph we will see how this GIS enables us to use a broader definition of time unit.

Commonly when musicians say time, they communicate in terms of beats and measures. Recall that time signature is a fraction with numerator being the number of beats in a measure and denominator being the length of a note that gets a beat. For example, 4/4 time has a beat of a quarter note, and four of them make up a bar, or a measure. Another example is 6/8 time, where an eighth note gets a beat and total six of them completes a bar. The beat and measure provides a convenient, flexible way to discuss time and rhythm in a certain piece (or a part of piece, if time signature changes in the middle of the piece).

David Lewin takes this notion one step further. He claims that since there are be passages in music where the beat of one part is different from another part, using beat as a time unit poses potential problems. He then suggests other ways to consider time units, saying that one could use a motif as one, or a tool to measure time, instead of using beats. This way is quite more restrictive compared to the traditional way, but it does provide some benefits since it allows theorists to focus on a motif and track how it transforms throughout the piece.

Even though this model cannot be used universally in the piece, it certainly has benefits, especially tracking a motif and analyzing its transformation throughout the piece.

With this model in mind, we construct a GIS that meets different needs at different times. If we use a motif as a time unit, the interval function would be $\text{int}((a, x), (b, y)) = ((a - b)/x, y/x)$, while keeping everything else the same from Example 6.0.6. With next set of lemmas and theorems, we use this new concept of time unit to create a formal GIS.

Lemma 6.0.7. *Let IVLS be a set of pairs (i, p), where i is a real number and p is a positive real number. Then IVLS forms a group under composition*

$$(i, p)(j, q) = (i + pj, pq).$$

The identity element of this group is $(0, 1)$ and the inverse element of (i, p) is $(-i/p, 1/p)$. Moreover, this group is not Abelian.

Theorem 6.0.8. *Let $\text{int}(\cdot, \cdot)$ be the function that maps TMSPS $\times$ TMSPS into the group IVLS of Lemma 6.0.7 accodring to the formula*

$$\text{int}((a, x), (b, y)) = ((b - a)/x, y/x).$$

Then $(TMSPS, IVLS, \text{int})$ is a GIS.

Theorem 6.0.9. *The GIS in Theorem 6.0.8 has following properties:*

(A) For any real number h, the interval from time span $(a + h, x)$ to time span $(b + h, y)$ is the same as the interval from (a, x) to (b, y).

(B) For any positive real number u, the interval from time span (au, xu) to time span (bu, yu) is the same as the interval from (a, x) to (b, y).

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Note that the time spans still rely numerically on an implied referential time unit and an implied time point zero. For example, (a, x) begins the a of referential units after the referential zero time point, and lasts the x of referential units. Properties (A) and (B) in Theorem 6.0.9 above shows that the numerical function int for the GIS under present discussion does not depend at all on the choice of time point zero, or on the choice of referential time unit. However, Theorem 6.0.9 does not hold true for the GIS in Example 6.0.6. Recall that the interval function for GIS in Example 6.0.6 maps by following formula: $\text{int}((a, x), (b, y)) = (b - a, y/x)$. If we replace the referential time unit such that the events (a, x) and (b, y) become (au, xu) and (bu, yu) with the new time unit, we can see that the interval does not hold true. With the new time unit $\text{int}((au, xu), (bu, yu)) = (bu - au, y/x)$, which is not equal to $(b - a, y/x)$.

In fact, it turns out that the GIS in Theorem 6.0.8 is essentially the only possible GIS involving time spans in which Theorem 6.0.9 holds true.

Theorem 6.0.10. *If there exist a $GIS = (S, G', \text{int}')$ such that S is the set of time spans and GIS satisfies the properties in Theorem 6.0.9, then the G' is isomorphic to G of the $GIS = (S, G, \text{int})$ in Theorem 6.0.8 by the mapping f such that $\text{int}' = f(\text{int})$.*

Note 6.0.11. In regards to GIS 6.0.8, the following are true:

(A) Given an interval (i, p) and a time span (a, x), the transposition of the given time span by the given interval is

$$T_{(i,p)}(a, x) = (a + ix, px).$$

(B) If we fix $(0, 1)$ as a referential time span in S, then for any $(a, x) \in G$,

$$L(a, x) = \text{int}((0, 1), (a, x)) = (a, x).$$

(C) For $(i, p) \in G$ and $(a, x) \in S$, there exists $(a, x) \in G$ such that

$$T_{(i,p)}(a, x) = (a, x)(i, p).$$

(D) For $(h, u) \in G$ and (a, x) both in S and G,

$$P_{(h,u)}(a, x) = (h, u)(a, x) = (h + ua, ux).$$

(E) The only element in G that is also in $Z(G)$ is the identity element.

(F) No transposition preserves intervals, and no interval preserving mapping is a transposition, except for the trivial mapping.

(G) For $(c, z), (d, w), (a, x) \in S$, the mapping $(c, z)/(d, w)$ inversion applied to (a, x) is the following:

$$I_{(c,z)}^{(d,w)}(a, x) = (d + \frac{(c - a)w}{x}, \frac{zw}{x})$$
$$= (d, w)(a, x)^{-1}(c, z)$$

(H) For time spans $s, t, s', t' \in S$, $I_{s'}^{t'} = I_{s}^{t}$ if and only if $s = s'$ and $t = t'$.

(I) There are no interval reversing mappings on the set of time spans.

7 Application of GIS

7.1 Analysis of Atonal Music

Generalized interval system is readily applicable to the analysis of atonal music. In this section, we take two twentieth century pieces that serializes various materials and attempt

to describe them in terms of GIS and Lewin's terms.

7.1.1 Luigi Dallapiccola: Quaderno Musicale di Annalibera, No. 3

The first piece we will look at is a piece from the collection Quaderno Musicale di Annalibera (1952), composed by Luigi Dallapiccola (1904 - 1975). Dallapiccola is an Italian composer famous for the lyrical twelve tone writing.[6] Quarderno Musicale di Annalibera is a collection of short compositions for a solo piano. Contrapunctus Primus is the third piece in the collection, written in twelve tone serial composition method, a composition style created by Arnold Schoenberg in 1921. Schoenberg sought after the equality among all twelve tones through this method. In twelve tone serial composition, a composer is not allowed to repeat a note until every other note has been played. Thus both the melody and the harmony are constructed without doubling any notes, unless there are two or more different rows stacked on each other. Since there is no one prominent note over another, pieces composed in twelve tone method are atonal, or not having a tonal center, in nature. These pieces are most commonly analyzed by using the twelve tone matrix. The twelve tone matrix is constructed by using the first row as the prime row and transposing, retrograding, inverting the row for all possible permutations while preserving the intervalic relationship between the notes in a row. In the analysis using a twelve tone matrix, the GIS is $(S, (\mathbb{Z}_{12}, +), \text{int})$ where S is a set of twelve pitch classes and int is number of half step in clockwise direction. We claim that Contrapunctus Primus in whole is thus a set of permutations of the prime row presented in the beginning.

Typically, the row that is presented in the very beginning is considered to be a prime row of some transposition. In the beginning of Contrapunctus Primus, Figure 7 shows that our prime row is in the right hand part of the piano, which is $11, 0, 4, 7, 9, 3, 2, 6, 8, 1, 10, 5$ in the integer notation. This is called to be P_{11} row, which is T_{11} function applied to P_0. We then construct aforementioned twelve tone matrix by permuting the row in all possible ways. The

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Figure 7: The first row that appears in Contrapunctus Primus, $P11$, in the red box.

completed matrix is shown in Table 3. Since we know that a row has been completed after twelve notes, we can continue this process of tracking down next twelve notes, finding out which functions has been applied to which row and repeat until the piece ends. As shown in Table 3, there are total four different types of row, prime, inversion, retrograde, and retrograde inversion. Any prime row P_i are simply the transposition function, T_i applied to P_0. Inversion rows, I_i, are inversion function applied to P_j, where $j = i^{-1}$. Retrograde rows, R_i, are the additive inverse of P_i, since $\mathrm{int}(s,t)^{-1} = \mathrm{int}(t,s)$. Lastly, Retrograde inverse rows, RI_i, are the inverse functions on R_j row, again where $j = i^{-1}$. Figure 8 shows how the permutations of the P_0 row composes the whole piece. From the beginning, yellow rows are P_{11}, orange rows are R_7, yellow and purple rows are RI_0, orange and green row is R_{10}, green rows are P_2, and blue rows are I_{10}. As shown in Figure 8, there is no note that does not belong in a row and since we know that any row is a permutation of another, we conclude that this piece is indeed a set of permutations of the first row.

Next we consider more intricate structures of the piece, including rhythm, articulation, and the relationships between rows. Notice, from Figure 8, that with the exception of R_{10} row, there exist two copies of every row that appears in the piece, both beginning and ending approximately at the same time. Let a GIS be $(S, G = (\mathbb{Q}^*, \cdot), \mathrm{int})$ such that S is the set of durations of notes, measured in quarter note beats, G is the group of rationals excluding

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Figure 8: Contrapunctus Primus, color coded by the types of row.

*	I_0	I_1	I_5	I_8	I_{10}	I_4	I_3	I_7	I_9	I_2	I_{11}	I_6	*
P_0	0	1	5	8	10	4	3	7	9	2	11	6	R_6
P_{11}	11	0	4	7	9	3	2	6	8	1	10	5	R_5
P_7	7	8	0	3	5	11	10	2	4	9	6	1	R_1
P_4	4	5	9	0	2	8	7	11	1	6	3	10	R_{10}
P_2	2	3	7	10	0	6	5	9	11	4	1	8	R_8
P_8	8	9	1	4	6	0	11	3	5	10	7	2	R_2
P_9	9	10	2	5	7	1	0	4	6	11	8	3	R_3
P_5	5	6	10	1	3	9	8	0	2	7	4	11	R_{11}
P_3	3	4	8	11	1	7	6	10	0	5	2	9	R_9
P_{10}	10	11	3	6	8	2	1	5	7	0	9	4	R_4
P_1	1	2	6	9	11	5	4	8	10	3	0	7	R_7
P_6	6	7	11	2	4	10	9	1	3	8	5	0	R_0
*	RI_6	RI_7	RI_{11}	RI_2	RI_4	RI_{10}	RI_9	RI_1	RI_3	RI_8	RI_5	RI_0	*

Table 3: The twelve tone matrix for Contrapunctus Primus. It lists all possible combination of inversions, retrogrades, and transpositions of the first row in the piece.

zero under multiplication, and $\text{int}(s, t) = t/s$. For example, the first three notes of the piece are $\{2, 2, 4\}$ and the set of intervals for the trichord is $\{1, 2\}$.

Now consider the pair of the first row, P_{11}. We can take the first trichord from each row, and label them TC_1 and TC_2, respectively, as shown in Figure 9. TC_1 is a subset of S, $\{2, 2, 4\}$ and TC_2 also is a subset of S, $\{3, 3/2, 3/2\}$. We can transform TC_1 to TC_2 by multiplying all elements in TC_1 by $3/4$, then retrograding it. This relation holds for the entire piece, the same trichords from the pair of the same row being retrograde of each other and the row that starts later being $3/4$ in value of the first row.

There also exists an interesting symmetry in the row itself. Consider again the P_{11} row, shown in Figure 10. The first two trichords have the value of $\{2, 2, 4\}$ and the last two trichords have the value of $\{4, 2, 2\}$, essentially mirroring itself. This pattern again holds for the entire piece, no matter what the value of each trichords are.

Now let us build a GIS that handles articulations. There are total three different articulation markings in Contrapunctus Primus, a staccato($\cdot$), a tenuto(_), and a staccato joined with a tenuto($\dot{-}$). Let S be the set of all articulation notation in Contrapunctus

Figure 9: Trichords TC_1 and TC_2, $TC_2 = R(3/4 \cdot TC_1)$.

Figure 10: Row P_{11}. Notice that the first two trichords are $\{2, 2, 4\}$ and the last two are $\{4, 2, 2\}$, later two retrograding the first two trichords.

Primus, $\{\cdot, \overline{\cdot}, \_\}$. In order to define the interval function, we need a way to measure these articulations, but they are relative in quality rather than being absolute. To solve this issue, we can arbitrarily assign length to the articulations, according to their relative lengths. Thus let the staccato($\cdot$) be length 1, the tenuto with staccato($\overline{\cdot}$) be length 2, and the tenuto($\_$) be length 3, lower number indicating shorter articulation. Then the interval function is $\text{int}(s, t) = t - s$, $s, t \in S$ and the group G is $(\mathbb{Z}, +)$, integers under addition. This GIS together with the duration GIS makes a useful tool to investigate the relationship between articulation and note durations.

Next, we create a new GIS, $(R \times S, (\mathbb{Q}^*, \cdot) \times (\mathbb{Z}, +), \text{int})$ such that $R \times S$ is the Cartesian product of the set of durations of notes and the set of articulations and $(\mathbb{Q}^*, \cdot) \times (\mathbb{Z}, +)$ is the direct product of rational numbers except 0 under multiplication and integers under addition. The interval function is $\text{int}((r, t), (s, u)) = (s/r, u - t)$ for $r, s \in R$ and $t, u \in S$.

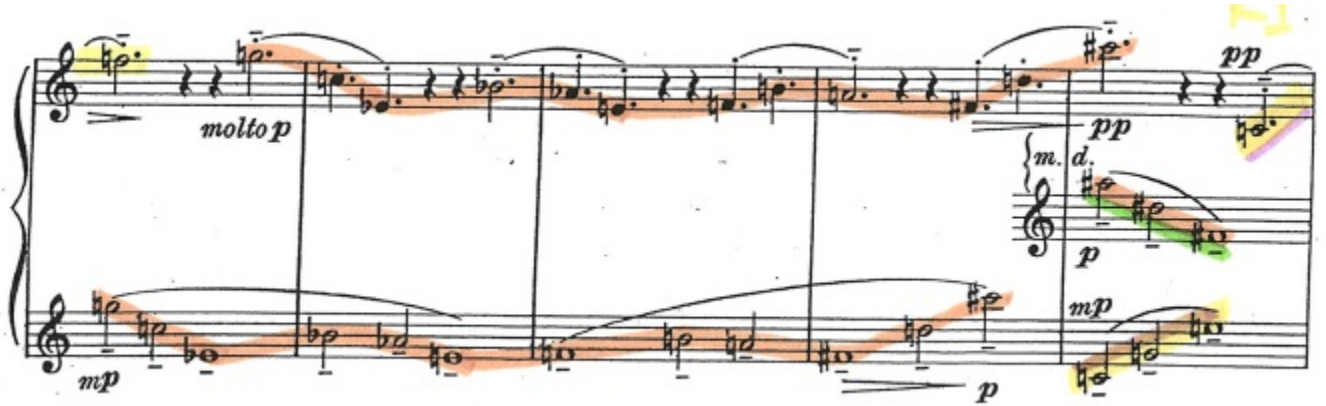In Figure 11, we have a pair of identical rows, both R_7. The first trichord in the top row

Figure 11: Row R_7. Observe the relationship between note duration and the articulation length.

is $\{(3, \div), (3/2, \cdot), (3/2, \cdot)\}$ and the first trichord in the bottom row is $\{(2, \_), (2, \_), (4, \_)\}$. We see that a certain note value is always matched with the same articulationi type. For example, notes with duration $3/2$ are always paired with staccatos, the ones with duration 3 are paired with staccato with tenuto, and the ones with duration 2 and 4 are always paired with tenutos. While there is no nice, linear relationship between the note duration and the articulations, we can see that the rows with shorter note length have shorter articulation and rows with longer note length have longer articulation, further emphasizing the duration differences.

7.1.2 Karlheinz Stockhausen: Kreuzspiel, Part 1

Karlheinz Stockhausen (1928 - 2007) is a leading German composer of his generation. He is also an influential composer of the post WWII avant-garde and redefined notions of serial composition. [7] Kreuzspiel, "crossplay" in German, is a chamber work written for piano, oboe, bass clarinet, and percussion. It is one of his first works and it is organized by serializing pitch, note duration, and register. In this section, we will examine how different musical objects are serialized in this piece by the terms of the GIS.
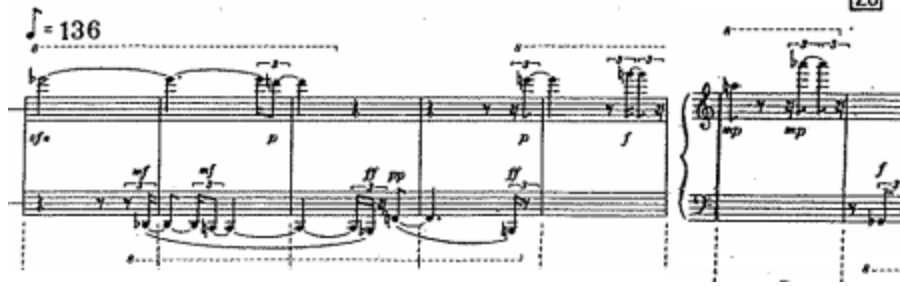
Figure 12: The first twelve notes in the piano part, after the introductory material.

In Kreuzspiel, the pitches are treated in a manner similar to twelve tone such that one pitch is not reused until after every other tone has been used. However, unlike the twelve tone serial composition where the ordered set is mapped by different interval preserving functions, the set of twelve tone is unordered thus the set is permuted.

Let a GIS (S, G, int) be the one discussed in Example 3.0.4, except the pitch classes will be discussed in integer notation. In measure 14, the first set of twelve pitches is introduced in the piano part after a brief introduction. The first tone row, shown in Figure 12 is an ordered set of pitches, which is $\{3, 1, 0, 2, 10, 5, 11, 4, 7, 9, 8, 6\}$. The next row is another ordered set with 12 elements, which is $\{1, 0, 2, 10, 5, 6, 3, 11, 4, 7, 9, 8\}$ If we take the interval function and apply it to these two sets, we get $\{-2, -1, 2, 8, -5, 6, 7, 3, 2, -1, -2\}$ for the first row and $\{-1, 2, 8, -5, 1, -3, 8, -7, 3, 2, -1\}$ for the second row. If the transformation that took the first row to the second row was interval preserving, the absolute value of elements in each set should be equal. However it is clearly not, so we know that the way Stockhausen went about permuting the first row is not interval preserving, thus it is not composed by twelve tone serial composition technique.

However, a broader look to the entire scheme of the pitch classes reveals an interesting pattern that is worth looking into. Let a GIS be $(\mathbb{Z}_{12} \times \mathbb{Z}_{12}, (\mathbb{Z}_{12}, +) \times (\mathbb{Z}_{12}, +), \text{int})$ such that $\mathbb{Z}_{12} \times \mathbb{Z}_{12}$ is the Cartesian product of the set of twelve pitch classes and the set of location on a tone row. For example, the first note in the first tone row would be $(3, 0)$, since it is
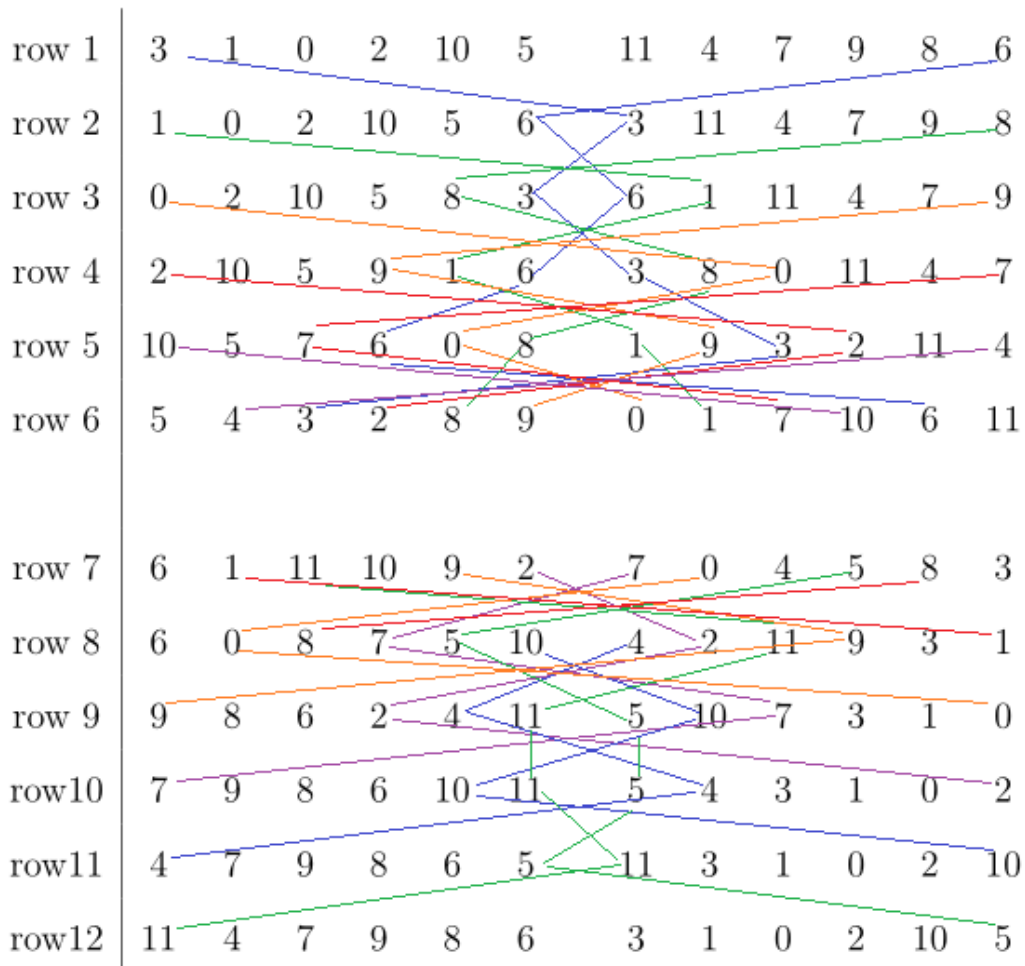
Figure 13: The table of all pitch classes in part 1 of the composition, listed by the row they belong to. It also tracks notable crossings from one row to the next. Note that from row 1 to 6, the crossings start at the outer edge, where as from row 7 to 12, the crossings start from inside, essentially *undoing* the crossings that has been done in rows 1 to 6.

pitch class 3 and it is the first note in that particular tone row. This enables us to track down the movement of certain pitch class from one row to another. Figure 13 is the complete table of all 12 rows in part 1. In the figure, we can easily see an algorithm taking place. For example, take pitch classes 3 and 6 from row 1 and if we track them to row 2 and to the halfway point, row 6, we get $(3,0), (3,6), (3,5), (3,6), (3,8), (3,2)$ for the pitch class 3 and $(6,11), (6,5), (6,6), (6,5), (6,3), (6,10)$ for the pitch class 6. If we consider the twelve tone row as two hexachords, positions 0 to 5 being the first and 6 to 11 being the second, we see that these two pitch classes start at the end of the respective hexachords and go back and forth between two hexachord, while being approximately symmetric to each other, since the positions of pitch classes 3 and 6 add up to 11 on every row, except for the one "glitch" in row 6. Figure 13 tracks the major crossings of pitch classes, from row 1 to 6 then 7 to 12, color coded by symmetric pairs. It is clear that from row 1 to 6, the crossings start at the edge of the row, folding in, and rows 7 to 12 undo the crossings done previously, with row 12 ending with the same hexachord as row 1. These pitch class crossings are only a tiny portion of the reflection of the name of the composition, "crossplay." The duration of pitches is also organized in this fashion, as well as the duration of percussion instruments.

Next we examine the registers. Let the GIS be $(\{1,2,3,4,5,6,7\}, (\mathbb{Z}_7, +), \text{int})$ such that $\{1,2,3,4,5,6,7\}$ is the set of octaves in piano, 1 being the lowest and 7 being the highest octave. Moreover, let $\text{int}(s,t) = t - s$. This GIS can track where the octave lies for different pitches. For example, we can fix a pitch, p, and follow p around from row 1 to 12, tracking the octave in which it lies. Doing so, we get following result shown in Table 4.

Table 4 displays a clear pattern showing how each pitch class moves through registers. All pitch classes start and end in either the octave 1 or 7, which are extreme octaves. As we progress to the middle, pitch classes come down to the middle octaves as well. Moreover, we see that all register rows are, regardless of pitch class, symmetric.

This obvious pattern also affects instrumentation. Since the oboe and bass clarinet cannot

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Pitch Class							
0	1	6	3	4	5	2	7
1	1	6	3	4	5	2	7
2	7	2	5	4	3	6	1
3	7	2	5	4	3	6	1
4	7	2	5	4	3	6	1
5	1	6	3	4	5	2	7
6	1	6	3	4	5	2	7
7	7	2	5	4	3	6	1
8	7	2	5	4	3	6	1
9	7	2	5	4	3	6	1
10	1	6	3	4	5	2	7
11	1	6	3	4	5	2	7

Table 4: A table tracking the register change of each pitch class in Part 1 of Kreuzspiel

Figure 14: The last system shown rightside up(top) and upside down(bottom), depicting how first and second violin players would see the same system differently.

play extreme registers, as pitch classes become closer to middle octaves, the bass clarinet and oboe part increase, and as the register becomes extreme again at the end, they slowly fade out.

7.2 Analysis of Tonal Music: Der Spiegel Duet

In the introduction of the book *Generalized Musical Intervals and Transfomations*, David Lewin claims that his theory can also be applied to tonal music as well as atonal music. In this section, we attempt to take a piece composed during the common practice period and use GIS to analyze and make sense out of it.

In this section, we examine Der Spiegel Duet, or the mirror duet, attributed to W. A.

Figure 15: The first system played by each instruments, put in order.

Mozart. This duet is special in the way that two violins read the same sheet music from opposite ends, creating two different parts from a sheet of music. The composer accomplishes this task by putting an upside down treble clef at the end of each line for the player reading upside down. Figure 14 shows how the two players see the same line differently.

Figure 15 shows how the piece would sound in the first couple bars. Note that the composer took care to sync the rhythm between two instruments and the distance between two instruments stick to octave and thirds, consonant intervals. This trend actually is visible through out the entire piece. Two parts are almost always in rhythmically together and they are most often an octave, 6th, or 3rd apart, all very consonant intervals. The use of perfect fifth, fourth, or other less consonant intervals, such as seconds or sevenths, is rather rare, other than occasional chromatic passages. The natural question arises: how did the composer manage to keep both parts from being dissonant?

To answer the question, we need to examine the underlying structure of the mirror duet. First, we can look at the staves and how the notes change when the orientation changes. Figure 14 clearly shows that one note in right side up orientation becomes something different when we flip the paper and look upside down. Further investigation reveals that there exists a bijective automorphism on the set of notes that appear in the mirror duet. Let S be the set of pitch classes that appear in the mirror duet, $S = \{D, G, B, E, C, A, F\sharp, A\sharp, C\sharp\}$. One thing to note is that in the staves, $F\sharp$ is written without $\sharp$ since it is implied by the key signature. This fact actually will save us from the potential headache that enharmonic

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x	$f(x)$
G	D
A	C
$A\sharp$	$C\sharp$
B	B
C	A
$C\sharp$	$A\sharp$
D	G
E	$F\sharp$
$F\sharp$	E

Table 5: Function $f : S \to S$ maps notes in right side up orientation to upside down orientation.

equivalence can cause. Now let $f : S \to S$ be the function that maps the notes in right side up orientation to upside down orientation. Table 5 shows how the function f maps the set S.

Using this function, we now know that to make the second violin part play an octave apart from the first part that plays $\{D, G, D, B, G\}$, we would put $\{G, D, G, B, D\}$ in the corresponding space for the second part. The function f along with the transposition function can be used to build the second violin part that maintains the same distance from the first part. For example, if we wanted to move the second violin part we wrote earlier such that it is either major or minor thirds apart using only the notes in the set S, we can transpose the original ordered set $\{D, G, D, B, G\}$ to $\{F\sharp, B, F\sharp, D, B\}$ and apply the function f, getting $\{E, B, E, G, B\}$. It is important to note that these two functions do not commute, thus $f \circ T_i$ is not equal to $T_i \circ f$. If we take the set $\{G, D, G, B, D\}$ which is $f(\{D, G, D, B, G\})$ and move up by thirds we would get $\{B, F\sharp, B, D, F\sharp\}$, while $f^{-1}(\{B, F\sharp, B, D, F\sharp\}) = \{B, E, B, G, E\} \neq \{E, B, E, G, B\}$.

So far we have covered how to work with two different orientations in terms of note placement. Furthermore, the second violin part of the mirror duet is the retrograde inversion of the first violin part, a side effect from looking at a sheet music upside down. Fortunately

the inversion portion is actually covered by the function f we have built earlier. However, the function does not take the retrograde into account, thus we need to make sure that the second violin part is retrograded in the music. This would make the last five notes in the piece $\{D, B, G, D, G\}$ if the first five notes of the piece are $\{D, G, D, B, G\}$. The composer certainly takes this into the account and works from both ends to carefully line the notes up so that there are no jarring dissonances. Moreover, this also means that the note durations needs to be retrograded as well in order for the two parts to play the same rhythm. This also can be easily discovered in the duet, since the set of durations of the first five notes measured by quarter notes is $\{1, 2, 1, 1, 1\}$ and the corresponding set in the last five notes is $\{1, 1, 1, 2, 1\}$.

8 Conclusion

In this paper, we looked at the generalized interval system, an integral part of Lewin's transformational theory. The use of group theory makes it easy for theorists to generalize this system beyond pitches, and it also successfully generalizes the usual transformations done to the pitches, such as transpositions and inversions. As a result, the generalized interval system works as a do-it-all tool that can tackle many different aspects in music, from pitch to timbre. However, in the application to real music, we found that the actual use of the GIS was rather limited to what we could already find without the use of the GIS. It was a nice tool to sum the results up and present to readers, but it did not work so well in discovering said results. Moreover, the sophistication that the GIS possesses due to its application in group theory was not present in the actual use but we were limited to the superficial use of it.

In the future, further investigation of the generalized interval system, especially with the focus to bring the mathematical sophistication to the actual analysis would make the use

of the GIS as an analysis tool more fruitful. Moreover, investigating the integration of the GIS with other parts of the theory, including the graph theory he presents later in the book, could further the ability of transformational theory beyond superficial level of analysis to discovering the inherent structure lying beneath the surface easier.

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Interval	Ratio
unison	1:1
minor third	6:5
major third	5:4
perfect fourth	4:3
perfect fifth	3:2
octave	2:1

Table 6: Ratio of commonly used intervals.

A   Just Intonation

There are two main methods of tuning in music, namely just intonation and equal temperament. Just intonation, also called pure intonation, is a tuning system developed by the Greek mathematician Ptolemy. Just intonation originated from the natural harmonic series and it tunes the intervals by those ratios. Table 6 shows the ratio of commonly used intervals. It is sometimes called Pythagorean tuning because Pythagoras expanded Ptolemy's original model from seven Greek modes to tune all twelve tones. Because these are naturally occurring ratios, our ears can tune them very accurately, and find the just tuned intervals, especially intervals like perfect fifths, more harmonious than equal or mean tuned intervals. [12] This tuning method fully benefits from different interval names because in just intonation, a perfect fourth is different from an augmented third.

However, there are a couple of flaws in this tuning system. One is that the octave does not close, which means the octave reached by this method of tuning will be wider than the frequency we reach by multiplying by two. This flaw makes instruments like the piano very difficult to tune since a piano cannot change tuning while playing, unlike string instruments. Pythagorean tuning works around this problem by using wolf fifth, a narrower fifth that is more dissonant, at the end to close the gap. However, this still does not solve another flaw of this tuning system. It is very hard to modulate, or to change key, during a piece without retuning, which is impossible for instruments like the piano. With a piano tuned in

just intonation, it is hard to play anything other than tonic, dominant, and sub dominant harmonies. As soon as we reach the chords like super tonic, it will sound out of tune since the chords are built on the second scale degree, not the first, the fourth, or the fifth scale degrees. Certain instruments, such as violins or trombones can adapt well to just intonation and many still tune this way when the situations permit, because it allows for more pure resonance. But for other instruments that cannot use this tunning method, they are tune by equal temperament, which essentially is a compromise of just intonation.