VOLUMES OF n-DIMENSIONAL SPHERES AND ELLIPSOIDS

MICHAEL JORGENSEN

Abstract. This paper starts with an exploration of the volume of sphere of radius r in n dimensions. We then proceed to present generalized results for the volume of a sphere under different p-norms or metrics also in n dimensions. We use a linear transformation to find the volume of an n dimensional ellipse, and use the Fundamental Theorem of Calculus in a clever way to find the surface area of a generalized spheroid. This paper is accessible to those familiar with calculus and linear algebra, but select parts of the paper use pieces of real analysis.

1. Introduction

What follows is a road map to this paper, and a outline of references. Section 2 defines an n dimensional sphere of radius r under the p-norm. Section 2.1 discusses how different p norms influence the shape of a sphere. In Section 2.2 we prove that the limit as p goes to infinity of the p norms is equivalent to the infinity norm. Section 2.3 extends the idea of volume to n dimensions and shows that a sphere is measurable using Lebesgue measure theory. In Section 3 we use a simple argument to show that the unit sphere in n dimensions is contained in a set of of thin boxes, and that the volume of these boxes goes to 0 as n goes to infinity.

This paper then outlines the properties of the gamma function in Section 4, since Γ is used in Section 5 where we use calculus in two different methods to prove an explicit formula for the volume of a n dimensional sphere. In Section 6 a formula for the volume of sphere as function of n, p, r is presented. Section 6.1 discuses p as a function of n, and finds function $p(n)$ that makes the volume of a sphere in the limit as n goes to infinity an arbitrary constant. In Section 6.3 the it is shown that $p(n)$ is well defined. We then explore the surface area of a sphere.

In Section 7 we prove a the volume formula in a third way that naturally leads to a surface area formula through the Fundamental Theorem of Calculus. In Section 8 we use basic linear algebra and a linear transformation of $\mathbb{R}^n$ to calculate the volume of a n dimensional sphere. Finally Section 9 shows that although the surface area of an n dimensional sphere is a manageable problem, the same techniques do not work on the ellipse.

The thin box proof comes from [4]. The first two proofs come from [2], which states the Gamma function properties with out proof. The proofs of the Gamma function properties mostly come from [5]. The volume as a function of n and p comes from [8] The third proof and surface area argument are from [11].

2. Definition of an n , p sphere

In this Section we denote an n dimensional sphere, of radius r with respect to a p-norm by $S_p^n(r)$. $S_p^n(r)$ is the set of all n dimensional points that which have a distance less then or equal to r from the origin when distance is measured under the p metric.

thanks to Albert Schueller for mathematical and latexical guidance.

Definition 2.1. *The p norm of $\mathbf{x} \in \mathbb{R}^n$ denoted $|\mathbf{x}|_p$ is*

$$|\mathbf{x}|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

for $p > 0$.

Definition 2.2. *The infinity norm of $\mathbf{x} \in \mathbb{R}^n$ is defined as*

$$|\mathbf{x}|_\infty = \sup\{|x_i| \big| 1 \leq i \leq n\}.$$

As in standard vector notation $|\mathbf{x}|$ means the norm or length of the vector $\mathbf{x}$. We will use $|\mathbf{x}|_p$ to denote the length of $\mathbf{x}$ under the p norm or equivalently the distance from $\mathbf{x}$ to the origin. Given this notation, for clarity, we make the following definition.

Definition 2.3. *$S_p^n(r)$ is the set of all $\mathbf{x} \in \mathbb{R}^n$ where $\mathbf{x} = (x_1, x_2, \ldots x_n)$ such that*

$$\sum_{i=1}^{n} |x_i|^p \leq r^p.$$

For example if $n = 2$ and $p = 2$ we recover have $S_2^2(r)$ which is the circle:

$$x_1^2 + x_2^2 \leq r^2.$$

Similarly $S_2^3(r)$ is the standard sphere. The standard (or Euclidean) norm is $p = 2$, throughout if we do not specify a p value then we will assume $p = 2$.

If we combine these two definitions then a point $\mathbf{x} \in S_p^n(r)$ sphere if an only if:

$$|\mathbf{x}|_p \leq r.$$

2.1. **Convexity of $S_p^2(1)$ for different values of p.** To develop intuition about p-norms we look at unit circles in 2 dimensions under the different p-norms. Here we discuss the effect of different p norms on a circle in 2 dimensions. In particular for $p \geq 1$ we have convex shapes, and for $p = \infty$ we have a square centered on the origin of side length two, which is also convex. FIGURES 1 2 3 4 are plots of $|x|_p = 1$ for different values of p. When $p = 2$ we have a circle, for $p = 1$ we have a square of side length $\sqrt{2}$, centered on the origin, with the y and x axis as its diagonals.

For $p \in (0, 1)$ the plots get more and more spiky as p decreases. For $p = \frac{1}{2}$ we get a cycloid. Although $p = 0$ is nonsensical, the geometric limits as p goes to zero would be a cross of line segments from $(-1, 0)$ to $(1, 0)$ and $(0, 1)$ to $(0, -1)$. When $p \in (1, 2)$ as p increases our shape bulges from a diamond to a circle. For $p \in (2, \infty]$ as p increases the plots bulge from a circle to the circumscribed square. As p increases from "0" to infinity our shape bulges out along the lines $y = x$ and $y = -x$.

These statement can be generalized to higher dimensions. For $p \geq 1$ we still have convex shapes, and $p = 1$ is a box inscribed in $S_2^n(1)$. Similarly $p = \infty$ is an n dimensional box that circumscribes $S_2^n(1)$. Furthermore we can sketch the process of p increasing. Though not technically defined, one can imagine (at least in the limit) starting at $p =$ "0" the shape is simply the unit axis from -1 to 1. As p increases our shape bulges along a basis of axis rotated $\pi/4$ from each of the original principle axis, (Analogous to $y = x$ and $y = -x$).

2.2. **Proof that the limit as p goes to infinity of $|x|_p$ is the sup norm.** The definition of the infinity norm is quite different from the other p-norms, here we show that it is a natural extension of the finite p

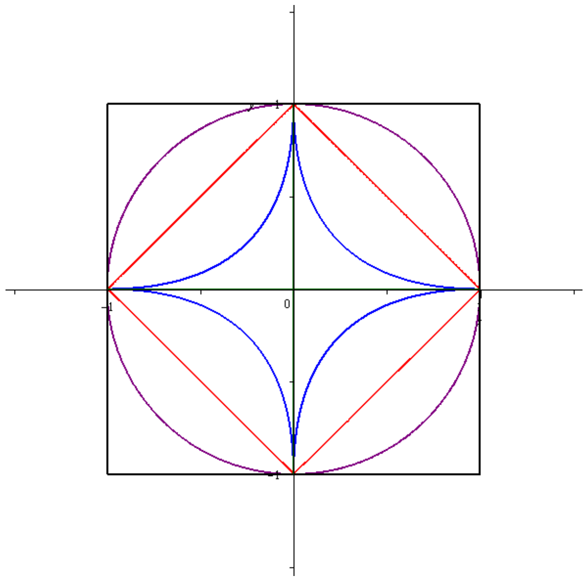


FIGURE 1. $S_p^2(1)$ for various values of p, $p =$ “0” in green, $p = 1/2$ is blue, $p = 1$ in red, $p = 2$ a circle in purple, and the infinity norm in black

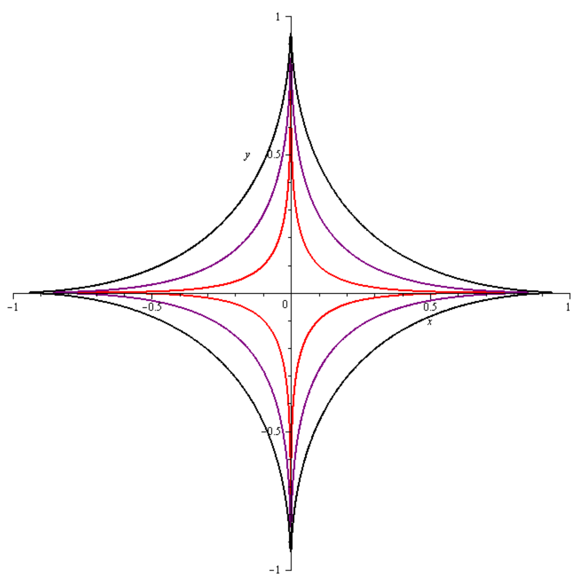


FIGURE 2. Here $p = .3$ is in red, $p = .4$ is in purple and $p = .5$ is in black

norms. we prove that the limit as p goes to infinity of the p-norm is equivalent to just choosing the component of $\mathbf{x}$ with the largest magnitude, or the supremum of $\{|x_i| \big| 1 \leq i \leq n\}$.

Theorem 2.1. *If $x \in \mathbb{R}^n$, then*

$$\lim_{p\to\infty} |x|_p = |x|_\infty = \sup\{|x_i| \big| 1 \leq i \leq n\}.$$

Proof. We use the proof found in [10]

For all $k = 1, 2, \ldots, n$, and $p \geq 1$, $|x_k| \leq \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} = \|x\|_p$, we have $\|x\|_\infty \leq \|x\|_p$. Thus, $\|x\|_\infty \leq \lim_{p\to\infty} \|x\|_p$

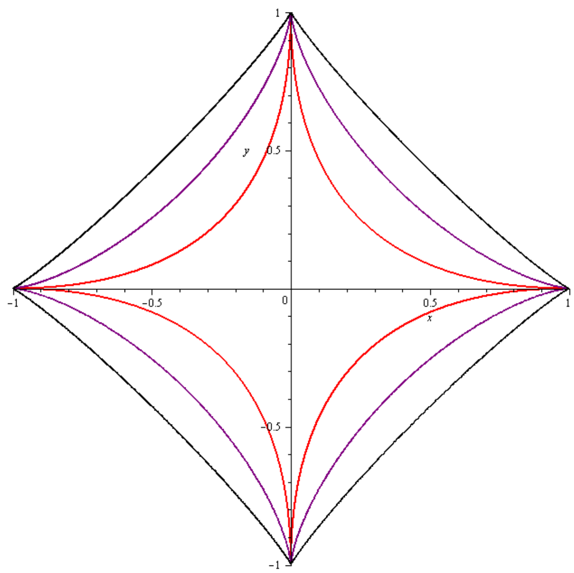


FIGURE 3. $p = .5$ in red, $p = .7$ in purple and $p = .9$ in black

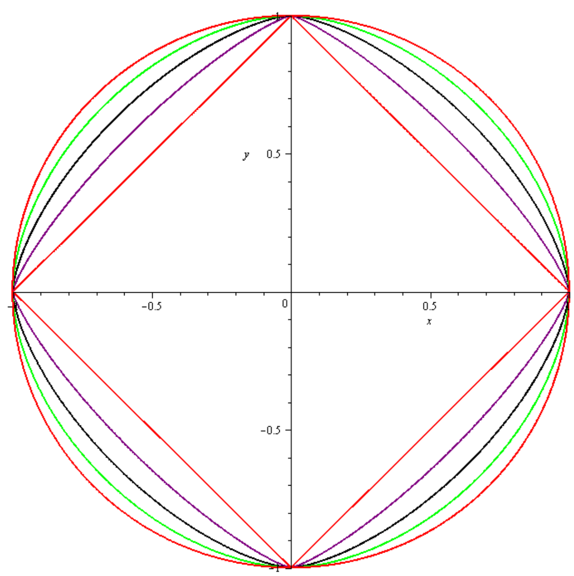


FIGURE 4. $p = 1$ in red, $p = 1.25$ in purple, $p = 1.5$ in black, $p = 1.75$ in green and $p = 2$ a circle in red

On the other hand, we know that

$$\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^{p-q} \cdot |x_j|^q\right)^{\frac{1}{p}} \leq \|x\|_\infty^{\frac{p-q}{p}} \cdot \left(\sum_{j=1}^{n} |x_j|^q\right)^{\frac{1}{p}} = \|x\|_\infty^{1-\frac{q}{p}} \cdot \|x\|_q^{\frac{q}{p}}$$

for all $q < p$ where we used $|x_j| \leq \|x\|_\infty$ for all integers $j = 1, 2, \ldots, n$ In the last equality we have rewritten the right sum as the q-norm raised to the $\frac{q}{p}$. The q cancels in the exponents of the sum leaving the $1/p$. Therefore,

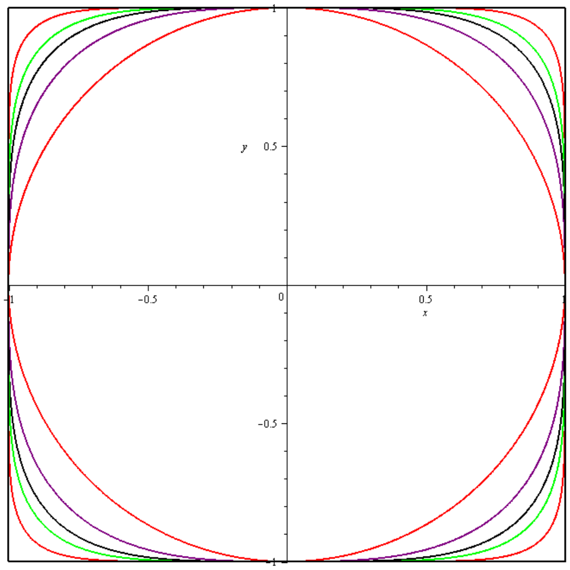


FIGURE 5. $p = 2$ in red, $p = 3$ in purple, $p = 4$ in black, $p = 5$ in green and $p = 8$ in red, and the infinity norm in black

$$\lim_{p\to\infty} \|x\|_p \leq \lim_{p\to\infty} \left(\|x\|_\infty^{1-\frac{q}{p}} \cdot \|x\|_q^{\frac{q}{p}} \right) = \|x\|_\infty \cdot 1$$

We conclude $\lim_{p\to\infty} \|x\|_p \leq \|x\|_\infty \leq \lim_{p\to\infty} \|x\|_p$ Then by the Squeeze Theorem for limits, $\lim_{p\to\infty} \|x\|_p$ exists and equals $\|x\|_\infty$. □

2.3. **An introduction to volume in n dimensions and measurable sets.** In this Section we use Lebesgue measure theory and associated theorems to extend a relatively easy and intuitive definition of volume to n dimensions. After defining the n dimensional sphere, we then show that a sphere has a measurable volume in any dimension. This Section is summarized from pages 141 to 163 of [3] .

Consider closed n-dimensional intervals of the form $I = \{\mathbf{x} : a_j \leq x_j \leq b_j, j = 1, ...n\}$ In one dimension this is simply an interval $[a, b]$, see FIGURE 6. In two dimensions this is a box or rectangle $[a_1, b_2] \times [a_2, b_2]$, see FIGURE 7. The "volume" of an interval is its length, and the "volume" of a box is its area. For $n = 3$,see FIGURE 8 this a rectangular prism, $[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, whose volume is:

$$V_{cube} = (b_1 - a_1)(b_2 - a_2)(b_3 - a_3).$$

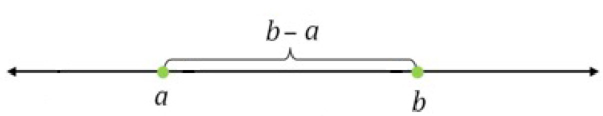


FIGURE 6. Volume of a 1 dimensional interval is its length $V = (b - a)$

In each case the volume is the product of the length of each of the 1 dimensional subintervals. Or formally:

$$V([a_1, b_1] \times \cdots [a_n, b_n]) = \Pi_{j=1}^{\infty}(b_j - a_j).$$

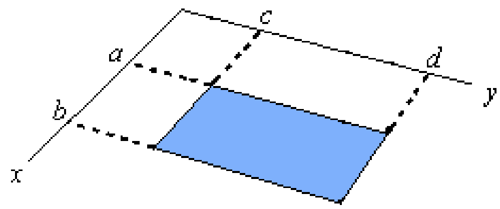


FIGURE 7. Volume of a 2 dimensional interval is an area, $V = (b - a)(d - c)$

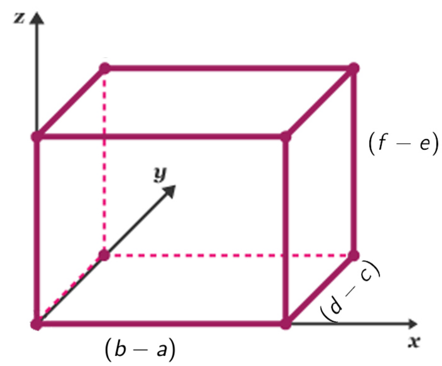


FIGURE 8. Volume of a 3 dimensional interval is traditional volume, $V = (b - a)(c - d)(f - e)$

Now suppose we are given an arbitrary subset E of $\mathbb{R}^n$. For any set E we can contain E in a countable union of intervals of $\mathbb{R}^n$. Hence cover E by a countable collection S intervals $\mathbf{I}_k$. Now we know the volume of intervals, hence let;

$$\sigma(S) = \sum_{\mathbf{I}_k \in S} v(\mathbf{I}_k)$$

Now $\sigma(S)$ is easy to understand. It is simply the sum of the volume of a bunch of intervals, each of whose volume is easy to understand and compute. But for any given S , $\sigma(S) \geq V(E)$, since the intervals in S cover E. But if we look at all possible sets S that cover E, then for E to be measurable it has to have a minimum cover. This coverS' minimizes $\sigma(S)$. We will define the Lebesgue outer measure to be volume, Hence $\sigma(S') = V(E)$.

Definition 2.4. *The Lebesgue outer measure of E, denoted $|E|_e$ is defined by*

$$|E|_e = \inf \sigma(S)$$

where the infimum (or greatest lower bound) is taken over every S that covers E.

In this paper we take volume and measure to be equivalent, hence we have a rigorous formulation of volume in n dimensions, but is not very useful for finding the volume of a given set, or a sphere. With a little effort in understanding the Lebesgue outer measure we may apply this theory to the volume of $S_p^n(r)$. First we need two definitions:

Definition 2.5. *The complement of a set E in $\mathbb{R}^n$, denoted by CE is:*

$$\mathbb{R}^n - E.$$

The complement of E is the set of all points in $\mathbb{R}^n$ that are not in E.

Definition 2.6. *A set E is open if for each point $\mathbf{x} \in E$, there exists a $\delta > 0$ such that a sphere centered on $\mathbf{x}$ of radius δ is contained in E. Or*

$$\{\mathbf{y} : |\mathbf{x} - \mathbf{y}| < \delta\} \in E$$

In other words a set is open if it does not contain any points that are precisely **on** its border. The familiar $[a, b]$ represents a closed set in $\mathbb{R}$. a and b are in this interval. Similarly (a, b) represents the open set, and a and b are not in the interval. Note that in our definition $(a, b]$ is is neither open nor closed.

Theorem 2.2. *A set E is closed if and only if CE is open.*

To return to a simple example, let $E = [a, b]$. Then $\mathrm{C}E = (-\infty, a) \cup (b, \infty)$. $\mathrm{C}E$ is open, and hence E is closed.

Theorem 2.3. *Every closed set is measurable.*

Recall that $S_p^n(r)$ is defined as all n dimensional points $|\mathbf{x}|_p \leq r$, hence $S_p^n(r)$ is closed. Therefore its is measurable and has a well-defined volume.

3. Thin boxes used to generate Upper Bound on Volume of Unit Sphere

Using only the above definition of the volume of n-dimensional boxes we prove that the volume of the unit sphere approaches zero as n approaches infinity. Perhaps this is surprising given that the volume of the circumscribed box (of edge length 2) goes to infinity as n approaches infinity. This is my own work and [4].

Theorem 3.1. *For $r = 1$,*

$$\lim_{n\to\infty} vol(S_2^n(r)) = 0.$$

Proof. Consider $n \geq 5$ and a point $\mathbf{x} = \{x_1, x_2, ...x_n\}$, with in the unit sphere. Now only 5 of the coordinates of the point $\mathbf{x}$ can have an absolute value greater than $\frac{1}{\sqrt{5}}$. If 6 or more did then:

$$|\mathbf{x}| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \geq \sqrt{6\left(\frac{1}{\sqrt{5}}\right)^2} > 1$$

Each point can have up to 5 coordinates greater then $\frac{1}{\sqrt{5}}$. There are $\binom{n}{5}$ ways to to pick these big coordinates. Up to some permutation of the Cartesian product, every point in the n dimensional sphere is contained in $\binom{n}{5}$ boxes of this form:

$$[-1, 1]^5 \times \left[-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]^{n-5}$$

To see this, suppose $\mathbf{y}$ is in the n dimensional unit sphere which will denote as S_2^n. Then $\mathbf{y}$ has at most 5 points with an absolute value greater then $\frac{1}{\sqrt{5}}$. Since there are $\binom{n}{5}$ boxes, by the pigeon hole principle, we can find a box that such that the big coordinates fall into $[-1, 1]$ interval. The other coordinates are all smaller than $\frac{1}{\sqrt{5}}$ and are contained in the $\left[-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]$.

The volume of a n dimensional unit sphere is bounded above by:

$$\begin{aligned}\operatorname{vol}(S_2^n(1)) &\leq \binom{n}{5}\left(\frac{2}{\sqrt{5}}\right)^{n-5} \\ &= \frac{n!}{5!(n-5)!}\left(\frac{2}{\sqrt{5}}\right)^{n-5} \\ &< n^5\left(\frac{2}{\sqrt{5}}\right)^{n-5} \\ &= \frac{n^5}{(\sqrt{5}/2)^{n-5}}.\end{aligned}$$

Since $\sqrt{5}/2 > 1$ the the last value goes to 0 as n goes to infinity.

□

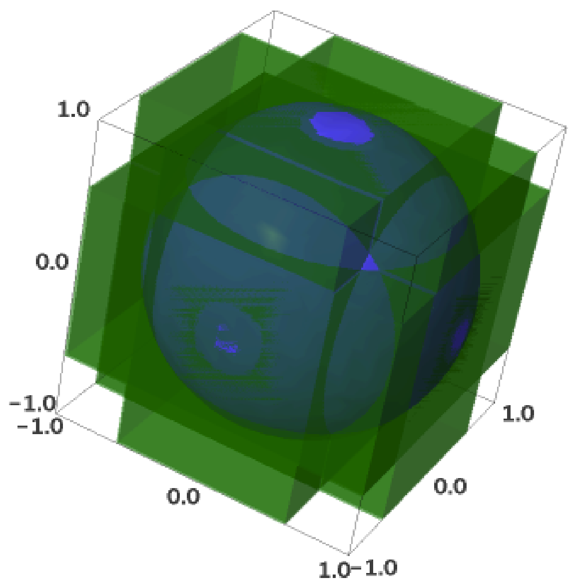


Figure 9. Three thin boxes cover a sphere for $n = 3$, this gives the general idea

4. Outline of the Gamma function and its properties

To get a formula for the volume of $S_2^n(r)$ we need to use the Gamma function. This Section defines the Gamma function and explores some its properties that we will need latter. [1]

Definition 4.1. *The gamma function is defined as follows:* [5]

$$\Gamma(t) \equiv \int_0^\infty x^{t-1}e^{-x}dx.$$

This definition is computationally challenging. Much of our use of the gamma function is limited to positive integer arguments. We will prove that for positive integers n, [2]

$$\Gamma(n) = (n-1)!.$$

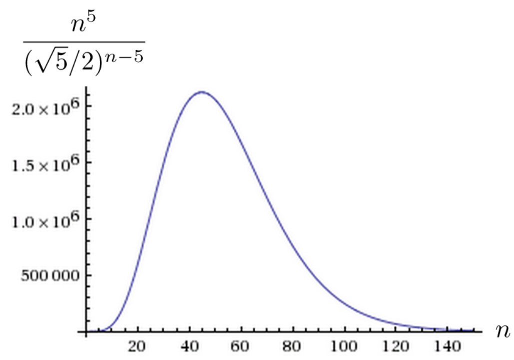


FIGURE 10. Upper bound volume of $S_n^2(1)$

This is just a factorial function shifted by 1. Γ is a "natural" extension of the factorial to all real numbers, except non-positive integers. It has applications in probability theory, mathematical physics, and in our case evaluating particular integrals.

4.1. **Properties of the gamma function.** Here we prove some properties of the gamma function that will be useful later.

Theorem 4.1. *The following are properties of* Γ. [2]*:*

(1) *For all* $t > 0$, $t\Gamma(t) = \Gamma(t+1)$.[2]
(2) *For all positive integers* n, $\Gamma(n) = (n-1)!$[2]
(3) $\Gamma(1/2) = \sqrt{\pi}$ [2]
(4) $\Gamma(t+1) \sim \sqrt{t}(\frac{t}{e})^x e$, $(t \to \infty)$

Proof. (1) Consider the left hand side of the identity:

$$t\Gamma(t) = t\int_0^\infty x^{t-1}e^{-x}dx$$

Then recall the integration by parts formula and let $dv = x^{t-1}$ and $u = e^{-x}$

$$\int_a^b udv = uv|_a^b - \int_a^b vdu$$

With $t > 0$, we have:

$$t\Gamma(t) = t\Big(e^{-x}\ \frac{x^t}{t}\Big|_0^\infty - \int_0^\infty \frac{x^t}{t}(-1)e^{-x}dx\Big)$$

The first term is zero, by L'hopital's rule, and the negative signs cancel in the second term. This leaves:

$$t\Big(\int_0^\infty \frac{x^t}{t}e^{-x}dx\Big)$$

Since t is a constant with respect to the integral

$$= \int_0^\infty x^t e^{-x}dx$$

$$= \Gamma(t+1)$$

(2) This will be a proof by induction. For the base case:

$$\Gamma(0) = \int_0^\infty x^0 e^{-x} dx$$

$$= \int_0^\infty e^{-x} dx = e^{-x}|_0^\infty = 1$$

Hence $\Gamma(0+1) = 1 = 0!$ this establishes our base case. Now suppose $k \in \mathbb{N}$, furthermore we assume the induction hypothesis that $\Gamma(k+1) = k!$, Now using the first result:

$$\Gamma(k+2) = (k+1)\Gamma(k+1) = (k+1) * (k!) = (k+1)!$$

(3)

$$\Gamma(1/2) = \int_0^\infty x^{-1/2} e^{-x} dx$$

We now make the substitution $u^2 = x$, and $dx = 2udu$

$$\Gamma(1/2) = \int_0^\infty \frac{1}{u} e^{-u^2} 2udu$$

$$= \int_0^\infty 2e^{-u^2} du$$

Since e^{-x^2} is an even function, and we know $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$, two times half the integral returns:

$$2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

(4) Since $\Gamma(x+1)$ is continuous for positive x, if $\Gamma(x+1)$ has a limit L as $(x \to \infty)$, then each sequence x_n that converges to ∞ , $\Gamma(x_n+1)$ converges to L. In particular we only have to consider $\Gamma(n)$ for $n \in \mathbb{N}$. We seek an approximation for $n!$. Since it works, we start with: [7]

$$\ln(n!) = \ln(1) + \ln(2) + \ln(3) + ... + \ln(n)$$

subtracting $\frac{1}{2}\ln(n)$ from both sides yields:

$$\ln(n!) - \frac{1}{2}\ln(n) = \ln(1) + \ln(2) + \ln(3) + ...\frac{1}{2}\ln(n)$$

The trapezoidal approximation for $\int_1^n \ln(x)dx$ with equal partitions each of length one, or $x_i = i, 0 \leq i \leq n$, equals:

$$= \frac{\ln(1)+\ln(2)}{2} + \frac{\ln(2)+\ln(3)}{2} + \frac{\ln(3)+\ln(4)}{2} + ... + \frac{\ln(n-1)+\ln(n)}{2}$$

$$= \ln(1) + \ln(2) + \ln(3) + \cdots + \frac{1}{2}\ln(n)$$

Combing this expression with our approximation for $\int_0^n \ln(x)dx$ yields

$$\ln(n!) - \frac{\ln(n)}{2} \approx \int_1^n \ln(x)dx = (x(\ln(x)-1))|_1^n$$

Plugging in the end points and using some properties of the natural logarithm yields:

$$\ln\left(\frac{n!}{\sqrt{n}}\right) \approx n\ln(n) - n + 1.$$

Now exponentiating both sides of this approximate equality:

$$\frac{n!}{\sqrt{n}} \approx e^{1-n}n^n.$$

In the positive limit of a continuous function we have:

$$\Gamma(x+1) = \Gamma(n+1) = n! \approx \left(\frac{n}{e}\right)^n e\sqrt{n}.$$

□

Definition 4.2. *The Beta function, $\beta(x, y)$ is defined as follows:* [2]

$$\beta(x,y) = \int_0^\infty t^{x-1}(1-t)^{y-1}dt.$$

Theorem 4.2. *The Γ and β functions satisfy the relationship* [6]

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \beta(x,y).$$

Proof. Notice that

$$\Gamma(x)\Gamma(y) = \int_0^\infty e^{-u}u^{x-1}du \int_0^\infty e^{-v}v^{y-1}dv.$$

Now by Fubini's Theorem, we rewrite this expression as a double integral in u and v.

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty e^{-u-v}u^{x-1}v^{y-1}dudv.$$

Use a change of variables, $u = zt$ and $v = z(t-1)$. When $v = 0$ then $z = 0$, and when $v = \infty$ then $z = \infty$. When $u = 0$ then $t = 0$. When $u = \infty$ then $t = 1$, since z ranges to ∞. We have:

$$\Gamma(x)\Gamma(y) = \int_{z=0}^\infty \int_{t=0}^1 e^{-z}(zt)^{x-1}((z(t-1))^{y-1}zdzdt.$$

We use Fubini's Theorem in the other direction to separate the integrals.

$$\Gamma(x)\Gamma(y) = \int_{z=0}^\infty e^{-z}z^{x+y-1}dz \int_{t=0}^1 t^{x-1}(1-t)^{y-1}dt.$$

By definition, the first term is $\Gamma(x+y)$ and the second term is $\beta(x,y)$.

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \Gamma(x+y)\beta(x,y)\\ \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} &= \beta(x,y).\end{aligned}$$

□

5. Explicit formula for the volume of $S_n^2(r)$ dimensions

Now that we have a basic grasp on the Γ, we will use some of its properties and induction to derive a closed form expression for the volume of $S_n^2(r)$. This Section is a combination my own simple mathematics and [2].

5.1. **Induction hypothesis.** We present two inductive proofs. Not surprisingly they both rely on the same induction hypothesis. Suppose for $1, 2, 3...n$ that the volume of $S_n^2(r)$ is:

$$V_n(r) = \frac{r^n \pi^{(n/2)}}{\Gamma\Big(\frac{n}{2}+1\Big)}.$$

Since it this is a rather abstract formula, we evaluate 2 base cases.

- For $n = 1$ we have:

$$V_1(r) = \frac{r^1 \pi^{(1/2)}}{\Gamma\Big(\frac{1}{2}+1\Big)}$$
$$V_1(r) = \frac{r\sqrt{\pi}}{\Gamma\Big(\frac{1}{2}+1\Big)}.$$

Now we need to evaluate $\Gamma\Big(\frac{3}{2}\Big)$. Using Corollary 1.1 and Theorem 1.2 we have:

$$x\Gamma(x) = \Gamma(x+1) \qquad (x > 0).$$

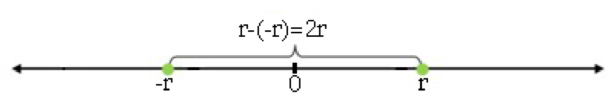
Now we let $x = 1/2$, and use the property that $\Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}$

$$\frac{1}{2}\Gamma\Big(\frac{1}{2}\Big) = \frac{1}{2}\sqrt{\pi} = \Gamma\Big(\frac{3}{2}\Big).$$

Substituting this into our original equation

$$V_1 = 2r.$$

In 1 dimension, there is only one degree of freedom. Volume is just length. Hence $V_1 = 2r$, says the length of all the points that are with in r of another point is $2r$. In other words the length of the interval $[r, r]$ is $2r$. It also has the correct units, of $length^n$.



- For $n = 2$ things will start to make sense. Our (as of yet unproven) formula yields:

$$V_2(r) = \frac{r^2\pi}{\Gamma(2)}$$
$$V_2(r) = r^2\pi \qquad \text{(since } \Gamma(2) = 1)$$

Which is the familiar formula for the area of a circle of radius r.

Theorem 5.1. *For $p = 2$, $r > 0$ and $n = 1, 2, \ldots$ volume of a sphere of radius r in n dimensions is:*

$$V_n(r) = vol(S_2^n(r)) = \frac{r^n \pi^{(n/2)}}{\Gamma\Big(\frac{n}{2}+1\Big)}$$

Proof. We have already stated our induction hypothesis and established several base cases. Now we view $\mathbb{R}^n$ as $\mathbb{R}^{n-2} \times \mathbb{R}^2$. By definition a point $(x_1, x_2, x_3 ... x_n)$ is in $S_n^2(r)$ if and only if $x_1^2, x_2^2, x_3^2 ... x_n^2 \le r^2$ Or rearranging for utility:

$$x_1^2 + x_2^2 + x_3^2 + \cdots + x_{n-2}^2 \le r^2 - x_{n-1}^2 - x_n^2$$

Converting to integrals:

$$V_n(r) = \int_{S_2^n(r)} dx_1 dx_2 \cdots dx_n$$

$$V_n(r) = \int_{S_2^2(r)} \left(\int_{S_2^{(n-2)}\left(\sqrt{r^2 - x_n^2 - x_{n-1}^2}\right)} dx_1 ... dx_{n-2} \right) dx_{n-1} dx_n$$

The interior integral we can evaluate with our induction hypothesis. By Fubini's Theorem we can do the "double" integral in either order. We do the $(n-2)$ integral by induction, and pull out constants. Our r for the $(n-2)$ integral is $\sqrt{r^2 - x_{n-1}^2 - x_{n-2}^2}$ and is not constant with respect to r and cannot be pulled out of the $S_2^2(r)$ integral. This leaves:

$$V_n(r) = \frac{\pi^{(n-2)/2}}{\Gamma\left(\frac{n-2}{2} + 1\right)} \int_{S^2(r)} \left(r^2 - x_{n-1}^2 - x_n^2\right)^{(n-2)/2} dx_n dx_{n-1}.$$

Now we change to polar coordinates, but since r is taken, we use t as our radius measurement in polar. Thus $-x_{n-1}^2 - x_n^2 = -t^2$ and $dx_n dx_{n-1} = t dt d\theta$. Simplifying our Γ argument yields:

$$\frac{\pi^{(n-2)/2}}{\Gamma(n/2)} \int_0^{2\pi} d\theta \int_0^r \left(r^2 - t^2\right)^{(n-2)/2} t dt.$$

The first integral is 2π. The second integral is done by a u substitution. We wish to evaluate

$$\frac{\pi^{(n-2)/2}}{\Gamma(n/2)} 2\pi \int_0^r \left(r^2 - t^2\right)^{(n-2)/2} t dt$$

Let $u = r^2 - t^2$. Now $du = -2tdt$, or $dt = -\frac{1}{2}du$. Similarly when $t = 0$, $u = r^2$. For our other end point, when $t = r$, $u = 0$. Now our integral is:

$$\begin{aligned}
V_n(r) &= \frac{\pi^{(n-2)/2}}{\Gamma(n/2)} - \pi \int_{r^2}^0 u^{(n-2)/2} du \\
&= -\frac{\pi^{(n/2)}}{\Gamma(n/2)} \frac{2}{n} (u^{n/2}|_{r^2}^0) \\
&= \frac{\pi^{(n/2)}}{\Gamma(n/2)} \frac{2}{n} (r^n) \\
&= r^n \pi^{(n/2)} \frac{1}{\Gamma(n/2) \frac{n}{2}}
\end{aligned}$$

Now by Theorem 1.1

$$\frac{1}{\Gamma(n/2)\frac{n}{2}} = \frac{1}{\Gamma(\frac{n}{2} + 1)}.$$

Substituting into the equation above yields:

$$V_n(r) = r^n \pi^{(n/2)} \frac{1}{\Gamma(\frac{n}{2} + 1)} = \frac{r^n \pi^{(n/2)}}{\Gamma\left(\frac{n}{2} + 1\right)}.$$

□

Now we prove the same result slightly differently.

Proof. Using the same hypothesis and base cases, but now we regard $\mathbb{R}^n$ as $\mathbb{R}^{n-1} \times \mathbb{R}$.

$$V_n(r) = \int_{S^1(r)} \left(\int_{S_2^{(n-1)}\left(\sqrt{r^2 - x_n^2}\right)} dx_1 ... dx_{n-1} \right) dx_n$$

Again we can do the $(n-1)$ integral by induction.

$$V_n(r) = \frac{\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2} + 1\right)} \int_{-r}^{r} \left(r^2 - x_n^2\right)^{(n-1)/2} dx_n$$

Since our integrand is even, we can multiply the integral by two, and take the positive half of the interval.

$$V_n(r) = \frac{\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2} + 1\right)} 2 \int_{0}^{r} \left(r^2 - x_n^2\right)^{(n-1)/2} dx_n$$

Now make the substitution $x_n = r\sqrt{t}$, and $dx_n = \frac{1}{2} r t^{-\frac{1}{2}} dt$. We must also adjust our limits. When $t = 0$, $x_n = 0$ and when $t = 1$, $x_n = r$. Our expression now reads:

$$V_n(r) = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2} + 1\right)} \int_{0}^{1} \left(r^2 - tr^2\right)^{(n-1)/2} t^{-\frac{1}{2}} \frac{1}{2} r dt.$$

Now we can factor an r^2 from the term in parenthesis, and pull our an r^n. We also cancel the 2 with the $\frac{1}{2}$.

$$V_n(r) = \frac{r^n \pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2} + 1\right)} \int_{0}^{1} t^{-\frac{1}{2}} \left(1 - t\right)^{(n-1)/2} dt$$

Now recall the definition of the Beta (β) function:

$$\beta(x, y) = \int_{0}^{1} t^{x-1} (1 - t)^{y-1} dt$$

We see that

$$V_n(r) = \frac{r^n \pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \beta\left(\frac{1}{2}, \frac{n+1}{2}\right).$$

Now Theorem 4.2 (a property of the β function)

$$V_n(r) = \frac{r^n \pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \beta\left(\frac{1}{2}, \frac{n+1}{2}\right) = \frac{r^n \pi^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \frac{\Gamma\left(\frac{n+1}{2}\right) \Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2} + 1)}$$

canceling the $\Gamma(\frac{n}{2} + 1)$ and remembering that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ We have:

$$\frac{r^n \pi^{(n-1)/2} \sqrt{\pi}}{\Gamma(\frac{n}{2} + 1)}$$

$$\frac{r^n \pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}.$$

□

There is another more direct brute force proof for this result. It is much more difficult and involves a Jacobian transformation with n terms and generalized spherical coordinates. It can be found in [2]

6. The Volume of $S_p^n(r)$ for different p-norms

As we discussed in Section 2 The traditional unit sphere is defined by a set points less than 1 unit-p-distance from the origin. This definition is dependent on a distance formula/definition. All the points $|\mathbf{x}|_p \leq 1$ are in the unit sphere. In this Section we consider the volume of $S_n^p(r)$ for different values of p. For $n > 3$ its unclear that an intuitive distance measurement exists. As a perhaps motivating aside, distance in special relativity, 3 spatial dimensions and one time dimension looks like Euclidean distance with an added term, it is $\sqrt{x_1^2 + x_2^2 + x_3^2 + (tc)^2}$. Let

$$D_p(\mathbf{x}) = \left(\sum_{i=1}^{n} x_i^p \right)^{1/p}$$

be the distance from $\mathbf{x} = (x_1, x_2, \cdots x_n)$ to the origin. Note that for $p = 2$ we recover the traditional distance formula, and therefore the traditional unit sphere.

In [8] the authors show that the volume of n dimensional sphere for a given value of p is:

$$V_{B_p^n} = \frac{\left(2\Gamma\left(\frac{1}{p}+1\right)\right)^n}{\Gamma\left(\frac{n}{p}+1\right)}$$

This formula leads to many different explorations. Is there a value of p or a function $p(n)$ that makes the volume of the sphere as n approaches infinity be a nonzero constant?
Suppose that p is constant. Recall that the gamma function is continuous and strictly increasing on $[2, \infty)$ and that $\Gamma(n-1) = n!$. Then:

$$\lim_{n\to\infty} \left(V_{B_p^n} = \frac{\left(2\Gamma\left(\frac{1}{p}+1\right)\right)^n}{\Gamma\left(\frac{n}{p}+1\right)} \right).$$

Now the numerator is just a number, call it k raised to the n. Note that $0 \leq k \leq 2$. This limit must be non-negative. For values of $n \geq j \cdot p$ were j is an integer we have:

$$\lim_{n\to\infty} \left(V_{B_p^n} \leq \frac{k^n}{j!} \right)$$

Despite the multiplicative shift, the factorial function grows faster then the 2^n. Hence this limit is zero for any fixed value of p.

6.1. **What if p is a function of n.**

Perhaps we utterly left the land of intuition a long time ago, but we may be near its border. It makes some sense for how distance is defined to vary with how many of degrees of freedom we have in our space. For example if the number of roads between Fargo and Sarasota increases (ie the degrees of freedom for getting between them) then the travel time (distance) would change. As n increases $D_p(\mathbf{x}_{Fargo} - \mathbf{y}_{Sarasota})$ would decrease. If that were the case then p would be a function of n.

For another example if $n = 1$, we have one degree of freedom and may be forced to travel along some train tracks that don"t go directly from point A to B, then if $n = 2$ we can get off the train tracks and cut a shorter distance between the end points.

Now suppose $n = 2$, and $p = 1$, otherwise known as the taxi cab norm. These are the n and p values of walking around a city on a grid. But if we change n to 3, and allow our pedestrian to fly (i.e. use the 3$^{\text{rd}}$ dimension or the z axis) then p will change to 2. (assuming the buildings are negligibly tall). Hence thinking about p as a function of arguably n makes sense.

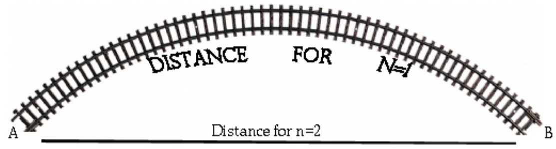


FIGURE 11. For $n = 1$ the distance is the curve, for $n = 2$ its the shorter straight line

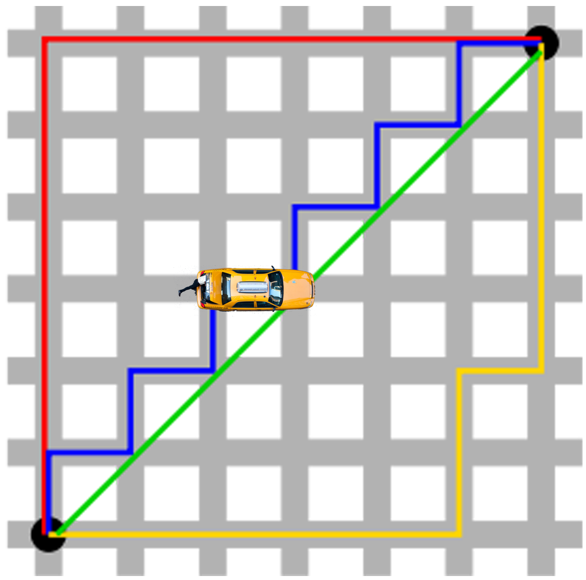


FIGURE 12. taxi cab norm is the distance on a grid, p changes if the taxi can fly

As an example suppose $p(n) = n$. Then:

$$V_{B_p^n} = \frac{\left(2\Gamma\left(\frac{1}{p}+1\right)\right)^n}{\Gamma\left(\frac{n}{p}+1\right)}$$

$$V_{B_p^n} = \frac{\left(2\Gamma\left(\frac{1}{n}+1\right)\right)^n}{\Gamma(2)}$$

$$V_{B_p^n} = \left[2\Gamma\left(\frac{1}{n}+1\right)\right]^n$$

Since $\Gamma(x) \geq .5$, for $x \in (1, 2)$ the limit as n goes to infinity of the above expression is infinity. Hence we need a more complex or subtle function $p(n)$.

6.2. **Finding $p(n)$ such that the limit of the volume of a n dimensional sphere as n goes to infinity is a nonzero constant.** We now seek a function $p(n)$ such that

$$\lim_{n\to\infty} V_{B_p^n} = c$$

It is shown below that $p(n) = \frac{n}{\Gamma^{-1}(c2^n)-1}$ makes the limit c for any constant c.

One easy way to do this is to set,

$$c\left(2\Gamma\left(\frac{1}{p}+1\right)\right)^n = \Gamma\left(\frac{n}{p}+1\right)$$

remembering that p is a function of n. For the moment assume that (or make the ansatz) $\lim_{n\to\infty} p(n) = \infty$. Since $\Gamma(1) = 1$ the left hand side is just $c2^n$. Then:

$$c2^n = \Gamma\left(\frac{n}{p} + 1\right)$$

$$\Gamma^{-1}(c2^n) = \frac{n}{p} + 1$$

$$\Gamma^{-1}(c2^n) - 1 = \frac{n}{p}$$

$$p\Gamma^{-1}(c2^n) - p = n$$

$$p(n) = \frac{n}{\Gamma^{-1}(c2^n) - 1}$$

Now we must check that our ansatz holds. That is does

$$\lim_{n\to\infty}\left(p(n) = \frac{n}{\Gamma^{-1}(c2^n) - 1}\right) = \infty?$$

The above equation is valid since $\Gamma(x)$ grows faster then 2^x, hence Γ^{-1} will grow very slowly. (Think $\ln(x)$ and e^x).

6.3. **Some comments on well definition of $p(n)$.** We are concerned primarily with where is Γ invertible. The Inverse Function Theorem says a strictly increasing function is invertible. [9] Since $\Gamma(x)$ is strictly increasing on $[2, \infty)$ as long as $c \neq 0$, $p(n)$ is eventually well defined. Since we wanted to avoid the limiting volume being 0 this is not a constraint, as $c = 0$ for any fixed value of p.

We must also require that $\Gamma^{-1}(c2^n) - 1 \neq 0$, which happens only when $c2^n = 1 or 2$. Hence for large enough values of n our distance metric will be well defined.

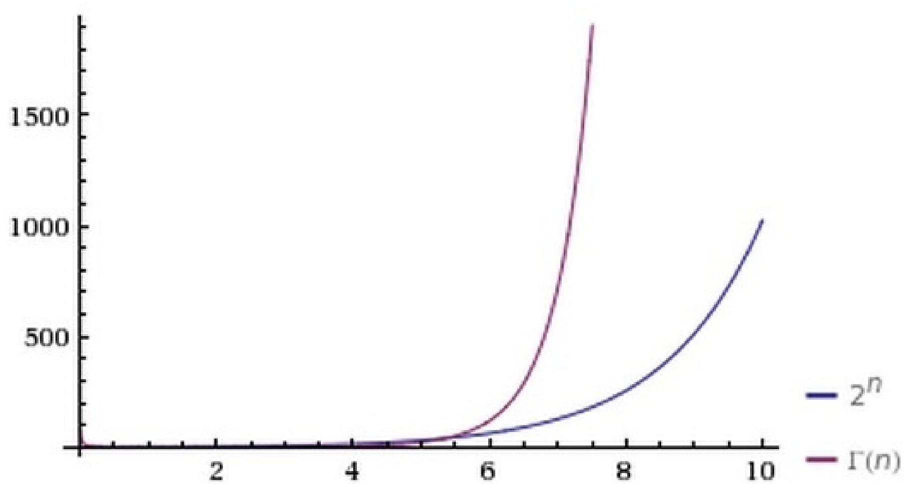


FIGURE 13. $\Gamma(n)$ vs. 2^n

7. Surface area of $S_2^n(r)$, denoted $\partial S_2^n(r)$

In this Section we present another proof of the volume formula for an n-dimensional sphere that leads naturally to a surface area formula for the n-sphere. What follows is from [11]

Definition 7.1. *The surface area of $S_2^n(r)$ is*

$$\partial S_2^n(r) = \{x \in \mathbb{R}^n \Big| |\mathbf{x}| = r\}.$$

Theorem 7.1. *The volume and surface area of $S_2^n(r)$ are given by:*

$$\partial S_2^{n-1}(r) = nC_n r^{n-1} = \frac{nr^{n-1}\pi^{n/2}}{\Gamma\left(1+\frac{n}{2}\right)}$$

$$V_n(r) = C_n r^n = \frac{r^n \pi^{n/2}}{\Gamma\left(1+\frac{n}{2}\right)}$$

Proof. To calculate the volume of the n dimensional sphere we must evaluate the following integral:

$$V_n(r) = \int \cdots \int_{x_1^2+x_2^2+\cdots x_n^2 \leq r^2} dx_1 dx_2 \cdots dx_n$$

Were $dx_1 dx_2 \cdots dx_n = dV$ the infinitesimal volume element for n dimensions. We claim that:

$$V_n(r) = \int \cdots \int_{x_1^2+x_2^2+\cdots x_n^2 \leq r} dx_1 dx_2 \cdots dx_n = C_n r^n$$

For some function C of n. The r^n results from dimensional analysis. The answer must be factorisable into a number, C_n times r^n, otherwise it would not have the correct units for volume in n dimensions. We must still calculate C_n.

The surface area of an n dimensional sphere is the set of points that correspond to $x_1^2 + x_2^2 + \cdots x_n^2 = r^2$. For a given sphere, we can construct it by adding infinitesimally thin shells of this form. If S_n is the surface area of a n dimensional sphere, then in integral form this reads:

$$V_n(r) = \int_0^r S_{n-1}(r')dr'.$$

Assuming for the moment that $S_n(r)$ is continuous for $r \in (0, \infty)$, then by the Fundamental Theorem of Calculus we have:

$$S_{n-1}(r) = \frac{dV_n(r)}{dr} = nC_n r^{n-1}$$

We we have used the fact the C_n is independent of r in the last equality. Now if we can calculate C_n we get a volume area formula and a surface area formula. We now have two expressions for $V_n(r)$, setting them equal yields:

$$\int \cdots \int_{x_1^2+x_2^2+\cdots x_n^2 \leq r} dx_1 dx_2 \cdots dx_n = \int_0^r S_{n-1}(r')dr'. = nC_n \int_0^r (r')^{n-1} dr'$$

To make things easier we want to write $dx_1 dx_2 \cdots dx_n$ in generalized spherical coordinates. Since we won't need an explicit formula for the spherical differentials we will bury then all in $d\Omega$. Equating these two Jacobian's yields:

$$dx_1 dx_2 \cdots dx_n = r^{n-1} dr d\Omega_{n-1}$$

Where $d\Omega$ contains the necessary trig functions of θ_1 through θ_{n-1}. Now if we let $r = 1$ and combine the two previous equations we see that:

$$\int \cdots \int d\Omega_{n-1} = nC_n.$$

Were the integral is taken over all (angular) space. This integral is still difficult. In typical mathematical fashion we do something seemly unrelated that solves the problem (a *hidenblau*). Consider the function defined as:

$$f(x_1, x_2, \ldots x_n) = e^{-(x_1^2+x_2^2+\cdots x_n^2)} = e^{-r^2}.$$

We now integrate this function over all space in both rectangular and spherical coordinates.

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(x_1^2+x_2^2+\cdots x_n^2)} = \int_0^{\infty} r^{n-1} dr \int d\Omega_{n-1} e^{-r^2}$$

The $d\Omega$ terms are independent of r. They are a bunch of angular terms integrated from 0 to π or from 0 to $\pi/2$. Hence we can do that portion of the integral. Recall $\int \cdots \int d\Omega_{n-1} = nC_n$. Hence:

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(x_1^2+x_2^2+\cdots x_n^2)} = nC_n \int_0^{\infty} r^{n-1} e^{-r^2} dr.$$

Then applying Fubbini's Theorem to the left hand side to separate the integral leaves:

$$\int_{-\infty}^{\infty} e^{-x_1^2} dx_1 \int_{-\infty}^{\infty} e^{-x_2^2} dx_2 \cdots \int_{-\infty}^{\infty} e^{-x_n^2} dx_n = nC_n \int_0^{\infty} r^{n-1} e^{-r^2} dr.$$

We can do all of these integrals.

$$\int_{-\infty}^{\infty} e^{-x_1^2} dx_1 = \sqrt{\pi}$$

$$\int_0^{\infty} r^{n-1} e^{-r^2} d = \frac{1}{2}\Gamma\left(\frac{n}{2}\right)$$

Plugging these results in yields:

$$\pi^{n/2} = C_n \frac{n}{2} \Gamma\left(\frac{n}{2}\right) = C_n \Gamma\left(1 + \frac{n}{2}\right).$$

In the last step we used the recursive property of Γ . Solving for C_n we obtain:

$$C_n = \frac{\pi^{n/2}}{\Gamma\left(1 + \frac{n}{2}\right)}.$$

Returning to our original definition of C_n we see that:

$$S_{n-1}(r) = nC_n r^{n-1} = \frac{n r^{n-1} \pi^{n/2}}{\Gamma\left(1 + \frac{n}{2}\right)}$$

$$V_n(r) = C_n r^n = \frac{r^n \pi^{n/2}}{\Gamma\left(1 + \frac{n}{2}\right)}.$$

□

Personally I find the result from the Fundamental Theorem of Calculus more interesting, or in other words the fact that:

$$S_{n-1}(r) = \left(\frac{n}{r}\right) \cdot V_n(r).$$

8. Volume of n dimensional Ellipses

We have computed the volume of $S_2^n(r)$. With a simple linear transformation of $\mathbb{R}^n$ and Pick's Theorem we can compute the volume of a n dimensional ellipse. The material here is taken from: [12] and [13].

Theorem 8.1. *Suppose $c_1, c_2, ..., c_n$ are positive constants. Consider the n dimensional ellipsoid, E_n given by*

$$E_n = \{(x_1, ..., x_n) | \sum_{k=1}^{n} \frac{x_k^2}{c_k^2} \leq 1\}.$$

Its n dimensional volume is

$$\frac{\pi^{n/2}}{\Gamma(n/2+1)} \prod_{k=1}^{n} c_k.$$

Proof. The less than 1 part is merely a convenience. By adjusting the constants c_k, we can make our ellipse as big or small as we want. To get an ellipse from the unit sphere all we have to do is a simple linear transformation of our coordinates. The n^{th} coordinate is scaled by c_n. This can produce any ellipse that has major axis aligned with the coordinate axis. Since a rotation leaves the volume unchanged, this will suffice for our purposes.

To make this transformation we must multiply a given vector point by a diagonal matrix, with diagonal entries $c_1, c_2, ..., c_n$. The determinant of a diagonal matrix is the product of its diagonal entries. From linear algebra we know that a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ multiplies the of volume an object by the determinant of the matrix corresponding to said linear transformation. Hence the volume of the n dimensional ellipsoid is the volume of unit sphere times the determinant of the diagonal matrix with entries $c_1, c_2, ..., c_n$.

If we multiply each element of the unit sphere by this matrix:

$$M = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & c_n \end{pmatrix}$$

We get the the n dimensional ellipsoid given by

$$\{(x_1, ..., x_n) | \sum_{k=1}^{n} \frac{x_k^2}{c_k^2} \leq 1\}.$$

Equivalently if a point $\mathbf{x}$ is in the unit sphere then the point $M\mathbf{x}$ is in the ellipsoid as formulated above.

$$E_n = \hat{T} \cdot S_n$$
$$V(E_n) = V(T\dot{S}_n)$$
$$V(E_n) = \det(M)V(S_n)$$
$$V(E_n) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} \prod_{k=1}^{n} c_k.$$

□

What follows are two simple examples. Suppose $\mathbf{x} = (1, 0, 0, \cdots, 0)$. Then $M\mathbf{x} = (c_1, 0, 0 \cdots, 0)$. Our original point $\mathbf{x}$ was on the surface of the sphere, and the final point $M\mathbf{x}$ is on the surface of the ellipse.

Suppose $\mathbf{y} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \cdots, 0)$, which is on the surface of a unit sphere in n dimensions. Now $M\mathbf{y} = (\frac{c_1}{2}, \frac{c_2}{2}, \frac{c_3}{2}, \frac{c_4}{2}, 0 \cdots, 0)$, which is also on the surface of the ellipsoid.

9. surface Area of n dimensional Ellipses

The surface area of an ellipse is a solved only via an infinite sum. In this Section we explain why the Fundamental Theorem of Calculus method and linear transformation method do not neatly solve the problem.

9.1. **Fundamental Theorem of calculus, surface area of n dimensional ellipses.** If we repeat the same calculations as before 7 but with an ellipse we get an answer, but it is wrong. It is wrong because the integral argument relied on the fact that the distance between successive spherical shells that make a sphere was the same everywhere on the sphere. This is not true for an ellipse.

9.2. **Linear transformation to generate surface Area of n dimensional Ellipses.** Surface area does not behave nicely under linear transformations. As an aside / example we deal with a cube.

We have a simple question with out a simple answer; if we have a shape whose surface area we know, and we apply a simple linear transformation of its vector space, what happens to the surface area of the the resulting shape? Namely if we have a sphere of radius 1, and we apply the matrix:

$$M = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ \cdots & 0 & 0 & c_n \end{pmatrix}$$

to our sphere, what is the resulting surface area? Since we cannot answer this question, we will answer an easier question. If we have a cube of side length 1 in the first quadrant that looks like FIGURE 14.
Now if we transform this sphere by a matrix

$$T = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$$

to this cube we get a rectangular based prism. Since T is diagonal each of our eigenvectors is just a unit vector. Hence our shape still has all right angles, just not all equal sides. Now the point $(1, 1, 0)$ is mapped to the point $(c_1, c_2, 0)$ Hence the surface area of the base is now c_1c_2. There is nothing special about the base of the cube, in general the new surface area is:

$$2(c_1c_2 + c_1c_3 + c_2c_3).$$

This form suggests that the surface area formula for a ellipsoid might look something like:

$$\sum_{j=1}^{n} \left(\prod_{i=1, i\neq j}^{n} c_i \right) * C,$$

were C is just an over all constant, similar to the 2 we got out for a cube.

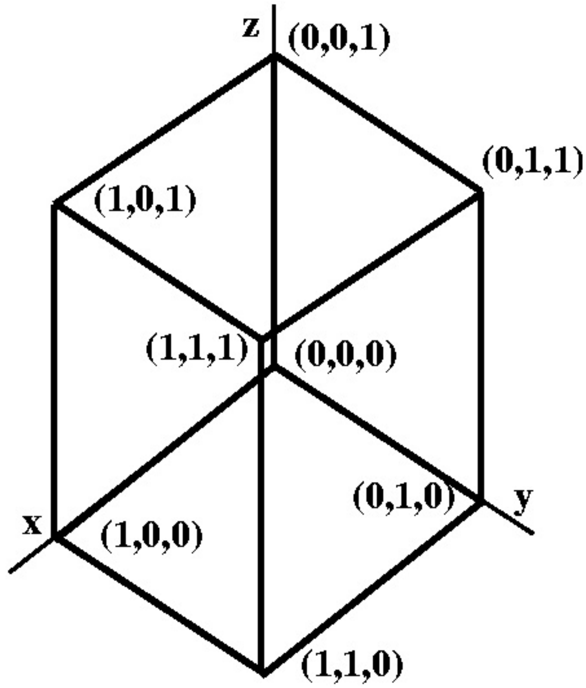


FIGURE 14. 3 dimensional cube of side length 1 [14]

10. Conclusion

There are still a few remaining questions. Could a linear transformation method lead to a simpler method or result for the surface area an ellipse? On another note, geodesics are the shortest path on a manifold (or surface), is there an interesting relationship between the geodesics of a $S_3^p(1)$ and p. In other words, we know that the geodesics of a normal sphere are great circles, but can the altering of distance calculation be built into the geodesic calculation? Something like a the geodesics of a $S_3^p(1)$ is a $S_2^p(1)$ path, which would be extremely elegant, but I don't know if its true. More generally can we find a function of $f(p)$ such that the geodesics of $S_3^p(1)$ are $S_2^{f(p)}(1)$? Or is there a nice relationship between p and the geodesics of $S_3^p(1)$?

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