INVENTORY THEORY

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ABSTRACT. This paper is an introduction to the study of inventory theory. The paper illustrates deterministic and stochastic models. We present the derivation of each model, and we illustrate each model through the use of examples. We also learn about quantity discounts, and use the aforementioned models to understand a real world situation involving firecrackers. Finally, some of the economic practices of Zappone Manufacturing are analyzed. It is shown how deterministic, stochastic and other simple models are not much help to this company. Also included in this paper is a derivation of Leibniz's Rule, which helps in deriving the stochastic model. This paper assumes the reader to have a basic understanding of mathematical statistics.

1. INTRODUCTION

Keeping an inventory (stock of goods) for future sale or use is common in business. In order to meet demand on time, companies must keep on hand a stock of goods that is awaiting sale. The purpose of inventory theory is to determine rules that management can use to minimize the costs associated with maintaining inventory and meeting customer demand. Inventory is studied in order to help companies save large amounts of money. Inventory models answer the questions: (1) When should an order be placed for a product? (2) How large should each order be? The answers to these questions is collectively called an inventory policy. Companies save money by formulating mathematical models describing the inventory system and then proceeding to derive an optimal inventory policy. This paper is an introduction to the study of inventory theory. We consider two models: deterministic continuous review models and stochastic models. First we learn that each model has a couple of variations to it. In addition, we learn how to derive the models, and use the models in examples. Next, we discuss quantity discounts and how these discounts affect the model. Then, we use the models to tackle a conceivable real world situation. Finally, we look at a company and see if we can use any of our newfound knowledge to help this company with its inventory policy. Also included in this paper is a derivation and example of Leibniz's Rule, which helps in the derivation of one of our models, and in section ten, there is a table of frequently used notation. Our information is from Frederick S. Hillier and Gerald J. Lieberman's textbook, *Introduction to Operations Research* [1]. This paper assumes the reader to have a basic understanding of elementary statistics. Some frequent terms used in this paper are: probability distribution, expected value, cumulative distribution function, and a uniform distribution. A good review for this is Richard J. Larsen and Morris L. Marx's *An Introduction to Mathematical Statistics and Its Applications* [2].

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2. Basic Terms that Describe Inventory Models

We begin by discussing in detail some important concepts used to describe inventory models. There are six components that determine profitability. These are:

(1) The costs of ordering or manufacturing the product
(2) Holding costs. This includes the cost of storage space, insurance, protection, taxes, etc.
(3) Shortage costs. This cost includes delayed revenue, storage space, record keeping, etc.
(4) Revenues. These costs may or may not be included in the model. If the loss of revenue is neglected in the model, it must be included in shortage cost when the sale is lost.
(5) Salvage costs. The cost associated with selling an item at a discounted price.
(6) Discount rates. This deals with the time value of money. A firm could be spending its money on other things, such as investments.

Inventory models are classified as either deterministic or stochastic. Deterministic models are models where the demand for a time period is known, whereas in stochastic models the demand is a random variable having a known probability distribution. These models can also be classified by the way the inventory is reviewed, either continuously or periodic. In a continuous model, an order is placed as soon as the stock level falls below the prescribed reorder point. In a periodic review, the inventory level is checked at discrete intervals and ordering decisions are made only at these times even if inventory dips below the reorder point between review times [1].

3. Continuous Review Model with Uniform Demand

The first model we look at is a continuous review model with uniform demand. Units are assumed to be withdrawn continuously at a known constant rate, a. We use this model to determine when to replenish inventory and by how much so as to minimize the cost. There are two forms to this model. In the first model, shortages are not allowed and in the second, shortages are allowed.

3.1. **Shortages are Not Allowed.** Let us use the following notation:

$$\begin{aligned}
a &= \text{demand for a product} \\
Q &= \text{units of a batch of inventory} \\
\frac{Q}{a} &= \text{cycle length or time between production runs} \\
K &= \text{the setup cost for producing or ordering one batch} \\
c &= \text{the unit cost for producing or purchasing each unit} \\
h &= \text{the holding cost per unit per unit of time held in inventory} \\
Q^* &= \text{the quantity that minimizes the total cost per unit time} \\
t^* &= \text{the time it takes to withdraw this optimal value of } Q^*.
\end{aligned}$$

With a fixed demand rate, shortages can be avoided by replenishing inventory each time the inventory level drops to zero, and this will also minimize the holding cost. Figure 1 illustrates the resulting pattern of inventory levels over time when

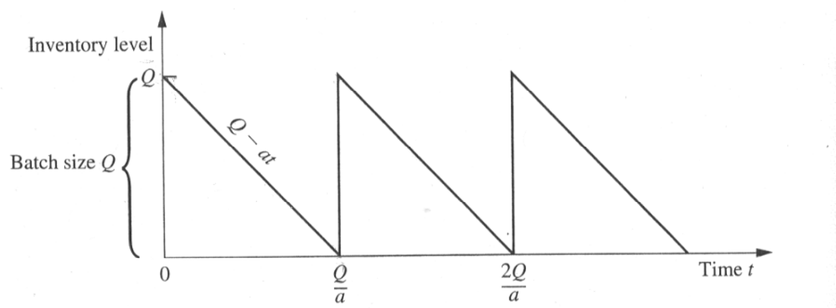

FIGURE 1. Diagram of inventory level as a function of time when no shortages are permitted ([1], pg.762).

we start at 0 by producing or ordering a batch of Q units in order to increase the initial inventory level from 0 to Q The total cost per cycle is equal to the total production cost per cycle plus the cost of holding the current inventory ([1], pg. 762).

The total production cost per cycle, PC, is given by the following equation:

$$PC = K + cQ.$$

The average inventory level during a cycle is $(Q + 0)/2 = Q/2$ units per unit time, and the corresponding cost is $hQ/2$ per unit time.. Because the cycle length is Q/a, the holding cost per cycle is given by the following:

$$\frac{hQ}{2}\frac{Q}{a} = \frac{hQ^2}{2a}.$$

Therefore, the total production cost per cycle is:

$$K + cq + \frac{hQ^2}{2a}.$$

However, we want the total cost per unit time, so we divide the total production cost per cycle by $\frac{Q}{a}$ to arrive at our total cost per unit time equation:

$$\frac{aK}{Q} + ac + \frac{hQ}{2}.$$

The value of Q^* that minimizes the total cost is found by taking the derivative of the total cost and setting it equal to zero, and solving for Q. After some algebra, we arrive at the following two equations which describe our model ([1], pg.763):

$$Q^* = \sqrt{\frac{2aK}{h}}, \tag{1}$$

$$t^* = \frac{Q^*}{a} = \sqrt{\frac{2K}{ah}}. \tag{2}$$

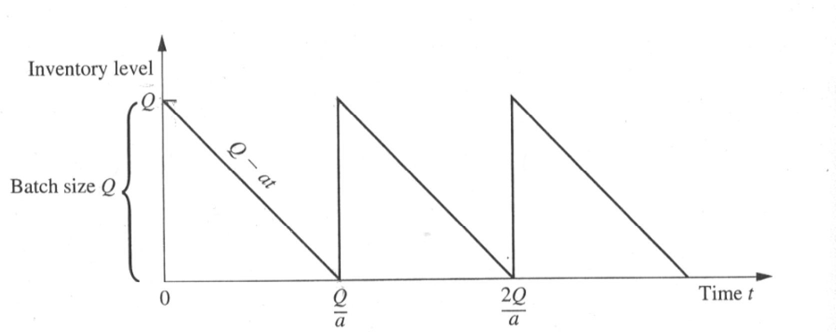

FIGURE 2. Diagram of inventory level as a function of time when shortages are permitted ([1], pg.763).

3.2. **Shortages are Allowed.** Sometimes it is worthwhile to permit small shortages to occur because the cycle length can then be increased with a resulting saving in setup cost. However, this benefit may be offset by the shortage cost. ventoryventoryventoryventoryventoryventoryTherefore, let us see the equations if shortages are allowed. First, we need to see some new notation:

$$\begin{aligned} p &= \text{shortage cost per unit short per unit of time short} \\ S &= \text{inventory level just after a batch of } Q \text{ units is added} \\ Q - S &= \text{shortage in inventory just before a batch of } Q \text{ units is added} \\ S^* &= \text{the optimal level of shortages} \end{aligned}$$

The resulting pattern of inventory levels over time is shown in Figure 2 when one starts at time 0 with an inventory level of S.

The production cost per cycle, PC, is the same as in the continuous review model without shortages. During each cycle, the inventory level is positive for a time S/a. The average inventory level during this time is $(S+0)/2 = S/2$ units per unit time, and the corresponding cost is $hS/2$ per unit time. Therefore,the holding cost per cycle is now given by:

$$\frac{hS}{2}\frac{S}{a} = \frac{hS^2}{2a}.$$

Also, shortages occur for a time $(Q - S)/a$. The average amount of shortages during this time is $(0 + Q - S)/2 = (Q - S)/2$ units per unit time, and the corresponding cost is $p(Q - S)/2$ per unit time. Therefore, the shortage cost per cycle is:

$$\frac{p(Q-S)}{2}\frac{Q-S}{a} = \frac{p(Q-S)^2}{2a}.$$

Again, we want the total cost per unit time. In order to determine this, we add up all of our costs and then divide by the cycle length (Q/a) to arrive at:

$$\frac{aK}{Q} + ac + \frac{hS^2}{2Q} + \frac{p(Q-S)^2}{2Q}.$$

In this model, there are two decision variables (S and Q), so the optimal values (S^* and Q^*) are found by setting the partial derivatives $\delta T/\delta S$ and $\delta T/\delta Q$ equal to zero. We solve for Q^* and S^* which leads to our models. Our three equations for this model are ([1], pg. 765):

$$S^* = \sqrt{\frac{2aK}{h}}\sqrt{\frac{p}{p+h}}, \tag{3}$$

$$Q^* = \sqrt{\frac{2aK}{h}}\sqrt{\frac{p+h}{p}}, \tag{4}$$

$$t^* = \frac{Q^*}{a} = \sqrt{\frac{2K}{ah}}\sqrt{\frac{p+h}{p}}. \tag{5}$$

3.3. **Example.** Suppose that the demand for a product is 30 units per month and the items are withdrawn at a constant rate. The setup cost each time a production run is undertaken to replenish inventory is \$15. The production cost is \$1 per item, and the inventory holding cost is \$0.30 per item per month ([1], pg 798, problem 17.3.1)

(1) Assuming shortages are not allowed, determine how often to make a production run and what size it should be.
Answer: We know that $a = 30$, $h = 0.30$, $K = 15$. Now, we use Equation 1 to get:

$$Q^* = \sqrt{\frac{2(30)(15)}{0.30}} = 54.77$$

Use Equation 2 to receive:

$$t^* = \frac{Q^*}{a} = \frac{54.77}{30} = 1.83$$

(2) If shortages are allowed but cost \$3 per item per month, determine how often to make a production run and what size it should be.
Answer: Now, $p = 3$. We use Equation 4 to find Q^*:

$$Q^* = \sqrt{\frac{2(30)(15)}{0.30}}\sqrt{\frac{3+0.30}{3}} = 57.4433$$

Finally, we use Equation 5 to find out how often we should place the order:

$$t^* = \frac{Q^*}{a} = \frac{57.4433}{30} = 1.914$$

4. Quantity Discounts

In the previous models, we assumed that the unit cost of an item is the same regardless of how many units were ordered. However, there could be cost breaks for ordering larger quantities.

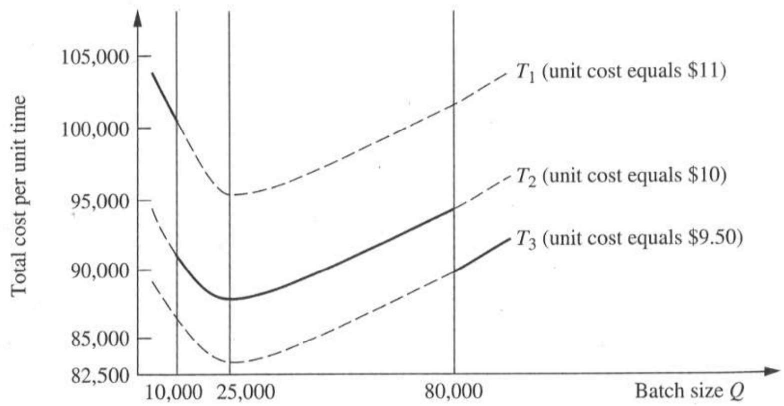

FIGURE 3. This is the graph of T_j versus Q. We need to examine the regions of the curves with solid lines ([1], pg. 766).

4.1. **Example.** Here is an example from Hillier and Lieberman ([1], pg. 766):

Suppose the unit cost for every speaker is $c_1 = \$11$ if less than $10,000$ speakers are produced, $c_2 = \$10$ if production is between $10,000$ and $80,000$ speakers, and $c_3 = \$9.50$ if more than $80,000$ speakers are produced. Demand for the speakers is $8,000$ per month and the speakers are withdrawn at a known constant rate. The setup cost each time a production run is undertaken to replenish inventory is $\$12,000$ and the inventory holding cost is \$0.30 per item per month. What is the optimal policy?

From Section 1, we are given from the derivation of the first model, that if the unit cost is c_j and $j = 1, 2, 3$, then the total cost per unit time, T_j, is:

$$T_j = \frac{aK}{Q} + ac_j + \frac{hQ}{2}. \tag{6}$$

The value of Q that minimizes T_j is found using Equation 1 from Section 3 (assuming shortages are not permitted). For $K = 12,000$, $h = 0.30$ and $a = 8,000$, we find that $Q^* = 25,298$:

$$\sqrt{\frac{(2)(8,000)(12,000)}{0.30}} = 25,298.$$

A plot of T_j versus Q is shown in Figure 3. The only regions that we need to examine are the regions of the curve shown by the solid lines. This is because the regions with the solid lines show the domain of that particular T_j curve. Looking at Figure 3, we see that $25,298$ is only in the domain of the curve T_2. Another way to see that $25,298$ is the optimal policy, we can evaluate the minimum cost for each T_j. The minimum feasible value of T_3 is $\$89,200$ (which can be seen in Figure 1 or computed using Equation 6 where $Q = 80,000$). The minimum feasible value of T_1 is $\$99,100$ (which is found using Equation 6 where $Q = 10,000$). Finally, the minimum value of T_2 evaluated at $25,298$ is $\$87,589$. Because $T_2 < T_3 < T_1$, it is better to produce in quantities of $25,298$ ([1], pg. 766).

5. Stochastic Single Period Model with No Set-Up Cost

We will first discuss the basic model, and then show two derivations of it. In one derivation, we will use calculus and in the other, we will not. Finally, we will look at a few examples of how to use our model.

5.1. The Model. There are two risks involved when choosing a value of y, the amount of inventory to order or produce. There is the risk of being short and thus incurring shortage costs, and there is a risk of having too much inventory and thus incurring wasted costs of ordering and holding excess inventory.

In order to minimize these costs, we minimize the expected value of the sum of the shortage cost and the holding cost. Because demand is a discrete random variable with a probability distribution function, $(P_D(d))$, the cost incurred is also a random variable. Let $P_D(d) = P\{D = d\}$.

We will now gather some background information about statistics. The expected value of some X, where X is a discrete random variable with probability function, $p_X(k)$, is denoted $E(X)$ and is given by ([2], pg. 192):

$$E(X) = \sum_{\text{all k}} k \cdot p_X(k).$$

Similarly, if Y is a continuous random variable with probability function, $f_Y(Y)$,

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y)dy.$$

By the Law of the Unconscious Statistician we can say that:

$$E(h(x)) = \int_{-\infty}^{\infty} h(x)f(x)dx.$$

Now, we return to analyzing our costs. The amount sold is given by:

$$\min(D, y) = \begin{cases} D & \text{if } D < y, \\ y & \text{if } D \geq y. \end{cases}$$

where D is the demand and y is the amount stocked. Now, let $C(d, y)$ be equal to the cost when demand, D is equal to d. Notice that:

$$C(d, y) = \begin{cases} cy + p(d-y) & \text{if } d > y, \\ cy + h(y-d) & \text{if } d \leq y. \end{cases}$$

The expected cost is then given by $C(y)$,

$$C(y) = E[C(D, y)] = cy + \sum_{d=y}^{\infty} p(d-y)P_D(d) + \sum_{d=0}^{y-1} h(y-d)P_D(d).$$

Sometimes a representation of the probability distribution of D is difficult to find, as in when demand ranges over a large number of possible values. Therefore, this discrete random variable is often approximated by a continuous random variable. For the continuous random variable D, let $\varphi_D(\xi)$ be equal to the probability density function of D and $\Phi(a)$ be equal to the cumulative distribution function of D. This means that

$$\Phi(a) = \int_0^a \varphi_D(\xi)d\xi.$$

Using the Law of the Unconscious Statistician, the expected cost $C(y)$ is then given by:

$$C(y) = E[C(D,y)] = \int_0^\infty C(\xi,y)\varphi_D d\xi.$$

This expected cost function can be simplified to $cy + L(y)$ where $L(y)$ is called the expected shortage plus holding cost. Now, we want to find the value of y, say y^0 which minimizes the expected cost function $C(y)$. This optimal quantity to order y^0 is that value which satisfies ([1], pg. 775):

$$\Phi(y^0) = \frac{p-c}{p+h}. \tag{7}$$

5.2. **Derivation of the Model Using Calculus.** To begin, we assume that the initial stock level is zero. For any positive constants, c_1 and c_2, define $g(\xi, y)$ as

$$g(\xi,y) = \begin{cases} c_1(y-\xi) & \text{if } y > \xi, \\ c_2(\xi - y) & \text{if } y \leq \xi, \end{cases}$$

and let

$$G(y) = \int_0^\infty g(\xi,y)\varphi_D(\xi)d\xi + cy.$$

where $c > 0$. By definition,

$$G(y) = c_1\int_0^y (y-\xi)\varphi_D(\xi)d\xi + c_2\int_y^\infty (\xi - y)\varphi_D(\xi)d\xi + cy.$$

Now, we take the derivative of $G(y)$ (see Appendix) and set it equal to zero. This gives us,

$$\frac{dG(y)}{dy} = c_1\int_0^y \varphi_D(\xi)d\xi - c_2\int_y^\infty \varphi_D(\xi)d\xi + c = 0.$$

Because,

$$\int_0^\infty \varphi_D(\xi)d\xi = 1,$$

we can write,

$$c_1\Phi(y^0) - c_2[1 - \Phi(y^0)] + c = 0.$$

Now, we solve this expression for $\Phi(y^0)$ which results in

$$\Phi(y^0) = \frac{c_2 - c}{c_2 + c_1}.$$

To apply this result, we need to show that

$$C(y) = cy + \int_y^\infty p(\xi - y)\varphi_D(\xi)d\xi + \int_0^y h(y-\xi)\varphi_D(\xi)d\xi,$$

has the form of $G(y)$. We see that $c_1 = h$, $c_2 = p$, and $c = c$, so that the optimal quantity to order y^0 is that value which satisfies

$$\Phi(y^0) = \frac{p-c}{p+h}.$$

5.3. **Without using Calculus.** We are going to arrive at the optimal policy thinking rationally about costs and without using calculus.

Suppose the current order level is y^0 and we are considering ordering one more unit. We are trying to decide if this a good idea or not.

The net average change in total cost is equal to the average extra cost on the holding side minus the average savings on the shortage side. An optimal policy is when this net average change in total cost is equal to 0.

The average extra cost on the holding side is the probability that demand is less than y^0, $(P(D < y^0))$, times the extra holding cost for one more unit (h) plus the extra purchase cost (c) or:

$$P(D < y^0)[h + c].$$

The average savings on the shortage side is the probability that demand is greater than or equal to y^0, $(P(D \geq y^0))$, times the shortage cost that we do not have to pay anymore (p) minus the cost of buying that extra unit (c) or:

$$P(D \geq y^0)[p - c].$$

Now, we solve the following equation for $\Phi(y^0)$, where $\Phi(y^0) = P(D < y^0)$ and consequently, $1 - \Phi(y^0) = P(D \geq y^0)$:

$$0 = P(D < y^0)[h + c] - P(D \geq y^0)[p - c],$$

or

$$0 = \Phi(y^0)[h + c] - (1 - \Phi(y^0))[p - c].$$

and we get that the optimal policy is:

$$\Phi(y^0) = \frac{p - c}{p + h}.$$

Therefore, we see that in this particular model, a single period model with no setup costs, we can arrive at our optimal policy without the use of calculus.

5.4. **Examples.**

(1) A baking company distributes bread to grocery stores daily. The company's cost for the bread is \$0.80 per loaf. The company sells the bread to the stores for \$1.20 per loaf sold, provided that it is disposed of as fresh bread (sold on the day it is baked). Bread not sold is returned to the company. The company has a store outlet that sells bread that is 1 day or more old for \$0.60 per loaf. No significant storage cost is incurred for this bread. The cost of the loss of customer goodwill due to a shortage is estimated to be \$0.80 per loaf. The daily demand has a uniform distribution between $1,000$ and $2,000$ loaves. Find the optimal daily number of loaves that the manufacturer should produce ([1], pg.801, problem 17.4.3).

Answer: $c = 0.80$, $h = -0.60$ and $p = 1.20 + 0.80 = 2.00$
Because demand has a uniform distribution, we need to solve $\varphi(z) = \int_a^z \frac{1}{b-a} dx$ where $a = 1000$ and $b = 2000$ to receive the following:

$$\varphi(z) = \int_{1000}^{z} \frac{1}{1000} dx = \frac{z - 1000}{1000}.$$

Now, we must substitute $z = y^0$ and solve the following:

$$\frac{y^0 - 1000}{1000} = \frac{2 - 0.8}{2 - 0.6}.$$

Therefore, the manufacturer should produce $1,857$ loaves of bread

(2) Suppose that the demand D for a spare airplane part has an exponential distribution with mean 50, that is,

$$\varphi_D(\xi) = \begin{cases} \frac{1}{50}e^{-\frac{\xi}{50}} & \text{for } \xi \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

This airplane will be obsolete in 1 year, so all production of the spare part is to take place at present. The production costs now are \$1,000 per item, but they become \$10,000 per item if they must be supplied at later dates-that is, $p = 10,000$. The holding costs, charged on excess after the end of the period are \$300 per item. Determine the optimal number of spare parts to produce ([1], pg. 800-801, problem 17.4.2).

Answer: We know that $c = 1,000$, $p = 10,000$ and $h = 300$. We solve the following integral for a:

$$\phi(a) = \int_0^a \frac{1}{50}e^{-\frac{\xi}{50}}d\xi = 1 - e^{\frac{-a}{50}}.$$

The optimal quantity to produce, y^0 is that value which satisfies:

$$1 - e^{\frac{-y^0}{50}} = \frac{10,000 - 1,000}{10,000 + 300} = 104.$$

Therefore, we have found an optimal policy of producing 104 spare parts.

6. Stochastic Single Period Model with a Set-up Cost

6.1. The Model. Now, we assume there is a set up cost incurred when ordering or producing inventory. The optimal inventory policy is the following ([1], pg. 781):

$$\text{If } x \begin{cases} < s & \text{order } S - x \text{ to bring inventory level up to } S, \\ \geq s & \text{do not order.} \end{cases}$$

We determine the value of S from

$$\varphi(S) = \frac{p - c}{p + h},$$

which is exactly the optimal policy from the stochastic model with no set up cost.

Also, s is the smallest value that satisfies the equation

$$cs + L(s) = K + cS + L(S).$$

Hence, this policy is referred to as an (s, S) policy.

6.2. Derivation of the Model. To begin, the shortage and holding costs are given by $L(y)$, where

$$L(y) = p\int_y^\infty (\xi - y)\varphi_D(\xi)d\xi + h\int_0^y (y - \xi)\varphi_D(\xi)d\xi.$$

Therefore, the total expected cost incurred by bringing the inventory level up to y is given by

$$\begin{array}{ll} K + c(y - x) + L(y) & \text{if } y > x, \\ L(x) & \text{if } y = x. \end{array}$$

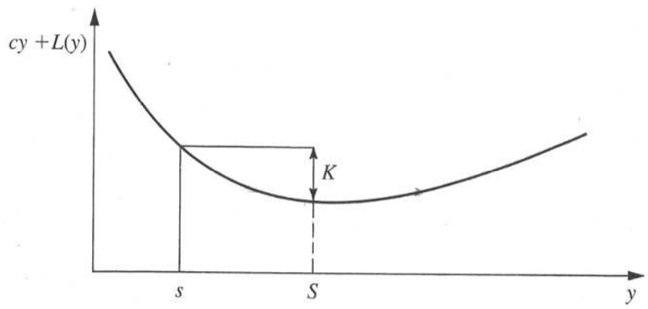

FIGURE 4. Graph of $cy + L(y)$ ([1], pg.780).

If $cy + L(y)$ is drawn as a function of y, it will appear as shown in Figure 4. Now we will define S as the value of y that minimizes $cy + L(y)$, and define s as the smallest value of y for which $cs + L(s) = K + cS + L(S)$. From Figure 4, it can be seen that

$$\text{If } x > S, \text{ then } K + cy + L(y) > cx + L(x), \text{ for all } y > x,$$

so that

$$K + c(y - x) + L(y) > L(x).$$

The left hand side of this inequality is the expected total cost of ordering $y - x$ to bring the inventory level up to y, and the right hand side of this inequality is the expected total cost if no ordering occurs. Therefore, the optimal policy says that if $x > S$, do not order.

From Figure 4, we note that, if $s \leq x \leq S$,then

$$K + cy + L(y) \geq cx + L(x), \text{ for all } y > x,$$

so that

$$K + c(y - x) + L(y) \geq L(x).$$

Again, we see that it is better not to order.

Now, if $x < s$, we can see from Figure 4 that

$$\min_{y \geq x}\{K + cy + L(y)\} = K + cS + L(S) < cx + L(x),$$

or rearranging terms we get:

$$\min_{y \geq x}\{K + c(y - x) + L(y)\} = K + c(S - x) + L(S) < L(x),$$

so that it pays to order.

Therefore, we get an optimal policy of the following:

$$\text{If } x \begin{cases} < s & \text{order } S - x \text{ to bring inventory level up to } S, \\ \geq s & \text{do not order.} \end{cases}$$

In addition, s is the smallest value which satisfies the equation

$$cs + L(s) = K + cS + L(S).$$

Thus, our policy is called an (s, S) policy.

7. Case Study: Tackling Newsboy's Teachings

7.1. **The Situation.** Howie Rogers wants to win a Corvette. In order to do this, he must establish a firecracker stand and purchase firecrackers from Leisure Limited, a large wholesaler. Howie will then resell the firecrackers to local customers for a higher price. He has until the Fourth of July. This is because after the holiday, no one will want firecrackers until New Year's Eve. He must return the leftover firecrackers to Leisure Limited, but Leisure Limited will only refund part of the cost of the returned firecrackers. Whoever sells the most firecrackers, wins a Corvette. Additionally, once Howie orders firecrackers, it takes 7 days for their delivery.

The question now is, how many firecrackers should he order? If he orders too few, he will not have time to place and receive another order before the holiday and therefore lose sales and his chance to win the Corvette. If he orders too many firecrackers, he will lose money since he cannot obtain a full refund for the extra firecrackers. Howie enlists the help of his sister Talia.

Talia calls Leisure Limited and obtains the following information: Howie will pay \$3.00 per firecracker set. The fees to place an order are approximately \$20.00 per order. After the Fourth of July, Leisure Limited returns only half of the cost for each firecracker set returned. In addition, Howie will have to pay shipping costs that average \$0.50 per firecracker set. Data compiled from last year's sales indicate that the firecracker sets sold for an average of \$5.00 per set. Also, data indicates that stands sold between 120-420 firecracker sets. Now, Talia makes a few assumptions. The most important one being that demand will follow a uniform distribution. Also, she decides to use the average of \$5.00 for the unit sale price.

7.2. **The Question.** This case study is basically a stochastic model without a set-up cost. There is no set up cost because Howie must place an order; he has no inventory on hand, so in order to start the business, there is no question as to whether or not to order based on the set up cost. Now, we must answer a few questions.

(1) How many firecracker sets should Howie purchase from Leisure Limited to maximize his expected profit?

    Answer: $c = 3$, $p = 5$, and $h = -1$. The value of h is determined by taking the storage cost minus the salvage value, $0 - (1.5 - 0.5)$.
    Because we are assuming demand follows a uniform distribution, we need to solve $\varphi(z) = \int_a^z \frac{1}{b-a}dx$ where $b = 420$ and $a = 120$. Plugging these numbers in and solving, we get

$$\varphi(z) = \int_{120}^{z} \frac{1}{300}dx = \frac{z-120}{300}.$$

    Now, we solve

$$\phi(y^0) = \frac{p-c}{p+h},$$

    where we plug in our known numbers and solve for y^0. This becomes:

$$\frac{y^0 - 120}{300} = \frac{5-3}{5-1}.$$

    Thus our answer is to order 270 firecrackers.

(2) How would Howie's order quantity change if Leisure Limited refunds 75% of the wholesale price for returned firecracker sets? How would it change if

Leisure Limited refunds 25% of the wholesale price for returned firecracker sets?

Answer: Now, Leisure Limited refunds 75% of the wholesale price. This means that Howie will receive 2.25 for every unsold firecracker set. This changes our holding cost value (h) from -1 to $-(2.25 - 0.50) = -1.75$. Everything else stays the same, so we solve the following equation for y^0:

$$\frac{y^0 - 120}{300} = \frac{5-3}{5-1.75}.$$

Now, Howie should order 280 firecrackers.

If Leisure Limited refunds only 25% of the wholesale price, Howie will receive only 0.75 for every unsold firecracker set. This changes the holding cost value to $-(0.75-0.50) = -0.25$. Now, we solve the following equation for y^0:

$$\frac{y^0 - 120}{300} = \frac{5-3}{5-0.25}.$$

Therefore, Howie should order 246 firecracker sets.

(3) Howie is not happy with selling the firecracker sets for $5.00 per set. Suppose Howie wants to sell the firecracker sets for $6.00 per set instead. What factors would Talia have to take into account when recalculating the optimal order quantity?

Answer: If Howie wants to sell the firecracker sets for $6.00 per set, then the shortage cost changes from 5 to 6. Therefore, we solve the following equation for y^0:

$$\frac{y^0 - 120}{300} = \frac{6-3}{6-1}.$$

Now, Howie should order 300 firecracker sets.

8. Zappone Manufacturing [3]

Zappone Manufacturing began in 1969 producing aluminum roofing shingles. It was not until the late 1970's that they began manufacturing copper shingles. This was because Joe Zappone and his wife Lynda went on a tour of Europe and Zappone noticed, that unlike in the US, most European roofs were designed to be permanent. Zappone decided to produce a roof that would match the quality of the roofing he saw in Europe. That is how Zappone became the first person to make shingles out of copper. Normally the roofs were made out of copper sheets, which were harder to work with, making them extremely expensive. He promoted his new product very heavily; one of the ways he did this was by putting the copper roof on the carousal in Spokane, Washington in 1983.

Zappone does not use an inventory model. Instead he has his own policy which we will now investigate. Zappone's policy depends heavily on the world commodity market prices of copper. Every day he checks the current price of copper and does some quick math in his head to figure out what the price will be once it reaches him. The world commodity market prices are the prices for which the mine sells copper to the mills. Then the mills have to add transportation costs, energy costs, and rolling costs to the commodity price and this total cost is what Zappone pays. For example, currently the Comex (or the commodity market) price is $2.08 per pound. However, Zappone anticipates his cost to be about $2.80. He adds on $0.455 per pound for rolling costs, $0.145 per pound for transportation costs, and $0.25 per pound for energy costs. This total cost actually comes to \$2.705 per pound, but he adds on about \$0.10 just to be careful because the price of copper fluctuates a lot from day to day. Although Zappone may place an order for copper today, expecting the price to be \$2.70, the copper will not be shipped for 5 weeks, and he will be charged the price of copper on the day it is shipped.

If the price of copper is low and steady, probably around \$1.80 to \$2.00 per pound, he bases his inventory policy on three different things:

(1) Availability of the copper, that is, how long it will take for the shipment to arrive, which is normally 5 weeks from the date of ordering.
(2) Projected Sales
(3) Current inventory, that is, when are they going to be out of their current stock.

Zappone is required to buy copper in truckloads; each truckload being $40,000$ pounds of copper. Normally he tries to buy 12 truckloads a year.

However, if the price of copper is pretty expensive, such as it is right now, Zappone does not want to have a lot of high price inventory. He will wait to order more inventory until his current inventory is low enough that he could not fulfill projected sales. When the price of copper is really high, Zappone must raise prices in order maintain his business. However, right now he is not raising the prices as high as he should, instead he is bearing part of the burden of the high priced copper.

Therefore, Zappone orders heavily when the prices of copper are low, and does not order as much when the prices of copper are high. Zappone's holding costs are pretty minimal. Although he owns the building where he stores the copper and machinery, he still pays insurance taxes on everything in the building. However, the higher insurance cost when he has more inventory is not high enough to outweigh the benefit of buying more inventory.

In order for this type of inventory policy to be successful, Zappone and his employees communicate often. He checks the level of his inventory and the price of copper daily, and discusses pending sales with his sales crew. All in all, the mathematical models in this paper cannot help Zappone's company. Because the price of copper fluctuates so much from day to day, it is hard to say when exactly to order. Perhaps, with more studying and a more complex model, we could formulate an optimal policy for Zappone. This would require more complex statistical analysis in order to deal with the fluctuating price of copper. Another reason we would need a more in depth model is that although Zappone orders the copper today, at today's prices, he will be charged the price of copper on the day it ships, roughly 5 weeks later. Even though he does not use a model, Zappone has done well for himself. He sells copper all over the world: Japan, South America, Europe, and all 50 states. In addition, he is environmentally friendly because about 80% of the copper he uses comes from recycled copper and only 20% comes from new copper being mined from the ground. However, the price of copper, whether it is reusable or new, does not differ, so this does not change his inventory policies. This shows that an inventory model is helpful but not necessary for all companies.

9. Conclusion

In this paper, we began the study of inventory theory. We examined two types models: deterministic continuous review models and stochastic models. In addition, we learned about quantity discounts and how these affected our models. We also looked at a few examples of how these models are used.

However, this paper only touches the surface of what inventory theory is all about. After learning the basics, we now can ask and study more complex questions. For example, what happens when customers place orders in advance for a future delivery? A company could choose to allow for four different levels of response time to customers: standard (five-day delivery), value (slower, but lower shipping cost), premium (faster, next day delivery), and precision (delivered on a specific date). How does this hypothetical company handle its inventory policy? If interested in the previous question, please refer to Wei Wei and Ozalp Ozer [4].

Another problem we can consider deals with a firm that supplies goods to two different types of customers: customers who have long-term supply contracts, and customers who request goods occasionally. The orders of the customers who have supply contracts are known in advance and must be fully met without delay every period. However, the unexpected requests from occasional customers are unknown and the company can either accept the order or reject it. How does a company deal with their inventory policy when it mixes deterministic and stochastic demand? If interested in this issue surrounding inventory theory, the reader is referred to Frank, Zhang and Duenyas [5].

10. Table of Notation

Notation	Meaning
a	the demand for a product
Q	units of a batch of inventory
$\frac{Q}{a}$	cycle length or time between production runs
K	set-up cost for producing or ordering one batch
c	unit cost for producing or purchasing each unit
h	holding cost per unit per unit of time held in inventory
Q^*	the quantity that minimizes the total cost per unit time
t^*	the time it takes to withdraw this optimal value of Q^*
p	shortage cost per unit short per unit of time short
S	inventory level just after a batch of Q units is added to inventory
$Q - S$	shortage in inventory just before a batch of Q units is added
S^*	the optimal level of shortages
y	the amount of inventory to order or produce
D	demand
$P_D(d) = P\{D = d\}$	the probability distribution of D
X	a discrete random variable with probability function $p_X(k)$
$E(X)$	the expected value of some X
Y	a continuous random variable with probability function $f_Y(Y)$
$\varphi_D(\xi)$	the probability density function of D
$\varphi(a)$	the cumulative distribution function of D

11. Appendix: Derivation of Leibniz's Rule

We are going to derive the formula for finding the derivative of an integral. In essence, we will find the derivative of

$$F(y) = \int_{g(y)}^{h(y)} f(x,y)dx.$$

11.1. **Rules to Recall.** First, we need to remember a few rules from calculus.

(1) The Fundamental Theorem of Calculus states that if f is continuous on the closed interval from a to b and differentiable on the open interval from a to b then

$$\frac{d}{dy}\int_a^y p(x)dx = p(y),$$

$$\frac{d}{dy}\int_y^b p(x)dx = -p(y).$$

(2) We must remember the rule for taking the derivative of an integral of a function of more than one variable. This rule is
$$\frac{d}{dz}\int_a^b f(x,z)dx = \int_a^b \frac{\delta f}{\delta z}[f(x,z)]dx.$$
(3) Finally, we must remember the chain rule for functions of 3 variables. Suppose a, b, and c, are each differentiable functions of j. Then $j(a,b,c)$ is a function of y and
$$\frac{dj}{dy} = \frac{\delta j}{\delta a}\frac{da}{dy} + \frac{\delta j}{\delta b}\frac{db}{dy} + \frac{\delta j}{\delta c}\frac{dc}{dy}.$$

Using these three rules, we can now derive the formula for finding the derivative of an integral with more than one variable.

11.2. **The Derivation.** Again, we want to find a formula for
$$F(y) = \int_{g(y)}^{h(y)} f(x,y)dx.$$
Now, let $h(y) = b$ and $g(y) = a$ and let
$$j(a,b,y) = \int_a^b f(x,y)dx.$$
Then, $F(y) = j(g(y), h(y), y)$.

$$\begin{aligned}
F'(y) &= \frac{dj}{dy} = \frac{\delta j}{\delta a}\frac{da}{dy} + \frac{\delta j}{\delta b}\frac{db}{dy} + \frac{\delta j}{\delta y}\frac{dy}{dy} \qquad \big(\text{By Rule } 3\big),\\
\frac{\delta j}{\delta a} &= -f(g(y),y) \qquad \big(\text{By Rule } 1\big),\\
\frac{da}{dy} &= \frac{d(g(y))}{dy} = g'(y),\\
\frac{\delta j}{\delta b} &= f(h(y),y) \qquad \big(\text{By Rule } 1\big),\\
\frac{db}{dy} &= \frac{d(h(y))}{dy} = h'(y),\\
\frac{\delta j}{\delta y} &= \int_{g(y)}^{h(y)} \frac{\delta f(x,y)}{\delta y}dx \qquad \big(\text{By Rule } 2\big),\\
\frac{dy}{dy} &= 1.
\end{aligned}$$

Therefore, our final formula is
$$\frac{d}{dy}\int_{g(y)}^{h(y)} f(x,y)dx = \int_{g(y)}^{h(y)} \frac{\delta f(x,y)}{\delta y}dx + f(h(y),y)h'(y) - f(g(y),y)g'(y).$$

11.3. **Example Using Leibniz's Rule.** Let $f(x,y) = x^2y^3$, $g(y) = y$ and $h(y) = 2y$, then
$$\frac{d}{dy}\int_y^{2y} x^2y^3dx = \int_y^{2y} 3x^2y^2dx + (2y)^2y^3(2) - y^2y^3(1) = 14y^5.$$

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