

# TRIANGLES IN HYPERBOLIC GEOMETRY

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ABSTRACT. This paper derives the Law of Cosines, Law of Sines, and the Pythagorean Theorem for triangles in Hyperbolic Geometry. The Poincaré model for Hyperbolic Geometry is used. In order to accomplish this the paper reviews Inversion in Hyperbolic Geometry, Radical Axes and Powers of circles and expressions for hyperbolic cosine, hyperbolic sine, and hyperbolic tangent. A brief history of the development of Non-Euclidean Geometry is also given in order to understand the importance of Euclid's Parallel Postulate and how changing it results in different geometries.

## 1. INTRODUCTION

Geometry aids in our perception of the world. We can use it to deconstruct our view of objects into lines and circles, planes and spheres. For example, some properties of triangles that we know are that it consists of three straight lines and three angles that sum to  $\pi$ . Can we imagine other geometries that do not give these familiar results? The Euclidean geometry that we are familiar with depends on the hypothesis that, given a line and a point not on that line, there exists one and only one line through the point parallel to the line. This is one way of stating Euclid's parallel postulate. Since it is a postulate and not a theorem, it is assumed to be true without proof. If we alter that postulate, new geometries emerge.

This paper explores one model of that Non-Euclidean Geometry—the Poincaré model of Hyperbolic Geometry. We will explore the Poincaré model, based on a system of orthogonal circles (circles which intersect each other at right angles), and learn about some of its basic aspects. Then we will study the characteristics of triangles in this geometry through discovering relationships between parts of triangles and deriving hyperbolic forms of the Pythagorean Theorem and the Laws of Sines and Cosines. While doing so we will also come across the concepts of Inversion and Radical Axes. First, however, we will have a brief review of the history leading up to the development of the Poincaré model.

## 2. DEVELOPMENT OF NON-EUCLIDEAN GEOMETRY

In the early third century B.C.E., Euclid gathered the existing knowledge about geometry and combined it with his new work to form a comprehensive textbook of geometry. He began with definitions and postulates (the fifth of which is the parallel postulate) and includes many theorems and propositions. The *Elements*, consisting of 13 books, is the oldest geometry book that has survived to modernity. This is partly because the *Elements* was comprehensive for the time, so few other geometry texts were needed or used. Consistent with his interest in spreading knowledge through teaching, Euclid also founded and taught at a school in Alexandria.

Although originally written in Greek, a translation (by Thomas Heath) of Euclid's five postulates is as follows [7].

**Definition 2.1** (Postulate 1). *A straight line segment can be drawn by joining any two points.*

**Definition 2.2** (Postulate 2). *A straight line segment can be extended indefinitely in a straight line.*

**Definition 2.3** (Postulate 3). *Given a straight line segment, a circle can be drawn using the segment as radius and one endpoint as center.*

**Definition 2.4** (Postulate 4). *All right angles equal one another.*

**Definition 2.5** (Postulate 5, the Parallel Postulate). *That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

A modern statement of this, along with a statement of the form of the parallel postulate used in Hyperbolic Geometry will be presented in the next section.

Mathematicians strove for 2,000 years to prove Euclid's parallel postulate from the first four postulates, but were unsuccessful. They attempted to prove it because they believed that the parallel postulate was not a postulate, but a theorem and thus could be proved. It was not until the 19<sup>th</sup> century that people began to accept that the parallel postulate could not be proved [2].

According to some dated letters, Karl Friedrich Gauss (1777-1855) began to develop a Non-Euclidean geometry around 1792. He first had to overcome his prejudices against a Non-Euclidean geometry and learn to accept a system of geometry that went against his intuition. Gauss eventually convinced himself of the validity of Non-Euclidean geometry and called the new geometry a series of names, Anti-Euclidean, Astral-Euclidean and finally, Non-Euclidean. He, however, did not publish any of his work in this area fearing ridicule and disbelief [2]. Already a prominent mathematician, he stood to lose his peers' respect for publishing controversial results.

Ferdinand Karl Schweikart (1780-1859) also developed a Non-Euclidean geometry and in 1818 sent a memorandum on his geometry to Gauss requesting his opinion. Gauss agreed with Schweikart's work and acknowledged that he had been working in that area for a long time. Schweikart's memorandum stated that there were two kinds of geometry, Euclidean and Astral and that in Astral Geometry the sum of the three angles in a triangle is less than two right angles. He also did not publish [2].

James Bolyai (1802-1860) also developed a Non-Euclidean geometry. Unlike Schweikart and Gauss, Bolyai took the brave step of publishing his findings. Although he was aware of the magnitude of Non-Euclidean Geometry, saying "out of nothing I have created a strange new universe," and held his work in high esteem, it received little acclaim and Bolyai grew depressed [2].

The man who published the first influential work on Non-Euclidean geometry was Nicolai Lobachevsky (1793-1856). He studied and later taught at the University of Kasan in Russia. In 1823-1825 he started developing a geometry independently of Euclid and published a paper in 1829 which showed that a Non-Euclidean geometry was logically consistent. He later published several books: *New Foundations of*

*Geometry* in 1838, *Geometrical Investigations on the Theory of Parallels* in 1840 and *Pangeometry* in 1855 [6]. Lobachevsky used the horocycle, a circle of infinite radius, for the lines and the horosphere, a sphere of infinite radius, for the plane [2].

It took a long time for people to accept the existence of a Non-Euclidean geometry, particularly because of the strong belief in Kantian philosophy in the late 18<sup>th</sup> and early 19<sup>th</sup> century. Immanuel Kant (1724-1804) believed that geometry, because it existed in space, was an absolute science. He believed that the axioms of geometry were true *a priori*. Therefore, the idea that different geometries could exist challenged Kantian philosophy. Finally people were convinced, mostly through Lobachevsky because he had the confidence and courage to publish his ideas, that a Non-Euclidean Geometry was logical.

The model of Non-Euclidean Geometry that we will be using was named after Jules Henri Poincaré (1854-1912). Poincaré was a French philosopher and mathematician. His work foreshadowed chaotic deterministic systems and algebraic topology. Poincaré developed his model of Non-Euclidean geometry in part to aid his work on solar systems.

### 3. POINCARÉ MODEL OF NON-EUCLIDEAN GEOMETRY

Once mathematicians and philosophers began accepting the validity of Non-Euclidean Geometries, more people began exploring and working on those geometries. Three different versions of the parallel postulate are possible, each leading to one or more unique geometries.

We will use the Poincaré model to consider Hyperbolic Geometry. The Poincaré model results in slightly different definitions for lines, distance and area. A useful tool when working in the Poincaré model is the transformation of inversion and its effects on objects in the Hyperbolic plane.

**3.1. Alternatives to Euclid's Parallel Postulate.** Depending on what axioms are used, different geometries can be developed. The geometry that can be proved using only the first four of Euclid's axioms is called Absolute Geometry. Other geometries rely on the parallel postulate. Different versions of the parallel postulate lead to different geometries as listed below.

**Parallel Postulate 1.** *If a point and a line not passing through it are given, there exists one and only one line which passes through the given point parallel to the given line.*

This axiom is equivalent to Euclid's original axiom and creates the Euclidean Geometry, also known as Parabolic Geometry, with which we are most familiar.

**Parallel Postulate 2.** *If a point and a line not passing through it are given, there exists no line passing through the given point parallel to the given line.*

It follows that every pair of lines intersect at two points. This axiom leads to an Elliptic Geometry. Spherical Geometry, which considers figures on the surface of a sphere and where lines are great circles, is a type of this geometry.

**Parallel Postulate 3.** *If a point and a line not passing through it are given, there exists at least two lines which pass through the given point parallel to the given line.*

Using this version leads to Hyperbolic Geometry, which the Poincaré model describes.

**3.2. The Poincaré Geometry.** To develop the Poincaré model, consider a fixed circle,  $\omega$ , in the Euclidean plane. The “plane” of this geometry is contained within  $\omega$ . Points and lines are defined below.

**Definition 3.1** (Orthogonal). *Two curves are orthogonal if, at every point of intersection, the tangent lines to the curves at that point are perpendicular.*

**Definition 3.2** (Points). *Points are Euclidean points inside  $\omega$ .*

This model of Hyperbolic Geometry is represented by orthogonal circles. The plane of this geometry is restricted to the interior of a circle,  $\omega$ . Points may lie anywhere inside  $\omega$  but not on the boundary of  $\omega$ . All lines in this geometry must be sections of circles orthogonal to  $\omega$ .

**Definition 3.3** (Lines). *Lines are either the arc that is inside  $\omega$  of a circle orthogonal to  $\omega$  or a diameter of  $\omega$ .*

**Definition 3.4** (Distance). *Let  $A$  and  $B$  be two points in  $\omega$ . Then  $A$  and  $B$  lie on a unique line. Let that line intersect  $\omega$  at points  $M$  and  $N$  such that  $A$  lies between  $M$  and  $B$ . Then the “distance” is given by:*

$$\widetilde{AB} = \left| \log \frac{AM \cdot BN}{AN \cdot BM} \right| = |\log[AB, MN]|$$

if  $A \neq B$  and zero if  $A = B$ . Furthermore,  $\widetilde{AB} = \widetilde{BA}$ .

We can prove that  $A$  and  $B$  lie on a unique line by finding it. We will not review that proof here because it is only an exercise in algebra once we make use of the following lemma.

**Lemma 1.** *The circle given by the equation  $x^2 + y^2 = r^2$  is orthogonal to the circle given by the equation*

$$(1) \quad x^2 + y^2 + ax + by + c = 0$$

if and only if  $c = r^2$ .

*Proof.* Express  $x^2 + y^2 + ax + by + c = 0$  in standard form. By completing the square.

$$\begin{aligned} \left(x^2 + ax + \left(\frac{a}{2}\right)^2\right) - \left(\frac{a}{2}\right)^2 + \left(y^2 + by + \left(\frac{b}{2}\right)^2\right) - \left(\frac{b}{2}\right)^2 + c &= 0 \\ \left(x^2 + \frac{a}{2}\right)^2 + \left(y^2 + \frac{b}{2}\right)^2 + c - \frac{a^2}{4} - \frac{b^2}{4} &= 0 \end{aligned}$$

So this circle has center  $\left(-\frac{a}{2}, -\frac{b}{2}\right)$  and radius  $r = \sqrt{\frac{a^2}{4} + \frac{b^2}{4} - c}$ . From the basic properties of orthogonal circles we know that if and only if two circles are orthogonal with radii of  $r_1$  and  $r_2$  and centers at  $O_1$  and  $O_2$ , then  $r_1^2 + r_2^2 = \overline{O_1O_2}^2$  since the radii intersect at a right angle. Since the first circle has center  $(0, 0)$  and radius  $r$ , the distance

$$\overline{AB} = \sqrt{\left(-\frac{a}{2} - 0\right)^2 + \left(-\frac{b}{2} - 0\right)^2} = \sqrt{\frac{a^2 + b^2}{4}}$$

So

$$r^2 + \frac{a^2 + b^2 - 4c}{4} = \frac{a^2 + b^2}{4}$$

$$r^2 = c$$

Since all of these steps are reversible, we can prove the converse the same way.  $\square$

We could then use this to solve the original problem by choosing any two points,  $[x_1, y_1]$  and  $[x_2, y_2]$  and making the circle given by equation (1) pass through those two points. Also note that this definition of distance satisfies the requirements of Euclidean distance. Letting  $x$ ,  $y$ , and  $z$  be points in the plane, and using  $D(x, y)$  to represent the distance between the points  $x$  and  $y$ , the requirements are:

- $D(x, y) \geq 0$ .
- $D(x, x) = 0$ . Furthermore, if  $D(x, y) = 0$ , then  $x = y$ .
- $D(x, y) = D(y, x)$ .
- $D(x, y) \leq D(x, z) + D(z, y)$ .

It is particularly easy to see, using the definition that  $\widetilde{AB} = \widetilde{BA}$  and that if  $A \neq B$ , then  $\widetilde{AB} > 0$ .

Consider the areas of polygons in the Poincaré model. Since any polygon can be made up of triangles and the sum of the area of the triangles that make up the polygon equals the area of the polygon we will only consider the area of a triangle [1].

**Definition 3.5** (Defect). *The defect of a triangle is the difference between the sum of its angles and  $\pi$ .*

**Definition 3.6** (Area). *The area of a triangle is equal to its defect.*

The area of any triangle is less than  $\pi$ . As the vertices approach the boundary of the fundamental circle, the length of the edges approaches infinity and the area of the triangle approaches  $\pi$ . It will never equal  $\pi$  since the boundary of the fundamental circle is not actually a part of the geometry so the vertices can never lie on the circumference. Interestingly, though the area of a triangle is bounded, the area of a polygon is unbounded as the number of sides approaches infinity and it grows to cover the entire circle  $\omega$ , the entire plane of the geometry.

In Figure 1 the vertices are close to the boundary, therefore the angles get small and the area gets closer to  $\pi$ . In Figure 2 we look at a triangle with vertices further from the boundary of  $\omega$ . This makes the angles larger and the area smaller.

**3.3. Inversion and Radical Axes.** Inversion is a technique used to move objects, such as lines and points, to a different space in the plane without changing some of the properties of the object. It allows for two dimensional rotation and translation with no distortion of objects. In the Poincaré model, moving an object, such as a triangle, makes it appear different in our Euclidean representation of it but we will claim that most of its properties, such as angle measure and area, are invariant under inversion.

Let the inversion transformation of point  $P$  be  $T(P)$  and let  $P$  be a point such that  $P \neq O$  and let  $O$  be the center of the circle  $[O, r]$ . See Figure 3, for an example of a point and its inverse point.

**Definition 3.7** (Inversion). *Each point  $P$  has a unique inverse point with respect to a circle  $[O, r]$ ,  $P' = T(P)$  on the line  $\overline{OP}$  such that,*

$$\overline{OP'} = \frac{r^2}{\overline{OP}}.$$

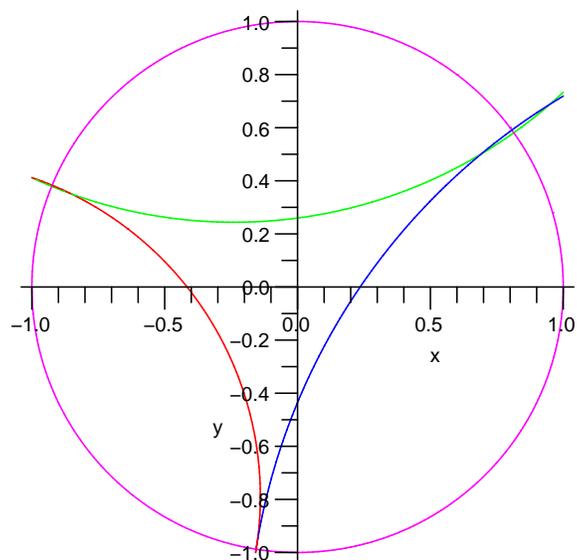


FIGURE 1. The triangle formed here has area of 2.83.

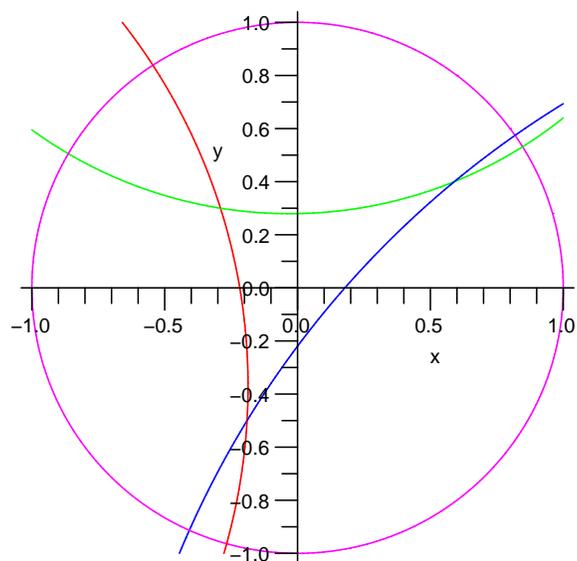


FIGURE 2. The triangle formed has area of 1.26.

Therefore,  $P$  and  $P'$  are **inverse points**. The inverse of point  $P$  lies on the extended line determined by the vector  $\overrightarrow{OP}$  as seen in Figure 3. To help us understand inversion we will consider two situations, one when  $P$  gets close to the origin and also when  $P$  gets close to the boundary of  $\omega$ . When  $P$  approaches  $O$ , the distance  $\overline{OP}$  approaches zero, therefore, from our formula in the definition, the length  $\overline{OP'}$

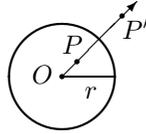


FIGURE 3.  $P'$  and  $P$  are inverse points on the circle of inversion  $[O, r]$ .

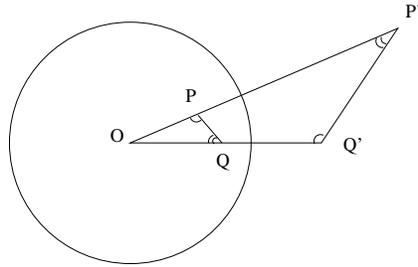


FIGURE 4. Another example of inversion for the points  $P$  and  $Q$ .

approaches infinity and therefore  $P'$  becomes infinitely far away from  $P$ . On the other hand, as  $P$  approaches the boundary of  $\omega$ , the distance  $\overline{OP}$  approaches  $r$  and therefore  $\overline{OP'} = \frac{r^2}{r} = r$ . So  $P$  and  $P'$  become very close to each other, merely on opposite sides of the boundary of  $\omega$ .

**Definition 3.8.** *The circle  $C$ , given by  $[O, r]$ , is the **circle of inversion** with center of inversion  $O$  and radius of inversion  $r$ .*

**Definition 3.9.** *A circle that does not contain its center point is a **Punctured Circle**.*

**Definition 3.10.** *If  $O$  is a point in a Euclidean plane,  $E$ , the set  $E - O$  is a **punctured plane**.*

Since we will use inversion later, here are some properties of inversion:

- If  $P \in C$ , then  $T(P) = P' = P$ .
- If  $OP < r$ , then  $OP' > r$  and conversely.
- For every  $P$ ,  $T(T(P)) = P$ .

Furthermore, for inversions across a line that is an orthogonal circle to the circle of inversion. That is, inversions across a line in the Poincaré geometry.

- Inversion preserves points.
- Inversion preserves the non-Euclidean distances between points.
- Inversion preserves lines in the Poincaré geometry.
- Inversion is conformal.
- Inversion preserve betweenness.
- Inversion preserves segments.
- Inversion preserves convexity.

We only touched on inversion at the end of our research so we will not prove that these properties are true. It is an area for future study.

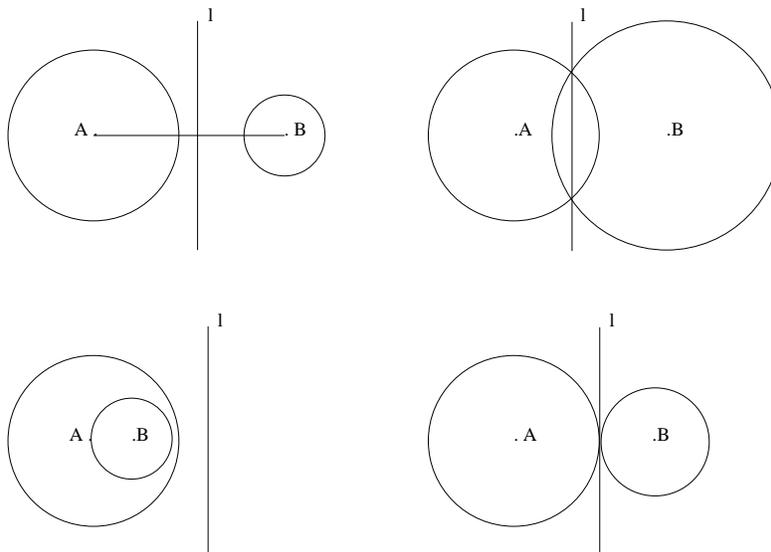


FIGURE 5. The line  $l$  is the radical axis of each of the two circles. In each case the line  $l$  is perpendicular to the line formed by  $\overline{AB}$ .

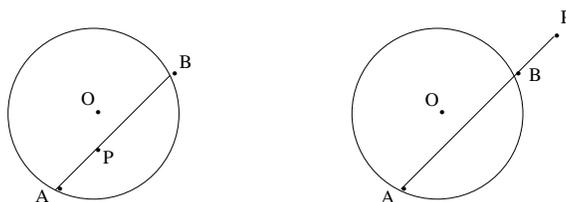


FIGURE 6. The power of point  $P$ .  $P$  can lie either inside the circle or outside.

The radical axes and powers of circles can be a useful tool in dealing with orthogonal circles. First we will describe radical axes and powers and then we will see how they relate to each other and to orthogonal circles.

**Radical Axis 1.** *The locus of points whose power with respect to two non-concentric circles are equal is a line perpendicular to the line of centers of the two circles. That line is the radical axis and contains the common chord of the two circles if they intersect and the common tangent if they touch.*

For the four examples in Figure 5, line  $l$  is the radical axis of the two circles.

**Power 1.** *Let  $PO$  is the Euclidean distance between the point  $P$  and the center of the circle,  $O$ . For any point,  $P$ , in the plane and any circle  $[O, r]$ , the number  $PO^2 - r^2$  is the power of  $P$  with respect to the given circle.*

**Power 2.** *Let the points  $A$  and  $B$  be on the circumference of the circle  $[O, r]$  and  $PA$  and  $PB$  be the Euclidean distances between  $P$  and  $A$  and  $P$  and  $B$ , respectively. For any point,  $P$ , in the plane and any circle  $[O, r]$ , the number  $PA \cdot PB$  is the power of  $P$  with respect to the given circle.*

Some things to note about the power of a point,  $P$ :

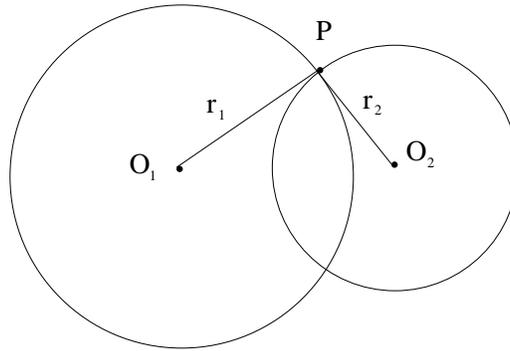


FIGURE 7. Radii drawn in from centers to point of intersection, P.

- If P is in the exterior of a circle, the power is positive.
- If P is on the circumference of a circle, the power is zero.
- If P is in the interior of a circle, the power is negative.

From a previous theorem the power is also the product of  $PA$  and  $PB$ . The line  $AB$  is any line intersecting the circle and point  $P$ . This can be expressed as:  $\text{Power} = PO^2 - r^2 = PA \cdot PB$ . See Figure 6.

The following theorem is from Wolfe, page 234 [8] and connects the orthogonality of two circles with the power of each center with respect to the other circle.

**Orthogonality of two Circles 1.** *If two circles are orthogonal, the square of the radius of each is the power of its center with regard to the other. Conversely, if the square of the radius of one circle is the power of its center with regard to another, the two circles are orthogonal.*

We will use the notation in Figure 7, where we have a circle with center  $O_1$  and radius  $r_1$  and a circle with center  $O_2$  and radius  $r_2$  intersecting at point  $P$ , to restate the theorem and prove the theorem.

**Orthogonality of two Circles 2.** *Two circles are orthogonal if and only if*

$$\overline{O_1O_2}^2 - r_1^2 = r_2^2$$

$$\overline{O_1O_2}^2 - r_2^2 = r_1^2.$$

*Proof.* Let

$$\overline{O_1O_2}^2 - r_1^2 = r_2^2$$

$$\overline{O_1O_2}^2 - r_2^2 = r_1^2$$

Then

$$\overline{O_1O_2}^2 = r_1^2 + r_2^2$$

Let the two radii be drawn to point  $P$  as in Figure 7. Then  $\angle O_1PO_2$  is a right angle and hence the two circles are orthogonal.

For the converse proof:

Let the two circles be orthogonal then

$$\overline{O_1O_2}^2 = r_1^2 + r_2^2$$

and hence

$$\overline{O_1O_2}^2 - r_1^2 = r_2^2$$

and

$$\overline{O_1O_2}^2 - r_2^2 = r_1^2$$

□

Since the radical axes and powers have interesting results when applied to orthogonal circles, we can use them to help us use the Poincaré model.

#### 4. TRIANGLE GEOMETRY

Now we will consider some properties of triangles in Hyperbolic Geometry using the Poincaré model. We will first find some laws of right triangles and then use those to derive results for any generic triangle in Hyperbolic Geometry.

**4.1. Hyperbolic Sine, Hyperbolic Cosine, and Hyperbolic Tangent.** The hyperbolic sine, hyperbolic cosine, and hyperbolic tangent are comparable to the sine, cosine, and tangent in Parabolic (Euclidean) Geometry; hence the "hyperbolic" in front of each term. So we will be using them extensively in our treatment of triangles. We will also need to use them to relate Hyperbolic distance to Euclidean distance. Recall that hyperbolic sine, hyperbolic cosine, and hyperbolic tangent are defined by the following,

$$\begin{aligned}\sinh(x) &= \frac{e^x - e^{-x}}{2} \\ \cosh(x) &= \frac{e^x + e^{-x}}{2} \\ \tanh(x) &= \frac{e^x - e^{-x}}{e^x + e^{-x}}.\end{aligned}$$

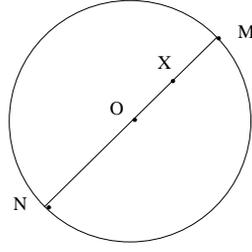
First we note some of the basic properties of hyperbolic sines and hyperbolic cosines that can easily be proved using their definitions,

$$\begin{aligned}\cosh^2(x) - \sinh^2(x) &= 1 \\ 1 - \tanh^2(x) &= \frac{1}{\cosh^2(x)} \\ \coth^2(x) - 1 &= \frac{1}{\sinh^2(x)} \\ \sinh(x \pm y) &= \sinh x \cosh y \pm \cosh x \sinh y \\ \cosh(x \pm y) &= \cosh x \cosh y \pm \sinh x \sinh y.\end{aligned}$$

Since  $\cosh^2(x) - \sinh^2(x) = 1$  is used later on in the paper we will prove it explicitly.

#### Hyperbolic Trigonometry Identity 1.

$$(2) \quad \cosh^2(x) - \sinh^2(x) = 1$$

FIGURE 8.  $\overline{ON} = r$  and  $\overline{OX} = a$ .

*Proof.*

$$\begin{aligned} \cosh^2(x) - \sinh^2(x) &= \left( \frac{e^x + e^{-x}}{2} \right)^2 - \left( \frac{e^x - e^{-x}}{2} \right)^2 \\ &= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{4} \\ &= 1 \end{aligned}$$

□

The other identities are proved in a similar manner using the definitions.

Consider point  $X$  on a diameter of our fundamental circle,  $\omega$  with radius  $r$  and center  $O$  so that  $\overline{OM} = \overline{ON} = r$ , as shown in Figure 8.

For simplicity, let  $a = \overline{OX}$  and  $a' = \widetilde{OX}$ . So

$$a' = \widetilde{OX} = \ln \left( \frac{OM \cdot XN}{ON \cdot XM} \right) = \ln \left( \frac{r \cdot (r + OX)}{r \cdot (r - OX)} \right) = \ln \left( \frac{r + a}{r - a} \right)$$

Then

$$e^{a'} = \frac{r + a}{r - a}$$

and

$$e^{-a'} = \frac{r - a}{r + a}$$

Therefore

$$\begin{aligned} \sinh(a') &= \frac{\left( \frac{r+a}{r-a} - \frac{r-a}{r+a} \right)}{2} = \frac{2ra}{r^2 - a^2} \\ \cosh(a') &= \frac{\left( \frac{r+a}{r-a} + \frac{r-a}{r+a} \right)}{2} = \frac{r^2 + a^2}{r^2 - a^2} \\ \tanh(a') &= \frac{\left( \frac{r+a}{r-a} - \frac{r-a}{r+a} \right)}{\left( \frac{r+a}{r-a} + \frac{r-a}{r+a} \right)} = \frac{2ra}{r^2 + a^2} \end{aligned}$$

Now we know how to find the hyperbolic trig functions for points on a diameter of the fundamental circle. This will be useful when we look at right triangles because we will use inversion to move two sides of a right triangle so that they are portions of diameters of  $\omega$ .

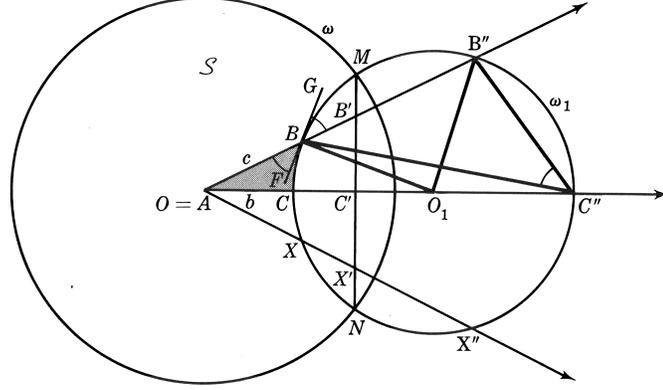


FIGURE 9. Any right triangle  $ABC$  with right angle at vertex  $C$ , inverted so that its vertex  $A$  is at the center of the circle.

**4.2. Properties of Right Triangles in Hyperbolic Geometry.** In order to derive a version of the Pythagorean Theorem for right triangles in Hyperbolic Geometry, we will consider the right triangle  $ABC$  in Figure 9. Because inversion maintains angle measure and length, we can move any right triangle so that one of its vertices lies at the center of the fundamental circle by inverting one point over a line chosen so that its inverse point lies on the center of the circle. This makes two of the edges lie on diameters of the circle and therefore appear to us as straight lines. Use the following notation:

$$a = \widetilde{BC}, \quad b = \widetilde{AC}, \quad c = \widetilde{AB}$$

$$A = \widetilde{\angle A} \quad B = \widetilde{\angle B} \quad C = \widetilde{\angle C} = \pi/2.$$

The Non-Euclidean line  $\widetilde{BC}$  is part of a circle,  $\omega_1$ , that has its center on the line defined by  $AC$  and has a radius of  $r_1$ . Since  $AC$  is perpendicular to  $\omega_1$ , it is a diameter of  $\omega_1$ . The chord  $\overline{MN}$  is part of the radical axis of the two circles,  $\omega$  and  $\omega_1$ . Therefore  $\omega_1$  can be considered as a circle of inversion and  $(X, X'')$ ,  $(B, B'')$ , and  $(C, C'')$  are inverse pairs with respect to  $\omega$ . Therefore,

$$OX'' = \frac{r^2}{OX}, \quad OB'' = \frac{r^2}{OB}, \quad OC'' = \frac{r^2}{OC}.$$

Let  $X'$ ,  $B'$ , and  $C'$  lie on the chord  $\overline{MN}$  and therefore on the radical axis of  $\omega$  and  $\omega_1$ . Therefore,

$$(3) \quad OB'^2 - r^2 = O_1B'^2 - r_1^2 = BB' \cdot B'B''$$

$$(4) \quad OC'^2 - r^2 = O_1C'^2 - r_1^2 = CC' \cdot C'C''$$

$$OX'^2 - r^2 = O_1X'^2 - r_1^2 = XX' \cdot X'X''.$$

Using equation (3) we can find an expression for  $OB'$ :

$$\begin{aligned}
OB'^2 - r^2 &= BB' \cdot B'B'' \\
&= (OB - OB')(OB'' - OB') \\
&= (OB - OB')\left(\frac{r^2}{OB} - OB'\right) \\
&= r^2 - OB \cdot OB' - OB' \cdot \frac{r^2}{OB} + OB'^2 \\
OB' \left( OB + \frac{r^2}{OB} \right) &= 2r^2 \\
OB' &= \frac{2r^2}{\frac{OB^2 + r^2}{OB}} \\
&= \frac{2r^2 OB}{OB^2 + r^2} \\
&= r \cdot \tanh(OB) \\
&= r \cdot \tanh(c).
\end{aligned}$$

The same procedure can be used on equation (4) to find

$$OC' = r \cdot \tanh(b)$$

Now we will find expressions for  $BB''$  and  $CC''$ .

$$\begin{aligned}
BB'' &= OB'' - OB \\
&= \frac{r^2}{OB} - OB \\
&= \frac{r^2 - OB^2}{OB} \\
&= \frac{2r}{\sinh(c)} \\
CC'' &= OC'' - OC \\
&= \frac{r^2}{OC} - OC \\
&= \frac{r^2 - OC^2}{OC} \\
&= \frac{2r}{\sinh(b)}.
\end{aligned}$$

Since  $b$  and  $c$  are straight lines, we can use Euclidean trigonometry to find

$$(5) \quad \cos(A) = \frac{b}{c} = \frac{OC'}{OB'} = \frac{\tanh(b)}{\tanh(c)}.$$

Now note from Figure 9, based on basic properties of chords of circles and Euclidean angle properties and noting that  $\overline{BG}$  is tangent to  $\omega_1$  at  $B$  that

$$\angle BO_1B'' = 2\angle BC''B'' = 2\angle GBB'' = 2\widetilde{\angle ABC} = 2B.$$

So  $B = \frac{1}{2}\angle BO_1B''$ . We can then find an expression for  $\sin(B)$  in hyperbolic trigonometric functions,

$$(6) \quad \sin(B) = \sin\left(\frac{\angle BO_1B''}{2}\right) = \frac{BB''}{2O_1C} = \frac{BB''}{CC''} = \frac{\sinh(b)}{\sinh(c)}.$$

Similarly, we can use the same procedures as in equations (5) and (6) to find that

$$\cos(B) = \frac{\tanh(a)}{\tanh(c)}$$

and

$$\sin(A) = \frac{\sinh(a)}{\sinh(c)}.$$

Now we will obtain the equivalent of the Pythagorean theorem for hyperbolic geometry.

From the Parabolic trigonometric identity  $\cos^2(A) + \sin^2(A) = 1$  we find

$$\begin{aligned} \frac{\tanh^2(b)}{\tanh^2(c)} + \frac{\sinh^2(a)}{\sinh^2(c)} &= 1 \\ \frac{\tanh^2(b)}{\tanh^2(c)} \cdot \sinh^2(c) + \sinh^2(a) &= \sinh^2(c) \\ 1 + \sinh^2(c) &= \cosh^2(c) \cdot \tanh^2(b) + 1 + \sinh^2(a) \\ \cosh^2(c) &= \cosh^2(c) \left( \frac{\sinh^2(b)}{\cosh^2(b)} \right) + \cosh^2(a). \end{aligned}$$

Taking equation (2) and multiplying through by  $\cosh^2(b)$  and simplifying gives the following.

$$\begin{aligned} \cosh^2(c) \cdot \cosh^2(b) &= \cosh^2(c) \cdot \sinh^2(b) + \cosh^2(a) \cdot \cosh^2(b) \\ \cosh^2(c)(\cosh^2(b) - \sinh^2(b)) &= \cosh^2(a) \cdot \cosh^2(b) \\ \cosh^2(c) &= \cosh^2(a) \cdot \cosh^2(b) \\ (7) \quad \cosh(c) &= \cosh(a) \cdot \cosh(b). \end{aligned}$$

Like the Pythagorean theorem in Euclidean geometry, Equation 7 relates all of the sides of a right triangle to each other. Note that side  $c$  is the hypotenuse. Finally, we can use Equation 7 to find an expression for the tangent of a vertex.

$$\begin{aligned} \tan(A) &= \frac{\sinh(a)}{\sinh(c)} \cdot \frac{\tanh(c)}{\tanh(b)} \\ &= \frac{\sinh(a)}{\sinh(c)} \cdot \frac{\frac{\sinh(c)}{\cosh(c)}}{\frac{\sinh(b)}{\cosh(b)}} \\ &= \frac{\sinh(a) \cdot \sinh(c) \cdot \cosh(b)}{\sinh(c) \cdot \cosh(a) \cosh(b) \cdot \sinh(b)} \\ &= \frac{\sinh(a)}{\cosh(a) \sinh(b)} \\ &= \frac{\tanh(a)}{\sinh(b)}. \end{aligned}$$

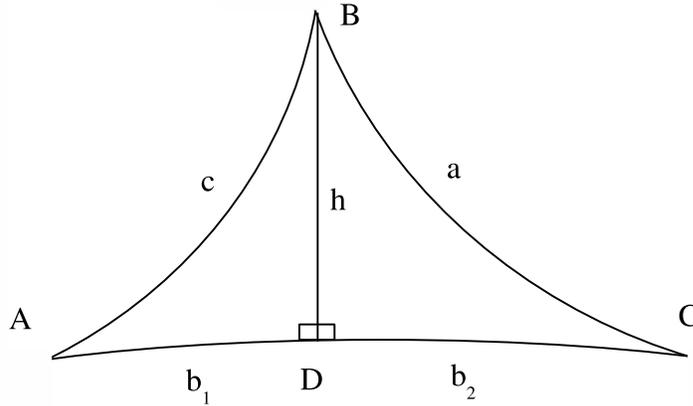


FIGURE 10. We will use triangle  $ABC$  with the altitude of  $h$  to prove the hyperbolic laws of sines and cosines.

Now we have expressions relating the various parts of a right triangle in hyperbolic trigonometry.

**4.3. Generic Triangles in Hyperbolic Geometry.** Now that we have some results for right triangles, we can divide any generic triangle into two right triangles by drawing in an altitude from one vertex to the opposite side so that it meets that side at a right angle. We will now derive the hyperbolic law of sines and the hyperbolic law of cosines. For the proofs of the hyperbolic laws of sine and cosine for a general triangle,  $\triangle ABC$ , we will want to use the relationships between the sides of the right triangles,  $\triangle ABD$  and  $\triangle BCD$  in Figure 10. Triangle  $ABC$  has an altitude of  $h$ . Let  $A$ ,  $B$ , and  $C$  denote the angle measure at the vertex as well as name the vertex, while  $a$ ,  $b_1$ ,  $b_2$ , and  $c$  represent the side lengths and  $h$  is an altitude. Let  $b$  be the distance  $b_1 + b_2$ .

**Theorem 4.1.** *The hyperbolic law of sines is the following relationships:*

$$(8) \quad \frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C}.$$

*Proof.* Using hyperbolic trig identities, we can rewrite the terms of Equation 8 as follows, where  $h$  is the altitude of the triangle in Figure 10.

$$\frac{\sinh a}{\sin A} = \sinh a \frac{\sinh c}{\sinh h} = \frac{\sinh c}{\frac{\sinh h}{\sinh a}} = \frac{\sinh c}{\sin C}.$$

Similarly, draw an altitude  $j$  in Figure 10 that starts at vertex  $A$  and intersects side  $a$  at a right angle. Now,

$$\frac{\sinh c}{\sin C} = \frac{\sinh c}{\frac{\sinh j}{\sinh b}} = \frac{\sinh b}{\frac{\sinh j}{\sinh c}} = \frac{\sinh b}{\sin B}.$$

Therefore,

$$\frac{\sinh a}{\sin A} = \frac{\sinh c}{\sin C} = \frac{\sinh b}{\sin B}.$$

□

**Theorem 4.2.** *The hyperbolic law of cosines relates the side lengths of a generic triangle  $\triangle ABC$ , with side lengths  $a$ ,  $b$ , and  $c$  in hyperbolic geometry. The equation of the hyperbolic law of cosines is the following,*

$$(9) \quad \cosh a = \cosh b \cosh c - \sinh b \sinh c \cos A.$$

*Proof.* Referring to Figure 10 and using our results on right triangles we can see that

$$\begin{aligned} \cosh a &= \cosh b_2 \cosh h \\ &= \cosh(b - b_1) \cosh h \\ &= (\cosh b \cosh b_1 - \sinh b \sinh b_1) \cosh h \\ &= \cosh b (\cosh b_1 \cosh h) - \sinh b \sinh b_1 \cosh h \\ &= \cosh b \cosh c - \sinh b \sinh b_1 \left( \frac{\cosh c}{\cosh b_1} \right) \\ &= \cosh b \cosh c - \sinh b \sinh c \left( \frac{\sinh b_1 \cosh c}{\cosh b_1 \sinh c} \right) \\ &= \cosh b \cosh c - \sinh b \sinh c \left( \frac{\tanh b_1}{\tanh c} \right) \\ &= \cosh b \cosh c - \sinh b \sinh c \cos A. \end{aligned}$$

So we have proved Equation 9, the hyperbolic Law of Cosines.

□

Therefore, we can see that the Hyperbolic Law of Sines and Hyperbolic Law of Cosines also exist in Non-Euclidean geometry.

## 5. CONCLUSION

Hyperbolic geometry is a useful tool in mathematics and physics. Einstein used it to help develop his theory of relativity; it is also a useful model for astrophysics. Hyperbolic geometry is also an interesting topic from a theoretical view. The Poincaré model of hyperbolic geometry provides a new way to consider geometry; we can see where Euclidean and Hyperbolic geometry parallel each other and how they differ.

This paper explored some of the attributes of Hyperbolic geometry using the Poincaré model, particularly the properties of triangles. Most important is the understanding of how Euclidean relations, such as the Law of Cosines, Law of Sines, and the Pythagorean Theorem, translate into relationships between sides and angles of triangles in Hyperbolic Geometry.

There are many more interesting specifics in this topic that could be pursued further. The radical axis and powers of orthogonal circles have the potential to provide many more interesting results. The families of orthogonal circles, or coaxial

systems, in addition to being aesthetically pleasing, might reveal more interesting results. In conclusion, it is fascinating to explore Non-Euclidean geometries and see how familiar, Euclidean properties change to work in a Non-Euclidean space.

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