FRACTAL CURVES

CHELLE RITZENTHALER

Abstract. Fractal curves are employed in many different disciplines to describe anything from the growth of a tree to measuring the length of a coastline. We define a fractal curve, and as a consequence a rectifiable curve. We explore two well known fractals: the Koch Snowflake and the space-filling Peano Curve. Additionally we describe a modified version of the Snowflake that is not a fractal itself.

1. Introduction

"Hike into a forest and you are surrounded by fractals. The inexhaustible detail of the living world (with its worlds within worlds) provides inspiration for photographers, painters, and seekers of spiritual solace; the rugged whorls of bark, the recurring branching of trees, the erratic path of a rabbit bursting from the underfoot into the brush, and the fractal pattern in the cacophonous call of peepers on a spring night."

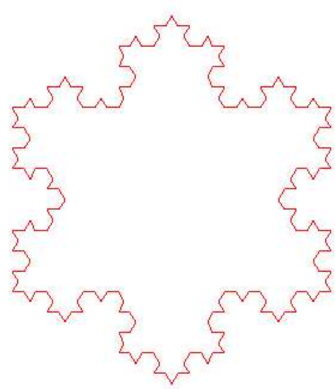

Figure 1. The Koch Snowflake, a fractal curve, taken to the 3rd iteration.

In his book "Fractals," John Briggs gives a wonderful introduction to fractals as they are found in nature. Figure 1 shows the first three iterations of the Koch Snowflake. When the number of iterations approaches infinity this figure becomes a fractal curve. It is named for its creator Helge von Koch (1904) and the interior is also known as the Koch Island. This is just one of thousands of fractal curves studied by mathematicians today.

This project explores curves in the context of the definition of a fractal. In Section 3 we define what is meant when a curve is fractal. This definition is found in Claude Tricot's book "Curves and Fractal Dimensions[1]" and the principles of this definition are continually used throughout the proofs and discussions of this report. In studying the definition of a fractal curve, it is necessary to understand the concept of a curve that is "nowhere rectifiable"(Section 3). To do this we must first fully grasp the concept of "rectifiability." In Section 2 we state the definition of this term and discuss its properties. Then following Tricot's process we negate it to create a nonrectifiable curve.

When we negate the properties of rectifiability and express this negation everywhere on the curve we create a curve that is nowhere rectifiable. After having defined "nowhere rectifiable," we then move into the definition of a fractal curve. We introduce and discuss the concept of "homogeneous,"the second and final attribute of a fractal.

In order to understand fractal curves and place the definition into context, Section 4 presents the studies of two fractal curves in depth. The Koch Snowflake curve and the space-filling Peano curve are the two curves used as examples to illustrate the properties of a fractal curve.

It is somewhat fascinating that the Peano curve actually meets all the requirements to be a fractal. When first created it was thought to be a paradox because an infinite curve, thought to be one-dimensional by nature, was creating a two-dimensional figure. Even so, it satisfies all the conditions to be a fractal curve.

We will also be revisiting the definition of a rectifiable curve in the discussion of a modified Snowflake curve (Section 5). In changing the sizes of the constructed triangles in each iteration we will create a non-rectifiable, non-fractal curve.

2. Rectifiability

In the definition of fractal curve(Section 3), one finds the term "nowhere rectifiable." To fully understand this term, it is imperative that we first understand the term **rectifiable**. For this purpose we present an overview of what it means for a curve to be rectifiable. Informally, for a point on a curve to be rectifiable, we say that it is "indistinguishable from a segment of a straight line in a neighborhood of this point." To characterize this concept, Tricot proposed four properties (All definitions, explanations and figures concerning rectifiability were taken from Tricot[1] 73 through 75).

2.1. **The Four Properties of Rectifiability.** In his book, Tricot describes four properties to define rectifiability curve at a point. A curve where all points on the curve are rectifiable is itself rectifiable. Let x_0 be a point on the curve Γ, where $x_0 = \gamma(t_0)$ for some t_0 in the interval I. We say $\gamma : I \to \Gamma$ is a parameterization of Γ over interval I.

P_1 : *There exists a tangent* $T(x_0)$ *at* x_0.

If a tangent does not exist at the point x_0 then the curve, in the area of that point, must not look like a line. Hence the point must be distinguishable from the rest of the line. For example, at the vertex of the graph $f(x) = |x|$, there is no tangent at $x = 0$ and it is very distinguishable from the points around it.

For the second property we use the concept of a local cone. If we let $\epsilon > 0$, when the curve is "smooth" near x_0 we can find a set of points on Γ whose distance to x_0 is less than ϵ and that lie within a cone with vertex x_0 and angle θ. The smoother the curve, the smaller the angle of the cone. So we let $\theta_\epsilon(x_0)$ define the minimal angle containing the whole curve within ϵ of x_0. We say that $\theta_\epsilon(x_0) = \pi$ if the cone does not exist, because we are claiming that no angle can contain the curve with the given radius of ϵ. Hence the function $\theta_\epsilon(x_0)$ is defined at every point on Γ and is always within $[0, \pi]$. As ϵ decreases, the function also decreases, so that the function has a limit as ϵ tends to zero. We can see this in Figure 2.

So our second property is:

$$P_2 : \lim_{\epsilon \to 0^+} \theta_\epsilon(x_0) = 0.$$

Next we will explore the relationship between the arc length and distance between x_0 and an arbitrary point on Γ, x. Recall that if Γ is a straight line, then the arc length, $L(x_0 \frown x)$, is equal to the distance between the points. For a rectifiable curve, as x approaches x_0 along Γ the ratio between $L(x_0 \frown x)$ and the distance between the points behaves as follows:

$$P_3 : \lim_{x \xrightarrow{\Gamma} x_0} \frac{L(x_0 \frown x)}{\mathrm{dist}(x_0, x)} = 1.$$

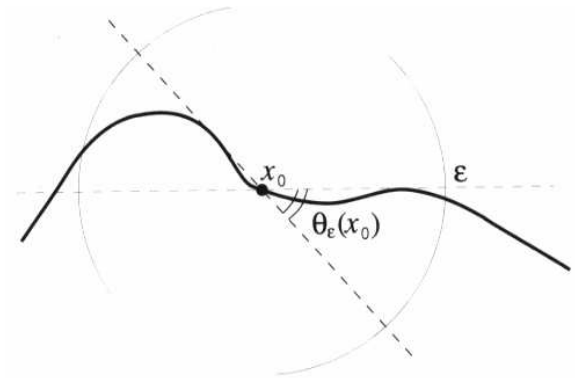

FIGURE 2. The function $\theta_\epsilon(x_0)$ as seen on an arbitrary curve. The cone is represented by the dashed lines and x_0 is its vertex, ϵ the length of its sides (figure from Tricot, page 72).

We can see in the case where the curve is a line that the ratio in P_3 will always be equal to 1. Thus a rectifiable curve, which looks like a line as x tends to x_0, will have a ratio between arc length and distance that approaches 1 as the curve more closely resembles a line.

The fourth and final property of rectifiability has to do with a local convex hull. Recall that a convex hull is the smallest convex set containing a given curve or set. Given $x_0 \frown x$, a subarc of Γ, let $K(x_0 \frown x)$ be the convex hull of the arc $x_0 \frown x$ on Γ.

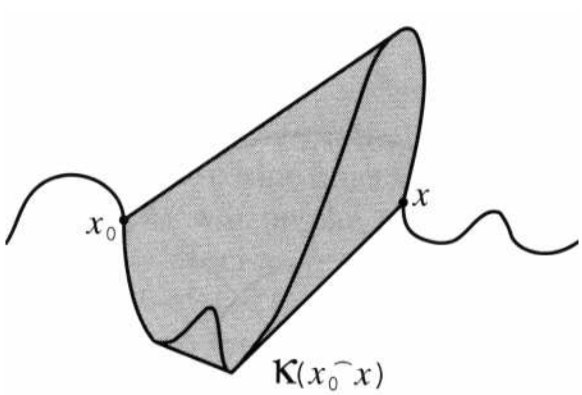

FIGURE 3. We can see how the area decreases greatly where the curve is less chaotic as x approaches x_0 (figure from Tricot, page 73).

Let $A(K(x_0 \frown x))$ be the area of the convex hull. We can deduce that the area is null only if $x_0 \frown x$ is a line segment. So as the arc becomes more linear, the area decreases and similarly the area increases if the arc becomes more chaotic.

$$P_4 : \lim_{x \to 0} \frac{A(K(x_0 \frown x))}{\mathrm{dist}(x_0, x)^2} = 0.$$

We see that in the case of a line, the convex hull between any two points will always be 0. Recall that a rectifiable curve is one that looks like a line as x tends to a point x_0 on the curve. We see this occurring in the behavior of the area of the convex hull as x tends to x_0. The area will be approaching zero, which is also the same as that of a line since a line has no area. In Figure 3, the convex hull between x_0 and x on an arbitrary curve is shown. It can be easily seen in this example how the curve begins to resemble a line more closely as x takes on positions very close to the point x_0, and therefore the area of the convex hull will approach 0 as x tends to x_0.

2.2. **Relating the Properties.** The four properties presented above are not independent. There are relationships between the properties which will be explained in this section.

Theorem 2.1. *P_1 and P_2 are equivalent.*

Proof. To prove this we first assume P_1 is satisfied, then at x_0 on Γ there exists a tangent to Γ. Let $T(x_0)$ be the tangent line at x_0 and let $C(x_0, x)$ be the chord of the curve between x and x_0. For every angle $\phi > 0$, you can find an ϵ such that $\mathrm{dist}(x_0, x) \leq \epsilon$ implies $\angle(C(x_0, x), T(x_0, x)) \leq \phi$. In other words, the angle between the chord, $C(x_0, x)$, and the tangent line is less than or equal to ϕ if you are within a certain distance of x_0, namely ϵ. We have now included every chord, $C(x_0, x)$, in some cone with a vertex at x_0 and $\theta_\epsilon < 2\phi$. But since our choice of ϕ was arbitrary, it may be as small as we wish. Thus P_2 holds for x_0.

For the second half of the proof, we assume that P_2 is satisfied at x_0. Let C_ϵ be the cone with vertex x_0, radius of ϵ, θ_ϵ being the angle containing the curve, and let D_ϵ be the axis of this cone. If $\epsilon' < \epsilon$, then $C_{\epsilon'} \subseteq C_\epsilon$. This is easily seen: if two cones share the same vertex, x_0, and one has a radius less than that of the second, then the first cone will be contained within the second. If this is true, then the angle created by the two axes D_ϵ and $D_{\epsilon'}$ will be contained within θ_ϵ, thus $\epsilon' < \epsilon \Rightarrow \angle(D_{\epsilon'}, D_\epsilon) \leq \theta_\epsilon$. Therefore as ϵ tends to 0, the line D_ϵ will tend to some fixed line D_0. But $\mathrm{dist}(x_0, x) \leq \epsilon$ implies that the angle between the chord and the axis is contained within the defining angle of the cone, $\angle(C(x_0, x), D_\epsilon) \leq \theta_\epsilon$. Therefore the chord $C(x_0, x)$ will also tend to D_0 as ϵ tends to 0, and thus D_0 is tangent to the curve at x_0 and P_1 is satisfied. □

Theorem 2.2. *P_3 implies P_4.*

Proof. Suppose that P_3 is satisfied. Then $\lim_{x\to x_0} \frac{L(x_0\frown x)}{\text{dist}(x_0,x)} = 1$ as x tends to x_0. From a previous result[1] not explored in this report, we know the following about convex hulls:

$$A(K(x_0 \frown x)) \leq L(x_0 \frown x)^{\frac{3}{2}} \sqrt{L(x_0 \frown x) - \text{dist}(x_0, x)}.$$

Where L is the arc length. It is easily seen that:

$$\frac{A(K(x_0 \frown x))}{(\text{dist}(x_0,x))^2} \leq \Big(\frac{L(x_0 \frown x)}{\text{dist}(x_0,x)}\Big)^{\frac{3}{2}} \sqrt{\frac{L(x_0 \frown x)}{\text{dist}(x_0,x)} - 1}.$$

Therefore, since the ratio $\frac{L(x_0\frown x)}{\text{dist}(x_0,x)}$ tends to 1, the two sides of this inequality will tend to 0 as x tends to x_0. □

One might wonder whether or not P_4 implies P_3. The following is a counter-example presented by Tricot (page 76).

We examine the function where $z(t) = t^2 \cos \frac{1}{t^2}$ for all $t > 0$ and $z(0) = 0$. We want to show that the $\lim x \to x_0 \frac{A(K(x_0\frown x))}{\text{dist}(x_0,x)^2} = 0$ is true at a point x on $z(t)$ but that $\lim x \to x_0 \frac{L(x_0\frown x)}{dist(x_0,x)} \neq 1$ at that point.

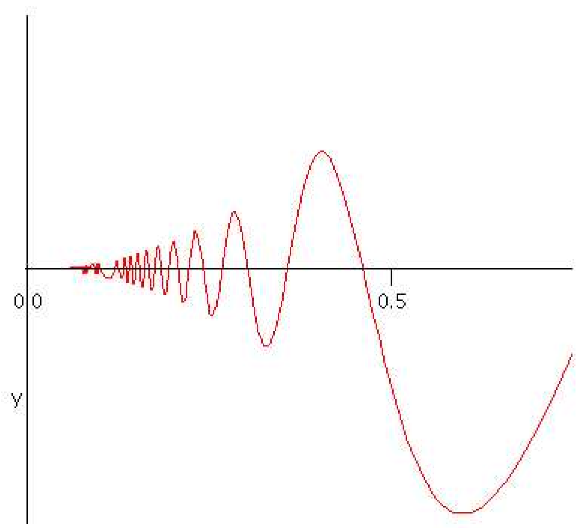

FIGURE 4. Graph of the function $y = z(t)$

Figure 4 shows us a close view of the function $z(t)$. If we look at the point $t = 0$ we see that it is a good candidate for satisfying P_4 but not P_3. For all points x with the coordinates $(t, t(z))$, the area of the convex hull of the arc $0 \frown x$ is less than the area (t^3) of the triangle with vertices $(0, 0)$, (t, t^2), and $(t, -t^2)$. Because the $\text{dist}(0 \frown x) \geq t$ then $(\text{dist}(0 \frown x))^2 \geq t^2$. Thus we can observe the following

$$\begin{aligned} A(K(0 \frown x)) &\leq t^3 \\ \frac{A(K(0 \frown x))}{t^2} &\leq t \\ \frac{A(K(0 \frown x))}{\text{dist}(0 \frown x)} &\leq t. \end{aligned}$$

Hence, P_4 is satisfied at 0.

Now we will see if the property P_3 is satisfied at the point 0. Looking at the section of our curve with abscissas values between $\frac{1}{\sqrt{2k\pi}}$ and $\frac{1}{2(k+1)\pi}$, we see that the length of it is larger than $z(\frac{1}{\sqrt{2k\pi}}) = (\frac{1}{\sqrt{2k\pi}})^2 \cos(\frac{1}{(\frac{1}{\sqrt{2k\pi}})^2} = \frac{1}{2k\pi})$. Given an integer, n we look at the sum of these lengths and see that it converges, as does the harmonic series. Therefore every arc $0 \frown x$ of the curve is of infinite length. Hence the curve does not satisfy P_3 at the point 0.

Now the term rectifiability has a definition, and we see how its properties are related to each other. The next step is to move into the definition of "nowhere rectifiable." In order to do this, we need to negate the properties of rectifiability. Once this is done, expressing this negation on the entire curve moves the curve from being nonrectifiable to being "nowhere rectifiable," one of the two characteristics required in the definition for a curve to be fractal.

3. Fractal Curves

We have defined what it means to be rectifiable. Now we can explore the idea of a curve that is nowhere rectifiable, a fractal curve. The following is the definition of a **fractal curve** given by Tricot.

Definition 3.1 (Fractal Curve). *A fractal curve is a curve with two main attributes: it is nowhere rectifiable and homogeneous everywhere on the curve.*

Thus we define these two terms in depth.

3.1. **Nowhere Rectifiable.** Having defined rectifiability, we can use its properties to define one of the attributes of a fractal curve: nowhere rectifiable[1]. The four properties of rectifiability allow us to define a curve, Γ, as locally rectifiable near a point, x_0. If we simply negate these statements then we have expressed local nonrectifiability. However if we also apply these negations to the entire curve the result is the three properties that define nowhere rectifiable, and the first attribute of a fractal curve.

The first property is:

Q_1 : *Every subarc of* Γ *longer than a single point is of infinite length.*

This property arises from negating P_3, though not directly. Negating this property we have $\lim_{x\to x_0} \frac{L(x_0 \frown x)}{dist(x_0,x)} \neq 1$. However, we cannot actually use the value $L(x_0 \frown x)$ because it becomes an infinite length. Thus we replace the concept of a negation of P_3 with $L(x_0 \frown x) = +\infty$. Let us think about this in relation to the Koch Snowflake. Every "line segment" is really a small version of the curve on which it lies. That is, if we were to magnify the segment what we would see appears to be the figure we took the line segment from. This is true for any magnification of the curve, and so even though a given segment may appear finite, it is actually of infinite length.

Our second property comes directly from the negation of P_2:

$$Q_2 : \textit{At every point } x_0 \textit{ of the curve, } \theta_\epsilon(x_0) \textit{ does not converge to 0,}$$
$$\textit{that is } \lim_{x\to x_0} \sup[\theta_\epsilon(x_0)] > 0.$$

By negating P_2, we are saying that the angle θ of the local cone with vertex x_0 does not converge to 0 as the radius of the cone decreases.

Our third property follows from the negation of P_4:

$$Q_3 : \textit{At every point } x_0 \textit{ of the curve, } \lim_{x\to x_0} \frac{A(K(x_0 \frown x))}{\operatorname{diam}(x_0,x)^2} \neq 0.$$

Initially there seems to be a slight discrepancy between the definitions found in rectifiability and nowhere rectifiable. One uses the distance between points and one the diameter. What is the difference, and is it significant to our definition? When a curve is rectifiable, the choice between $\operatorname{diam}(x_0, x)$ and $\operatorname{dist}(x_0, x)$ is trivial because their ratio approaches 1 nearly everywhere. However, with fractal curves the function of the distance can have very irregular behavior as x approaches x_0 due to the fact that the curves themselves are often very erratic. Thus Tricot prefers to use the diameter of the convex hull described on the subarc. It is an easier value to work with because it has the advantage of being a constantly increasing function of arcs. That is for every point y of $x_0 \frown x$, $\text{diam}(x_0 \frown y) \leq \text{diam}(x_0 \frown x)$. As opposed to the distance function, which may swing rapidly between 0 and very large numbers due to the nature of fractal curves[1].

Now we have three properties that define a curve that is nowhere rectifiable. The second condition for a fractal curve is that it is homogeneous, this condition is defined by the fourth property of a fractal curve.

3.2. **The Fourth Property.** Before introducing Q_4, we will present a function that is used in the property itself.

In order to create a property regarding the arc length of a curve, we look at an arbitrary segment of the curve $[a, b]$. On this segment, we define a function whose input is a given a point t on the interval and returns an output of the length of a small subsegment of the interval on which is this point is found. Once we define this function, we can then look at its behavior as the length of the subsegments change.

Let a curve be continuous on $[a, b]$. For all t in $[a, b]$ and all τ in $(0, b - a/2]$ we have the function,

$$T(t,\tau) = \begin{cases} \text{size}(\gamma(a) \frown \gamma(a+2\tau)), & \text{if } t-\tau \leq a \\ \text{size}(\gamma(t-\tau) \frown \gamma(t+\tau)), & \text{if } a \leq t-\tau < t+\tau \leq b \\ \text{size}(\gamma(b-2\tau) \frown \gamma(b)), & \text{if } b \leq t+\tau \end{cases}$$

Where Γ is the parameterized curve and $\gamma(t)$ is a continuous function that maps values from the interval $[a, b]$ to Γ. Thus the function $T(t, \tau)$ is taking input values from the interval $[a, b]$ and giving us the corresponding local arc on which $\gamma(t)$ lies in Γ.

The first and third parts of this piecewise defined function may look a little odd, but the idea is to find the arc length from subsegments of length 2τ on $[a, b]$. For some values of t, namely those near the endpoints of $[a, b]$, centering a subsegment of length 2τ on t will cause us to take subsegments of $[a, b]$ which go outside of the interval endpoints a and b. Thus for these cases, instead of centering around t, we take the subsegment of $[a, a+2\tau]$ or $[b-2\tau, b]$, depending on which endpoint t is near. Thus we are always finding arc lengths of subarcs containing $\gamma(t)$ corresponding to a subsegment of length 2τ from $[a, b]$ that contains t.

Thus, using this function we can define the fourth property of a fractal curve. We say that a curve is **fractal** if the first three properties discussed above hold and the homogeneous property holds true

Q_4 : *The ratio* $\dfrac{T(t,\tau)}{\tau} \to \infty$ *uniformly with respect to* t *when* $\tau \to 0$.

This property is comparing, using a ratio,the length of the subsegments of the parameterized curve to the variable τ, which is determining their length. It is easy to see that if $T(t, \tau)$ remains sufficiently positive, or decreases at a slow enough rate than that of τ as it approaches zero, then the property holds true. Thus, returning to an older statement, if the length of any subsegment of the curve is of infinite length, this will be true.

Another way to approach and state this property is to say that for A chosen to be as large as we wish, we can find a $\tau_0 > 0$ such that for all $\tau \leq \tau_0$ and all $t \in [a, b]$, the lengths of the subarcs are greater than or equal to $A\tau$. That is,

$$T(t, \tau) \geq A\tau.$$

This implies, as stated above, that for any subinterval of $[a, b]$, the average corresponding local arc length on Γ, $T(t, \tau)$, tends to infinity. Thus any subarc of the curve Γ is of infinite length and Γ is nowhere rectifiable. Thus, because for all $\tau \leq \tau_0$, all the subarcs of measure 2τ satisfy the inequality $2\tau \geq A\tau$, Tricot's definition of a curve that is homogeneous.

4. Examples of Fractal Curves

In order to understand the properties of fractal curves and nowhere rectifiability we explore two curves: the Koch Snowflake curve and the Peano Curve. The largest qualitative difference you will notice in these two curves is that the Snowflake is continually building out, creating larger islands in each iteration (though this growth is bounded by the circumcircle of the original triangle). On the other hand, the Peano curve is not increasing an enclosed area, in fact it is filling a two-dimensional space, a diamond shape, as the iterations continue toward infinity. Munkres[2] presents a proof of another Peano curve showing that it is indeed space-filling. Since the two curves are analogous we reference the reader to this proof that the curve addressed here is also a space-filling curve. This fact makes the Peano curve a particularly interesting fractal to study, and the reason it was considered a paradox for many years after its creation.

4.1. **The Koch Snowflake.** The Koch snowflake is a fractal curve. In this section we show this curve to be nowhere rectifiable and homogeneous.

First we examine how the Koch Snowflake violates the first two properties of rectifiability.

Theorem 4.1. *There does not exist a point of tangency on the Koch Snowflake curve and the angle between any two points on the curve containing the arc between them will never be smaller than* $\frac{\pi}{6}$.

Another way to think of this theorem is to view it in terms of the function used in P_2 of rectifiability. That is the function $\theta_\epsilon(x_0) \geq \frac{\pi}{6}$ (Section 2).

Proof. Let x_0, and x be an arbitrary points on the Koch Snowflake curve, creating arc $x_0 \frown x$. We claim that if x_0 and x are not on the same line segment, then they were at some iteration k, with the exception of the case where x_0 and x were on different sides of the 0th iteration, the equilateral triangle. So we move point x to a new point q such that on the kth iteration, x_0 and q lie on the same line segment of length l.

On the $(k+1)$st iteration an equilateral triangle is erected on the segment with side lengths of $\frac{l}{3}$ and each angle of the triangle is $\frac{\pi}{3}$. We draw line j from the point x_0 to the peak of the triangle, which we call y_1 and we call the intersection with the original segment n and m which we can see in Figure 5.

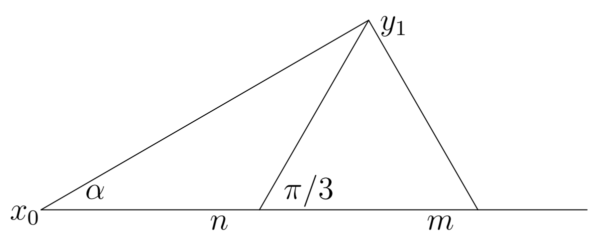

Figure 5. The first iteration of the Koch Snowflake. Joining the peak y_1 of the triangle with the original endpoint, x_0, by line j we have created the isosceles triangle $\triangle x_0 y_1 n$ whose base angles are $\frac{\pi}{6}$.

The lengths $\overline{x_0 n}$ and $\overline{n y_1}$ are equal because in our construction, we constructed an equilateral triangle over the middle third of the original segment, so each line is one third the length of the original segment. Hence $\triangle x_0 y_1 n$ is an isosceles triangle. We can find the base angles of the triangle, because the opposite exterior angle to this triangle is one of the angles of the equilateral triangle and therefore is $\frac{\pi}{3}$, and since our base angles are equal, thus we have $\alpha = \frac{\pi}{6}$. Therefore we have two points of the curve, y_1 and n, that are contained by an angle no smaller that α.

Now we continue to iteration $(k+2)$. In this iteration an equilateral triangle is erected on each segment with side lengths one-third of the segment, $\frac{l}{9}$. Again, each angle of the triangle is $\frac{\pi}{3}$ and we will label peak y_2 on the triangle erected on $\overline{x_0 n}$, with base points s and t, and draw line k from x_0 to y_2. Using an analogous argument as above, $\angle s x_0 y_2 = \frac{2\pi}{3}$ and $\overline{x_0 s} = \overline{s y_2}$. Therefore $\triangle x_0 y_2 x$ is isosceles with base angles that sum to $\frac{\pi}{3}$, and hence each are equal to $\frac{\pi}{6}$. Thus $\angle y_2 x_0 s = \frac{\pi}{6} = \alpha$ and the points two points on the curve, s and y_2, are contained by an angle no smaller than $\frac{\pi}{6}$.

We can see that this argument is applicable to every iteration of the Koch snowflake. So as the number of iterations continues towards infinity, there curve contains an infinite pair of points that cannot be contained by an angle smaller than $\frac{\pi}{6}$. Further more, as a variant point x tends toward x_0 on the curve it is not true that the cone angle, $\theta_\epsilon(x_0)$, does not tend to zero. This violates the second property of rectifiability. □

Now we address the third and fourth properties of rectifiability. For these properties to be satisfied the length of any subarc curve must be finite. Therefore, if we can prove that the distance between any two arbitrary points on the curve tends to infinity, then the curve will violate P_3 and P_4 (Section 2).

Theorem 4.2. *Every subarc of the Koch Snowflake curve has infinite length.*

Proof. Given the Koch Snowflake on the kth iteration, choose two arbitrary points P_1 and P_2 on the curve. If P_1 and P_2 are not on the same segment then P_1 lies on some segment RQ. Find the endpoint of the segment on which P_1 lies, Q, and consider the subsegment P_1Q as the your segment P_1P_2. In this way you have "moved" the point P_2 to Q. This does not change the arbitrary nature of our choice because in proving that $\|P_1Q\|$ is infinite then the entire segment P_1P_2 is also infinite because it contains an infinite subarc.

Because P_1 and P_2 lie on the same segment on the kth iteration, then their arc length, which is also equal to their distance, is $l \leq (\frac{1}{3})^k$, assuming the side length of the original triangle to be 1. As the iterations continue, the distance between the points does not change; however, the arc length between them does. For every iteration, the curve gains a third of the previous length, and we see the following pattern:

$$\begin{aligned} l_0 &= \left(\frac{1}{3}\right)^k \\ l_1 &= \left(\frac{1}{3}\right)^k\left(\frac{4}{3}\right) \\ l_2 &= \left(\frac{1}{3}\right)^k\left(\frac{4}{3}\right)\left(\frac{4}{3}\right) \\ l_3 &= \left(\frac{1}{3}\right)^k\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{3}\right) \\ \vdots &= \vdots \\ l_n &= \left(\frac{1}{3}\right)^k\left(\frac{4}{3}\right)^n \end{aligned}$$

where n is the number of iterations after the kth iteration and l_i is the length of the segment at the ith iteration. So as the number of iterations increases, the arc length is being multiplied by a number greater than one for each iteration, and the length is infinitely increasing. Therefore the Koch Snowflake violates the third and fourth properties of rectifiability. □

We have shown the Koch Snowflake violates all the properties of rectifiability such that it is nowhere rectifiable. In proving that it violated properties P_3 and P_4, we see that every subarc of the curve is of infinite length. Thus if we map an interval to the Snowflake, no matter the size of the subinterval which we are mapping, the corresponding subarc of the curve will be infinite. Hence if we let τ be the length of the subinterval, we can see that $\lim_{\tau\to 0}\frac{T(t,\tau)}{\tau}$ will indeed approach infinity. Therefore the Koch Snowflake is also homogeneous. Since it is a nowhere rectifiable curve and homogeneous the Koch Snowflake is, by definition, a fractal curve.

4.2. **The Peano Curve.** The Peano Curve is a space filling fractal curve created by building squares on both sides of the middle third of a line segment. We can see the first three iterations of this curve in Figure 6 below.

We prove that the Peano curve is in fact a fractal by showing a violation of the properties of rectifiability and that it is homogeneous. Like the Snowflake proof, this is done by proving that the length of any arbitrary segment is infinite.

Theorem 4.3. $\lim_{\epsilon\to 0^+}\theta_\epsilon(x_0) \neq 0$ *for all* x_0 *on the Peano curve*

Proof. Let x_0 be an arbitrary point on the Peano curve. Then x_0 became part of the curve on some segment $\overline{cd}$ created in the kth iteration.

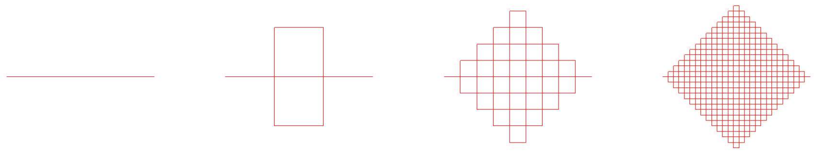

Figure 6. The initial line and first three iterations of the Peano curve (note how the Peano curve appears to be a space-filling curve). It is easy to see how the arc length of the curve grows at a fast rate towards infinity. What we prove is that this is true for every sub-arc of the curve.

As the number of iterations continues towards infinity, the segment $\overline{cd}$ will have a "smaller" version of the curve created on it and thus a diamond will form over the segment. We know that the Peano curve is a space-filling curve[2] and thus the area within ϵ of x_0 will all be filled with the curve. Thus there is no cone with a vertex at x_0 that will contain the entire curve if x_0 lies within the boundaries of the diamond. Thus the function $\theta_\epsilon(x_0) = \pi$ (Section 2) for all values of x_0 with the exception that if x_0 happens to be a corner of the diamond, in which case $\theta_\epsilon(x_0) = \frac{\pi}{2}$. In all cases we see that $\lim_{\epsilon \to 0^+} \theta_\epsilon(x_0) \neq 0$. Therefore the Peano curve violates P_2 and P_1. □

Theorem 4.4. *Every subarc of the Peano curve is of infinite length*

Proof. Take an arbitrary arc $\overline{x_0 y}$ of the Peano curve. The point x_0 joins the curve on some iteration, we call it the kth iteration, as a point on a segment $\overline{pq}$. Without loss of generality, let the endpoint q be in the direction of y. Then $\overline{x_0 q}$ is a subsegment of $\overline{x_0 y}$, thus if $\overline{x_0 q}$ is of infinite length, then so is $\overline{x_0 y}$.

After the $(k+1)$th iteration, the Peano curve squares are constructed on $\overline{pq}$. When this happens one of two cases occurs on the segment $\overline{x_0 q}$, one being that the construction occurs on a part, or all, of the segment. If this is true, then as the iterations continue toward infinity each one produces 9 segments for each previous segment and they have a length that is one-third of the preceding segment length. Thus the arc length after $(k+n)$ iterations is $L = \frac{9^{k+n}}{3^{k+n}} = \frac{3^{2(k+n)}}{3^{k+n}} = 3^{k+n}$. It is easy to see that L is increasing without bound as the number of iterations approaches infinity.

Now if the $(k+1)$th iteration does not produce a Peano curve square on $\overline{x_0 q}$ then the segment is still of finite length. We have a new question to address: is it possible for $\overline{x_0 q}$ to be so small that no Peano curve activity ever occurs on it? No, it is not. Let $\|x_0q\| = \delta$. Because δ is a constant, it is possible to find a number of the form $\frac{1}{3^m}$ such that $0 < \frac{1}{3^m} < \delta$. Because the number of iterations of the Peano curve is going to infinity and the segments have a length of $\frac{1}{3^k}$ on the kth iteration, eventually the segments will be of length $\frac{1}{3^m} < \delta$ and thus, the Peano squares will be constructed on part or all of $\overline{x_0q}$. When this occurs, we can follow a proof analogous to the above showing that $\overline{x_0q}$ is of infinite length. □

Thus we have proved that any arbitrary subarc of the Peano curve is of infinite length. This means that the Peano curve cannot satisfy the 3rd and 4th properties of rectifiability. Similarly to the Koch Snowflake, we can use this fact to show that the Peano curve is homogeneous. Since we have proved that any arbitrary subarc is of infinite length, if we let the Peano curve be our parameterized curve Γ discussed in Section 3, then the ratio $\frac{T(t,\tau)}{\tau}$ will tend to infinity as τ approaches zero. The Peano curve therefore satisfies the fourth property and is homogeneous, thus it is, by definition, a fractal curve..

5. Modified Curves

Having explored fractal curves, we now return to the idea of rectifiability. What would it look like to have curves that only satisfy one or two of the properties of nowhere rectifiability? In order to examine this question we will discuss a modified Koch Snowflake curve as well as one of the curves presented by Tricot in his book.

5.1. **The Modified Koch Snowflake.** The first curve we examine is a modification of the Koch Snowflake. Given a segment, the first iteration looks exactly like the Koch snowflake: an equilateral triangle erected on the middle third of the segment. The second iteration will erect an equilateral triangle over the middle fifth of each segment, then the third will be over the middle seventh and so on. So on the kth iteration, you are constructing an equilateral triangle of side length $\frac{1}{2k+1}$ on the middle "$(2k+1)$th" of each segment. We can see the first three iterations in Figure 7.

Theorem 5.1. *This Modified Snowflake does not violate property P_2 of rectifiability.*

Proof. Given a segment $\overline{x_0z}$, let the peaks of the triangles constructed on the first and second iterations be labeled y_1 and y_2 respectively and the intersections of the triangles' sides closest to x_0 with $\overline{x_0z}$ be m and n on the first and second iterations respectively. As we have

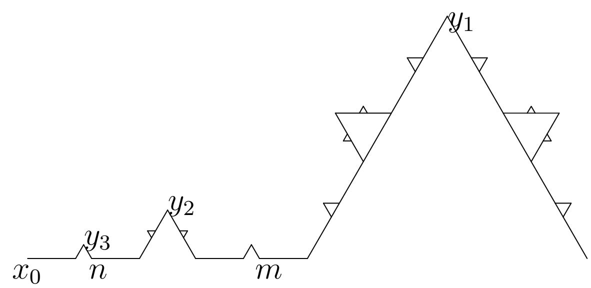

FIGURE 7. The first three iterations of the modified snowflake as described in this section. Note that compared to the Koch Snowflake, the peaks of this curve do not seem to follow any set pattern or line.

shown previously, if you draw a line from x_0 to y_1 you have created an isosceles triangle, $\triangle x_0 y_1 m$ where $\angle y_1 x_0 m = \frac{\pi}{6}$. Drawing a line from x_0 to y_2, the peak of the second iteration's triangle, we create $\triangle x_0 y_2 n$, however this triangle is not isosceles because $\overline{x_0 n} = 2\overline{n y_2}$. Since the angle between these two sides is $\frac{2\pi}{3}$ then the remaining angles will have a sum of $\frac{\pi}{3}$. Thus, because they are unequal, one angle will have a measure greater than $\frac{\pi}{6}$ and the other will be less than $\frac{\pi}{6}$. Euclid tells us that in any triangle, the smallest side lies opposite the smallest angle, thus $\angle y_2 x_0 n < \frac{\pi}{6}$. Therefore the peak of the second triangle does not fall on $\overline{x_0 y_1}$, but below it. This pattern continues to repeat as the number of iterations continues towards infinity. As the length of the sides of each triangle decreases, the height decreases proportionally. Therefore the side length at iteration k is $\frac{1}{2k+1}$, and the so the height of the erected triangle will be $\sqrt{(\frac{1}{2k+1})^2 - (\frac{1}{2(2k+1)})^2}$. Simplified, the height will be $\frac{\sqrt{3}}{2(2k+1)}$ in the kth iteration and will tend to zero as the number of iterations tends to infinity. Thus the angle containing the entire curve tends to zero as x tends to x_0. Therefore this curve does follow one property of rectifiability. □

As we have previously shown, if the curve satisfies the second property of rectifiability, it will also satisfy the first. Thus the curve satisfies both properties.

However the curve is still of infinite length because every subarc is of infinite length.

Theorem 5.2. *Every subarc of the Modified Snowflake curve is of infinite length.*

Proof. Let the length of our original segment be 1. Then the length after the first iteration is $1 + \frac{1}{3} = \frac{4}{3}$, after the second it is $\frac{4}{3} + \frac{1}{5} = \frac{6}{5}\frac{4}{3}$, and after the third it is $\frac{4}{3}\frac{6}{5} + 17 = \frac{8}{7}\frac{6}{5}\frac{4}{3}$. We see a pattern developing. On the nth iteration, the length of the curve will be $\prod_{n=1}^{\infty} \frac{2n+2}{2n+1}$. Taking the natural log of both sides, we have

$$
\begin{aligned}
L &= \prod_{n=1}^{\infty} \frac{2n+2}{2n+1} \\
\ln(L) &= \ln\Big(\prod_{n=1}^{\infty} \frac{2n+2}{2n+1}\Big) \\
\ln(L) &= \sum_{n=1}^{\infty} \ln\Big(\frac{2n+2}{2n+1}\Big)
\end{aligned}
$$

Hence, we can see that each term of the infinite sum will be $\ln \frac{2n+2}{2n+1}$. To prove that this value will diverge, we introduce a lemma.

Lemma 5.1. *For all* $x \in (0, 1)$, $\ln(1 + x) > \frac{x}{2}$.

Proof. Let $f(x) = \ln(1 + x) - \frac{x}{2}$. We note that $f(0) = 0$ and that $f'(x) = \frac{1}{2(1+x)} > 0$ for all x on $(0, 1)$. Thus $f(x)$ is strictly increasing on that interval and $f(0) = 0$, it is easily deduced that $f(x) > 0$ for all x in $(0, 1)$. Thus we can conclude the following for all x in the interval

$$
\begin{aligned}
f(x) = \ln(1 + x) - \frac{x}{2} &> 0 \\
\ln(1 + x) &> \frac{x}{2}
\end{aligned}
$$

□

Having proved the lemma, we now look at our addend term $\ln(\frac{2n+2}{2n+1})$ and rewrite it as $\ln(1 + \frac{1}{2n+1})$. Since $0 < \frac{1}{2n+1} < 1$ for all $n > 0$ we can replace the x-value in our function from the lemma $f(x)$, with the value $\frac{1}{2n+1}$ and find that $\ln(\frac{2n+2}{2n+1}) = \ln(1 + \frac{1}{2n+1}) > \frac{\frac{1}{2n+1}}{2} = \frac{1}{4n+2}$ for all $n > 0$. Thus we can make a comparison term by term as $n \to \infty$ to that of the harmonic-like series, $\frac{1}{4n+2}$. Since the sum of this diverges as $n \to \infty$ and the inequality above holds for all $n > 0$, our sum diverges as well. Since our sum is diverging, the summands ($\ln \frac{2n+2}{2n+1}$) are not approaching zero fast enough to converge. Thus the infinite product of $\frac{2n+2}{2n+1}$ will also diverge as $n \to \infty$. Therefore the length of the curve tends to infinity as the number of iterations increases. □

6. Conclusion

This report introduced us to what it means to be a fractal curve. A fractal curve is defined by two terms: nowhere rectifiable and homogeneous. In order to understand and create the properties of a nowhere rectifiable curve, we explored the properties of a rectifiable curve and then negated them. We then explored the concept of a homogeneous curve.

To place the definition of a fractal curve in context we worked with two known fractal curves. The Koch Snowflake curve and the Peano curve were proved to violate the properties of rectifiability in such a way that they were nowhere rectifiable and homogeneous. In order to further explore these properties we examined a modified iterative process of the Snowflake that rendered it no longer a fractal curve.

While these subjects were intriguing, they only scratch the surface of the study of fractal curves. One of the many topics associated with fractal curves that we did not explore here are fractal dimensions. These are dimensions used to define fractals and they allow nonintegral values.

Today fractal curves are used in research in seemingly unrelated fields such as psychology and medical research. The more we discover about fractals, the more we find them defining processes and sights in our world. The possible directions we may go in continuing beyond the bounds of this paper are truly infinite.

References

[1] Tricot,Claude: *Curves and Fractal Dimension*, Springer-Verlag, New York (1995).

[2] Munkres,James R.: *TOPOLOGY: A First Course*, Prentice-Hall, New Jersey (1975).