SIMSON'S THEOREM

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ABSTRACT. This paper is a presentation and discussion of several proofs of Simson's Theorem. Simson's Theorem is a statement about a specific type of line as related to a given triangle. The theorem has interesting implications for lines in general position, but our concern here is to examine several different methods for proving the theorem. We present one analytic proof of the theorem and the converse, three synthetic proofs of the theorem, and one synthetic proof of the converse.

1. INTRODUCTION

imson's Theorem is a statement that relates a specific set of lines with the sides and circumcircle of a given triangle. The circumcircle is the unique circle that intersects the triangle at the three vertices and no other points. Simply stated, Simson's Theorem reads: *Given a triangle $\triangle ABC$ and a point P on its circumcircle, the feet X, Y, and Z of the perpendiculars dropped from P to the lines BC, AC, and AB respectively are collinear.* The point P is called the pole and the line XYZ is the Simson Line generated by P denoted SL_P. An example of the set up of the theorem can be seen in Figure 1. Any proof of Simson's Theorem relies upon Euclid's Parallel Postulate, and the theorem does not apply in a non-Euclidean Geometry. It is worth noting that the converse of Simson's Theorem is also true, and thus the points X, Y, and Z are collinear if and only if the point P lies on the circumcircle.

Simson's Theorem is attributed, perhaps erroneously, to Robert Simson a Scottish mathematician whose main contributions to the body of mathematics seem to have been his translation of Euclid's *Elements* and his reconstruction of the lost works of Euclid and Apollonius.[1]

Evidence suggests that the true innovator behind Simson's Theorem may in fact be William Wallace, another Scottish mathematician born in the year of Simson's death. Wallace is known to have published the theorem in 1799 while no evidence exists to support Simson's having studied or discovered the lines that now bear his name.[2]

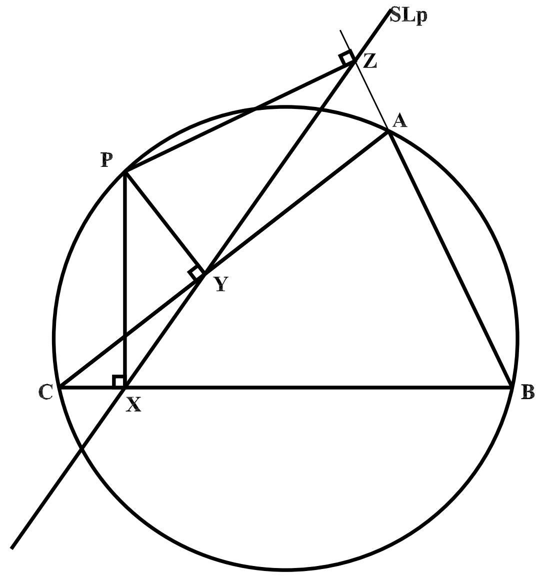

FIGURE 1. This Figure contains an example setup of Simson's Theorem.

There are several proofs available for the theorem, each having its own set of benefits and drawbacks. In this paper we present four proofs of Simson's Theorem and one of its converse. We discuss the background needed to understand each proof, and each proof is accompanied by one or more figures illustrating key steps in the proof. We use the same triangle in each figure to help identify the parts involved in each proof. The triangle we use is acute (that is it has three acute angles), but each of the proofs can be applied to, or modified slightly to apply to, an obtuse or right triangle.

Simson's Theorem can be proved in any number of ways using both analytic and synthetic geometry. The proofs we present have been collected from a number of sources including Martin I. Isaacs's *Geometry for College Students*[3], David Kay's *College Geometry*[5], and a geometry website hosted by Antonio Gutierrez.[4] Although each proof shares similarities with the others, each is unique in its approach and is presented in its entirety. For this paper we use the three volume translation by Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* for any reference to a specific numbered proposition. Such references are called out as (Euclid, Book number.Proposition number).

We begin in Section 2 with a general introduction to the theorem. Section 3 is an analytic proof of Simson's Theorem. Section 4 contains a synthetic proof by parallel lines. Section 5 is a synthetic proof using vertical angles to prove collinearity. Section 6 is a synthetic proof applying Menelaus's Theorem. A proof of the converse of Simson's Theorem is presented in Section 7. Each section begins with a discussion of the method of the proof and any prerequisite material needed for the proof. The proof itself follows. Finally, the proofs of any intermediate results used in the section are presented at the end of that section. In this way each section presents a complete proof of the theorem, or in the case of Section 7, its converse. It should be noted that the analytic proof in Section 3 proves both Simson's Theorem and its converse, in this way the paper presents four proofs of the theorem and two of its converse.

2. Introduction to Simson's Theorem

Once again, Simson's Theorem stated as an if and only if relationship, says that given a triangle $\triangle ABC$ and a point P, called the pole of the Simson Line, the three feet of the perpendiculars from P to the sides or sides extended of $\triangle ABC$ are collinear if and only if P is on the circumcircle of $\triangle ABC$. That is, we can show that the pedal triangle of a point P, the triangle whose vertices are the feet of the perpendiculars to the sides extended of the original triangle from the point P, degenerates into a line exactly when P is on the circumcircle. Examples of pedal triangles resulting from points outside and inside the circumcircle of a given triangle are shown in Figure 2. We note that for a point either exterior or interior to the circumcircle of a given triangle, the three feet from that point are not collinear, but rather form a new triangle. This suggests that something interesting happens at the circumcircle of a triangle. In discussing each proof it becomes apparent that they all rely on the properties of angles inscribed in circles and that we need the circle to even begin proving the theorem.

There are a few special situations that we need to look into before we begin any of the proofs of the theorem. Our first question is what happens if P is one of the vertices of the triangle? It turns out that, in this case, Simson's Theorem is trivial. To see why this is true we begin by finding the feet of the perpendiculars from P. We must recall that a perpendicular from a point on a line intersects that line at only that initial point. Thus, if P is a vertex of the triangle and consequently on two of the sides of the triangle, the feet of the perpendiculars to those sides are the same point as P. We know that two points determine a unique line. It follows that, as two of our three feet fall on the same point, and the third is the foot of the altitude from that point, that the Simson Line with pole P at one of the three vertices of the triangle is the altitude of the triangle from that vertex. We note that this is the only case where a pole P generates less than three points.

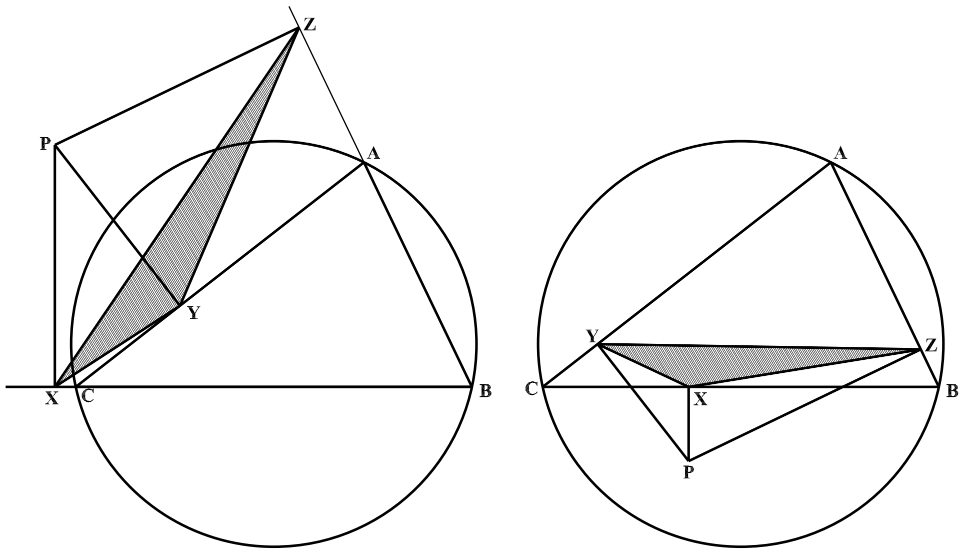

FIGURE 2. Pedal triangles generated by points P outside of the circumcircle and inside the circumcircle.

This instance of the Simson Line being an altitude of a triangle leads to another interesting question, that is, can a point P be on the circumcircle *and* be the orthocenter, the intersection of all three altitudes? That is, is there a case where the orthocenter of a triangle is on the circumcircle, and if so what happens if we pick P to be the orthocenter? There are three cases we need to consider: an acute triangle, an obtuse triangle, and a right triangle. Again, the orthocenter is the point of intersection of the three altitudes of a triangle. We know from previous study of triangles, that for an acute triangle the orthocenter is in the interior of the triangle as all three altitudes pass through the interior. As such the orthocenter of an acute triangle cannot fall on the circumcircle. For an obtuse triangle two of the three altitudes are entirely exterior to the triangle and as such so is the orthocenter. The orthocenter falls on the altitude from the obtuse vertex, extended beyond the triangle. Although we do not include the proof here, it is relatively easy to show that the orthocenter falls beyond the circumcircle as well. The only type of triangle whose orthocenter may fall on the circumcenter is a right triangle. In the case of a right triangle, the two "legs" of the triangle are also two of the altitudes. The orthocenter of the triangle is therefore the vertex at the right angle. If we take P to be this point it is both the orthocenter and on the circumcircle, but it also satisfies the conditions of our first question as a vertex of the triangle and therefore this case is also trivial.

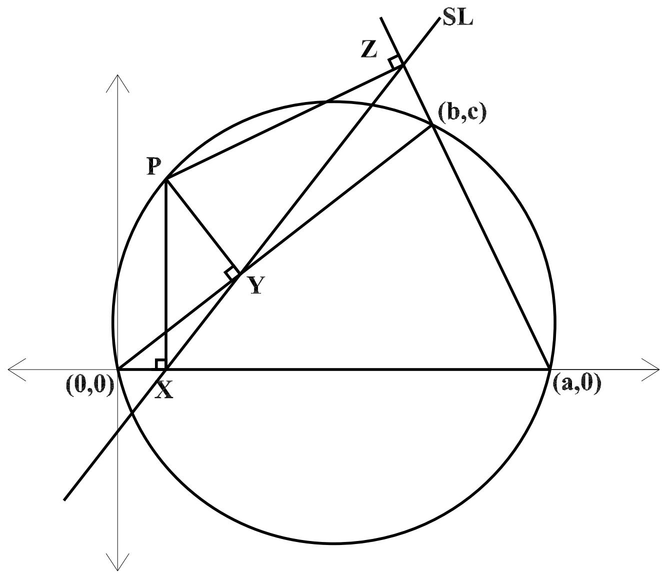

Figure 3. Coordinate system for analytic proof of Simson's Theorem

Given these two results, we assume for each proof that P does not fall on any of the three vertices and that there is at least one altitude of the triangle such that P is not on the altitude extended. We will see why this last assumption is important in our proof by parallel lines in Section 4.

3. Analytic Proof

In designing this proof, the goal is to avoid as much as possible any need to have studied synthetic geometry in the past. That is, beginning with known values for a triangle, we want to prove Simson's Theorem using algebra without appealing to specific geometric properties. In order to accomplish our goal, we need to have a few basic tools. First, we note that any triangle can be placed in the first quadrant with one side along the x-axis and an acute angle formed with that side at the origin. If our triangle is a right triangle, we will place it with its hypotenuse on the x-axis to avoid the potential problem of needing to divide by zero at any point in our proof. Our proof proceeds from this set-up. In order to begin our proof, we need to be aware of the equation for a circle in the Cartesian plane, and we need to be comfortable using basic algebraic manipulations.

If, as suggested above, we orient the triangle such that its vertices are $(0, 0)$, $(a, 0)$, and (b, c) with a, b, and c positive real constants and $a \neq b$, the problem is as illustrated in Figure 3. To prove the theorem, we first locate the center of the circumcircle of the triangle, by intersecting the perpendicular bisectors of the sides.

Finding the distance between this point and the origin, one of our vertices, we can find the radius of the circumcircle and define an equation for the circumcircle. Once we have this equation we can choose a generic point P. We then locate the feet of the perpendiculars from this point to the sides by finding equations for the sides in terms of a, b, and c. Once we have the three feet generated by the pole P, we can proceed with the proof. We assume that the three points are collinear, and using a set of logically or algebraically justified steps, we prove that the pole P must satisfy the equation for the circumcircle. That is, if the points are collinear then P must be on the circumcircle. Each step may be reversed to prove that if P is on the circumcircle then X, Y, and Z are collinear. In this way we will be proving Simson's Theorem and its converse.

Proof. The proof has three key steps: locating the circumcircle, finding the points X, Y, and Z, and proving that P is on the circumcircle if and only if X, Y, and Z are collinear.

- *Finding the circumcircle of our triangle.*

In order to locate a point P on the circumcircle of our triangle we need to locate the circumcenter of the triangle and determine its circumradius. The circumcenter is found by locating the intersection of the perpendicular bisectors of the sides as depicted in Figure 4. We use the perpendicular bisectors to find the circumcenter, as the circumcenter is the point equidistant from the three vertices, and the perpendicular bisector of a side is the locus of all points equidistant from the two end points. As the base of the triangle coincides with the x-axis, we note that the circumcenter of the triangle must lie on the line $x = \frac{a}{2}$, which is the perpendicular bisector of that side. The line from $(0, 0)$ to (b, c), has midpoint $(\frac{b}{2}, \frac{c}{2})$ and slopes $\frac{c}{b}$.

Noting that the perpendicular bisector of this side has slope $-\frac{b}{c}$ we can use the point slope equation to find the equation for its perpendicular bisector. Substituting $x = \frac{a}{2}$ into the equation allows us to solve for the y-coordinate of the circumcenter of the triangle. The circumcircle of the triangle is labelled O in Figure 4 and has coordinates $\left(\frac{a}{2}, \frac{b^2+c^2-ab}{2c}\right)$.

Using the distance formula to find the length of the line segment from $(0, 0)$ to the circumcenter, we find the circumradius of the triangle to be

$$r = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b^2 + c^2 - ab}{2c}\right)^2}$$

The circumcircle of our triangle has equation

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b^2 + c^2 - ab}{2c}\right)^2 = r^2.$$

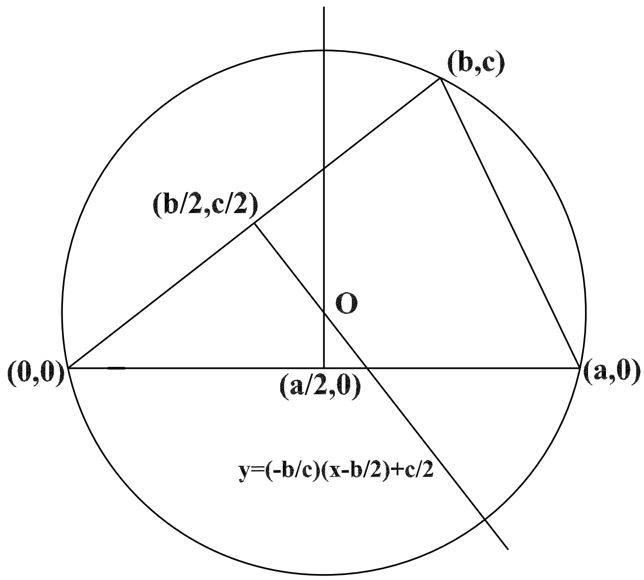

FIGURE 4. Generating the circumcircle of the triangle using two of the perpendicular bisectors of the sides.

The equation of the circumcircle can be slightly modified into a more useful form using basic algebra to

$$x^2 + y^2 = a \cdot x + \frac{b^2 + c^2 - ab}{c} \cdot y. \tag{1}$$

This modified form of the equation is the form that will be the most useful to our proof.

• *Locating the feet of the perpendiculars from P.* Our next step is to use an arbitrary point $P = (P_x, P_y)$ in the plane to locate perpendiculars to the sides extended of our triangle and to locate the feet of those perpendiculars. The first of the three perpendiculars is the line $x = P_x$ as one of our three sides coincides with the x-axis and the perpendicular is simply the vertical line through P. We therefore get one of our feet for free due to the way in which we have chosen to set-up the problem. Our first foot is the point $X = (P_x, 0)$.

We have also already done work on finding the point Y as we have all the information needed to describe side m and we have the slope of the perpendicular to m. We therefore have the equations for the lines m and n.

$$\begin{aligned} m: \quad y &= \tfrac{c}{b} \cdot x \\ n: \quad y &= -\tfrac{b}{c} \cdot (x - P_x) + P_y \end{aligned}$$

Solving these equations simultaneously for x and y we find that the point Y has coordinates

$$\left(\frac{b^2P_x + bcP_y}{b^2+c^2}, \frac{bcP_x + c^2P_y}{b^2+c^2}\right).$$

Following a similar process we find the slope of the third side of the triangle and the perpendicular to that side. As we have a point on each of these lines, we can use the point slope equation to find the equations for the lines labelled l and k.

$$\begin{aligned} l: \quad y &= \tfrac{a-b}{c} \cdot (x - P_x) + P_y \\ k: \quad y &= \tfrac{c}{b-a} \cdot (x - a) \end{aligned}$$

Again solving simultaneously for x and y we find that the foot Z has coordinates

$$\left(\frac{ac^2 + (b-a)^2P_x + c(b-a)P_y}{c^2 + (b-a)^2}, \frac{(b-a)cP_x + c^2P_y - ac(b-a)}{c^2+(b-a)^2}\right).$$

We should note that as we discussed in our setup, if working with a right triangle, it should be placed with hypotenuse along the x-axis so $a \neq b$. Now that we have our three feet, we can calculate the slopes between pairs of the points.

- *Using equal slopes to derive the equation for the circumcircle.*

In this final step we assume that the three points X, Y, and Z are collinear. As a result of this assumption, the slopes of the segments XY and XZ are equal. Using this assumption, we prove that the point P satisfies Equation 1. Reversing each step we find that if P is on the circumcircle, then the slopes of XY and XZ are equal and thus the three points are collinear.

We begin by assuming that the slopes are equal. As the coordinates of points Y and Z are messy, we start by renaming the coordinates in order to simplify as much as possible.

If for a moment we let

$$\alpha = \frac{b^2P_x + bcP_y}{b^2+c^2} \quad \text{and} \quad \sigma = \frac{ac^2 + (b-a)^2P_x + c(b-a)P_y}{c^2+(b-a)^2}$$

then from our equations, $Y = (\alpha, \frac{c}{b}\alpha)$ and $Z = \left(\sigma, \frac{c}{b-a}(\sigma - a)\right)$. If we assume that the slopes of XY and XZ are equal, we are assuming that

$$\begin{aligned} \frac{\frac{c}{b}\alpha}{\alpha - P_x} &= \frac{\frac{c}{b-a}(\sigma - a)}{\sigma - P_x} \\ \frac{\alpha}{b}(\sigma - P_x) &= \frac{\sigma - a}{b-a}(\alpha - P_x) \end{aligned}$$

Then re-substituting for α, and σ in this equation,

$$\frac{bP_x + cP_y}{b^2+c^2} \cdot \frac{ac^2 - c^2P_x + c(b-a)P_y}{c^2+(b-a)^2} = \frac{(b-a)P_x + cP_y - a(b-a)}{c^2+(b-a)^2} \cdot \frac{bcP_y - c^2P_x}{b^2+c^2}.$$

As both sides of this equation have the same denominator, the numerators must be equal. We can also cancel out a factor of c from both sides. Expanding the products we have

$$-bcP_x^2 + abcP_x + c(b-a)P_y^2 + (ac^2)P_y + [b(b-a) - c^2]P_xP_y =$$
$$-c(b-a)P_x^2 + ac(b-a)P_x + bcP_y^2 - ab(b-a)P_y + [b(b-a) - c^2]P_xP_y.$$

Combining like terms where possible, we arrive at the following equation:

$$acP_x^2 - a^2cP_x + acP_y^2 = [a^2b - ab^2 - ac^2]P_y = 0$$

If we rearrange the terms of this equation, and simplify in terms of P_x and P_y we arrive at the following:

$$\begin{aligned} ac(P_x^2 + P_y^2) &= [ac^2 + ab^2 - a^2b]P_y + a^2cP_x \\ P_x^2 + P_y^2 &= \frac{c^2 + b^2 - ab}{c} \cdot P_y + a \cdot P_x \end{aligned}$$

This is Equation 1 for the point $P = (P_x, P_y)$. Thus the point P is on the circumcircle of the triangle if X, Y, and Z are collinear. If we were instead to assume that P is on the circumcircle, we could work backward through the steps of our proof to arrive at the slopes of XY and XZ being equal and thus the three points being collinear. Therefore, the feet X, Y, and Z of the perpendiculars from a point P to the sides of a triangle are collinear if and only if the point P is on the circumcircle of that triangle. □

4. Proof by Parallel Lines

This synthetic approach to the proof of Simson's Theorem is inspired by a proof found in I. Martin Isaac's *Geometry for College Students*.[3] In this section we use general properties of triangles, inscribed angles, parallel lines, and similar triangles to prove Simson's Theorem. Our goal will be to prove that both XY and XZ are parallel to a specific constructible line. The reasoning behind this proof is similar to that of the analytic proof presented in Section 3. In order to complete this proof of Simson's Theorem we need several results. We need our assumption from Section 2 that P is not a vertex of the triangle and further that it is not the orthocenter and therefore does not fall on all three altitudes. We also need the equality of inscribed angles that subtend the same arc of a circle and that the angle between a chord and a tangent at one of its points of intersection with the circle is equal in degrees to half of the measure of the central angle that subtends the same arc. We also use the fact that inscribed angles are right angles if and only if they subtend a semicircle, that is, the segment connecting the ends of the two chords is a diameter of a circle. Our proof will proceed in two cases: first, if the perpendicular PX

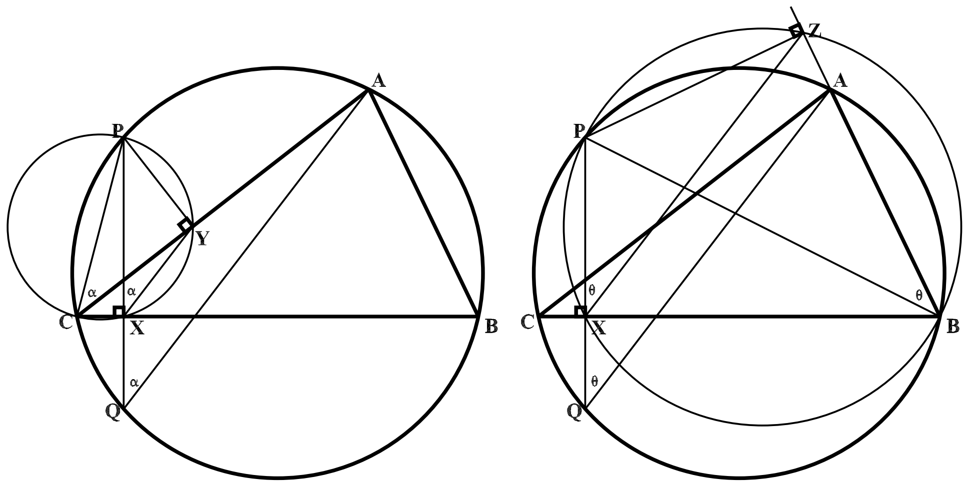

FIGURE 5. On the left, the circle with diameter PC leads to XY parallel to QA, while on the right, the circle with diameter PB leads to XZ also parallel to QA.

meets the circumcircle at another point Q, and second, if PX is tangent to the circle at P. The second case may be thought of as the limiting case of the first, but we present the modified proof for completeness. We assume for our proofs that P is not a vertex of the triangle and that it is not on the altitude from A. The proofs are similar for any of the three altitudes on which P does not fall.

Case 1:

Proof. Assume that the perpendicular PX is not tangent to the circumcircle at P. We begin by constructing the perpendiculars PX and PY and extending the perpendicular PX so that it intersects the circumcircle again at the point Q. Our next step is to construct the circle with diameter PC and note that as $\angle PXC$ and $\angle PYC$ are right angles by construction, the points X and Y fall on this new circle. As seen on the left in Figure 5 we have that $\angle PCY = \angle PXY$ as both angles subtend the arc $\overset{\frown}{PY}$ of the new circle. Similarly $\angle PCA = \angle PQA$ as they subtend the same arc of the circumcircle. It follows that $\angle PXY = \angle PQA$ and the lines XY and QA are therefore parallel by corresponding angles. The same steps on the circle with diameter PB, seen on the right in Figure 5, lead to XZ also parallel to QA. Since $XY \parallel QA$ and $XZ \parallel QA$ then XY and XZ must be the same line.

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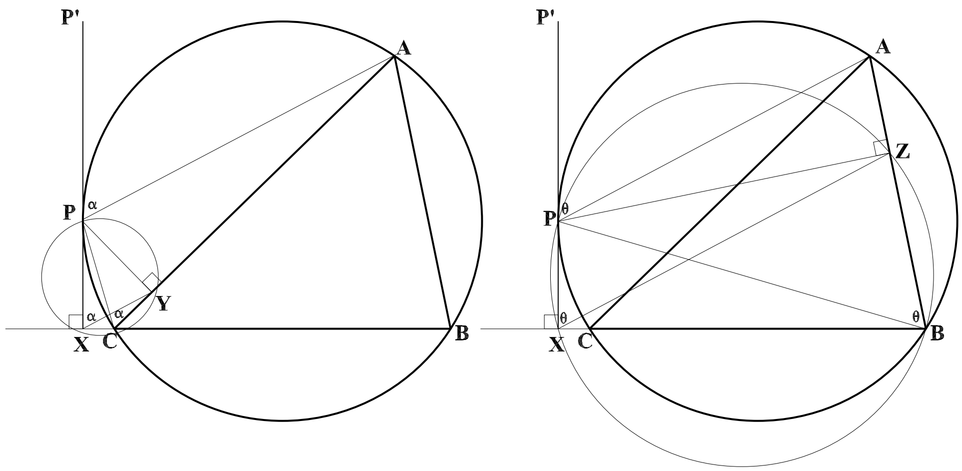

FIGURE 6. Modification to our proof if PX is tangent to the circle at P.

Case 2:

Proof. We know by assumption that the point P is not one of the vertices of the triangle, and that the perpendicular PX is tangent to the circle at P. We consider the unique circle with diameter PC to begin our proof. Since $\angle PXC$ and $\angle PYC$ are right angles by construction, the points X and Y are on this new circle. The angles $\angle PXY$ and $\angle PCY$ are therefore equal as they subtend the same arc. We know that the measure of an inscribed angle in a circle is equal in degrees to one half the measure of its subtended arc, (Euclid, III.20) and that the angle between a chord and the tangent at one of its endpoints is equal in degrees to half the subtended arc in degrees, which is the limiting case of the angle between two chords. Thus if we extend the line segment XP to some point P' the angles $\angle P'PA$ and $\angle PCY$ are equal as they subtend the same arc $\overset{\frown}{AP}$. Thus we have $\angle P'PA = \angle PCY = \angle PXY$, and PA and XY are parallel by corresponding angles. Similarly we can show that XZ and PA are parallel and thus that X, Y, and Z are collinear. Both parts of this proof can be seen in Figure 6.

□

5. PROOF BY VERTICAL ANGLES

This synthetic proof is a proof by vertical angles.[4] We are going to show that two angles, one on either side of a given straight line, formed by two other lines are equal and therefore that the two cutting lines are in fact the same line. In order to accomplish this goal we need several results from geometry. The first is that opposite angles of a quadrilateral, inscribed in a circle, equal two right angles (Euclid, III.22.) This is a consequence of an inscribed angle being equal to half the measure of the subtended arc (Euclid, III.21). We use that the sum of the angles of a triangle is equal to two right angles and by extension that the sum of the angles of a quadrilateral is equal to four right angles (Euclid, I.32). Of further use is that given a triangle, the angle at any of the three vertices is right if and only if the opposite side is a diameter of the circumcircle of the triangle (Euclid, III.31). We also need the fact that if two different lines terminate at the same point on a given line making adjacent angles supplementary, then the two lines are in fact the same line (Euclid, I.14). Using just these basic properties of planar geometry, all of which can be found in the first three books of Euclid's *Elements*, we can prove Simson's Theorem.

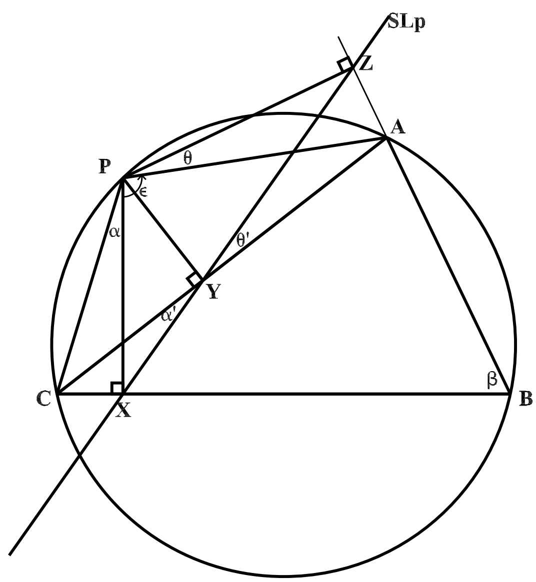

FIGURE 7. Setup for an alternative proof of Simson's Theorem using vertical angles to prove linearity.

The setup for this proof requires that we label certain angles and construct certain lines in addition to those explicitly mentioned in Simson's Theorem, as illustrated in Figure 7.

We begin by noting since opposite angles of an inscribed quadrilateral are equal to two right angles, $\beta + (\alpha + \epsilon) = 180°$ (Euclid, III.32). Also, as the sum of the angles of a quadrilateral is equal to 360° and as the angles $\angle PXB$ and $\angle PZB$ are right by construction, $\beta + (\theta + \epsilon) = 180°$.

Using these observations, we can conclude that $\alpha = \theta$. We next recognize that the pairs of angles α and α', and θ and θ' are equal. To prove this we first note that the points X and Y are on the unique circle with diameter PC as the angles $\angle PXC$ and $\angle PYC$ are right. Thus as α and α' subtend the same arc $\overset{\frown}{CX}$ of that circle, they are equal. Similarly θ and θ' are equal.

It follows that $\alpha' = \theta'$, and applying Proposition 14 of Book I of the *Elements*, as the lines XY and YZ are such that adjacent angles are supplementary, they must be the same line, thus completing our proof of Simson's Theorem. This proof relies only on P not being a vertex of the triangle, and no other discussion of cases is needed.

6. Proof by Menelaus's Theorem

In this proof we use an existing theorem called Menelaus's Theorem to prove Simson's Theorem as suggested in David C. Kay's *College Geometry*.[5] We begin by examining how the setup of Simson's Theorem satisfies the conditions of Menelaus's Theorem. The proof by Menelaus's Theorem relies on the sum of the angles of triangles and quadrilaterals, the supplementary nature of opposite angles of quadrilaterals, and properties of similar triangles. We present a proof of Menelaus's Theorem at the end of the section. In order to apply this approach, we first need the statement of Menelaus's Theorem.

Menelaus's Theorem *Given $\triangle ABC$, let points D, E, and F lie on lines BC, CA, and AB respectively, and assume that none of these points is a vertex of the triangle. Then D, E, and F are collinear if and only if an even number of them lie on the segments BC, CA, and AB and if the lengths of the resulting segments satisfy*

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1.$$

To explain the negative value in this equation, we establish the concept of positive direction in terms of measuring the side lengths of the triangle. We define as the positive direction that which follows the perimeter of the triangle clockwise from A to B to C and back to A. That is, on the line that is the sided extended of AB, we measure distance from A as positive in the direction of B and negative in the oposite direction. Applying this to all three sides of the triangle, a line segment on the sides extended of a triangle have both magnitude and sign.

6.1. Proving Simson's Theorem using Menelaus's Theorem.

Proving Simson's theorem in terms of Menelaus's Theorem is relatively straight forward. We

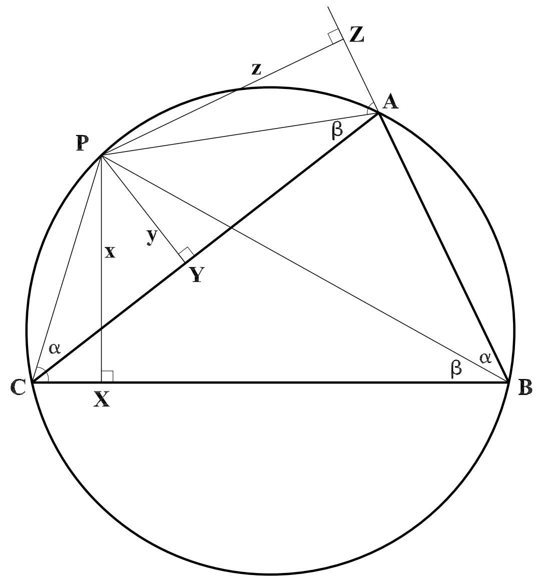

FIGURE 8. Setup to prove Simson's Theorem using Menelaus's Theorem

need to prove that the feet X, Y, and Z of the perpendiculars from Simson's Theorem satisfy the conditions of Menelaus's Theorem, and if they do, then Simson's Theorem follows. To begin applying Menelaus's Theorem to Simson's Theorem we connect the vertices of our triangle $\triangle ABC$ to the point P on the circumcircle as seen in Figure 8.

Using some elementary properties of lines and circles we can locate three pairs of equal angles. The pairs of angles labeled α and those labeled β are equal as each pair subtends the same arc. The angles $\angle PAZ$ and $\angle PCX$ labeled in the figure with a small arc are equal, as the opposite angles of an inscribed quadrilateral are supplementary, that is, $\angle PCX + \angle PAB = 180°$ but as BAZ is a straight line, $\angle PAZ + \angle PAB = 180°$. It follows that the angles $\angle PAZ$ and $\angle PCX$ equal. Using these pairs of equal angles along with the right angles formed at each of the points X, Y, and Z we have three pairs of similar triangles as seen in Figures 9, 10, and 11.

To reiterate, the positive direction is that which follows the perimeter of the triangle from A to B to C, so using properties of similar triangles we arrive at the following result:

$$\frac{BX}{YA} \cdot \frac{AZ}{XC} \cdot \frac{CY}{ZB} = \frac{x}{y} \cdot \frac{-z}{x} \cdot \frac{y}{z} = -1.$$

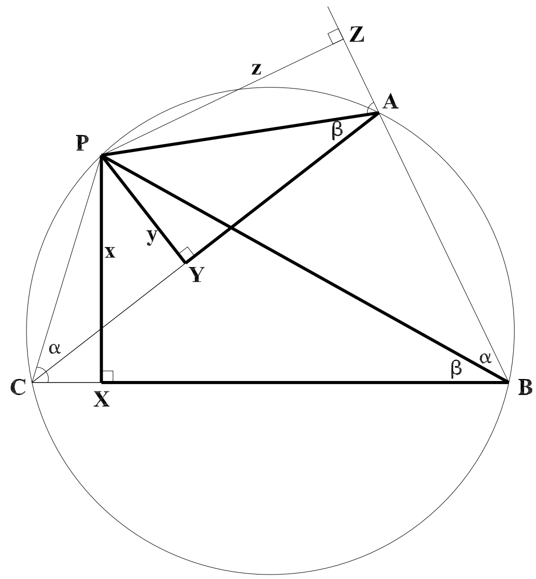

FIGURE 9. By AA, $\triangle PYA \sim \triangle PXB$ and consequently, $\frac{BX}{YA} = \frac{x}{y}$

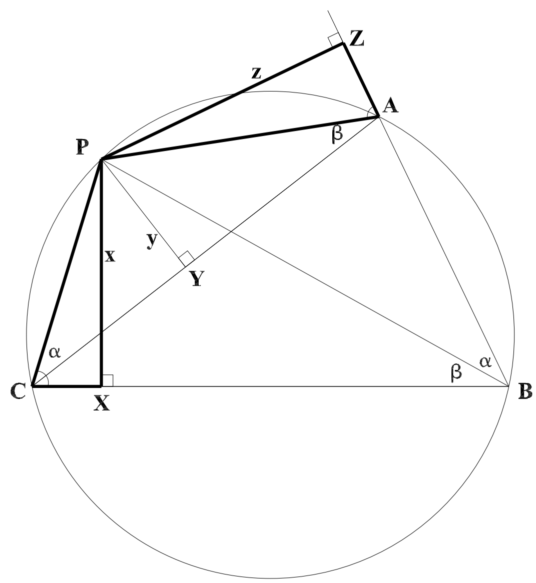

FIGURE 10. By AA, $\triangle PZA \sim \triangle PXC$ and consequently, $\frac{AZ}{XC} = \frac{-z}{x}$

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Therefore the conditions of Menelaus's Theorem are satisfied by the setup of Simson's Theorem and X, Y, and Z are collinear, thus proving Simson's Theorem.

6.2. **Proof of Menelaus's Theorem.**

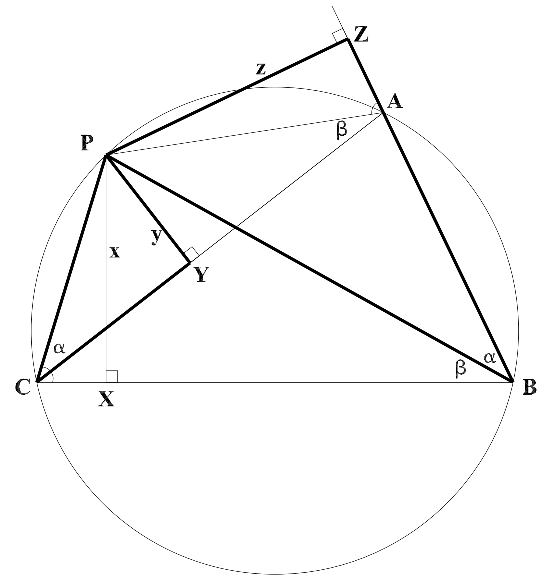

FIGURE 11. By AA, $\triangle PZB \sim \triangle PYC$ and consequently, $\frac{CY}{ZB} = \frac{y}{z}$

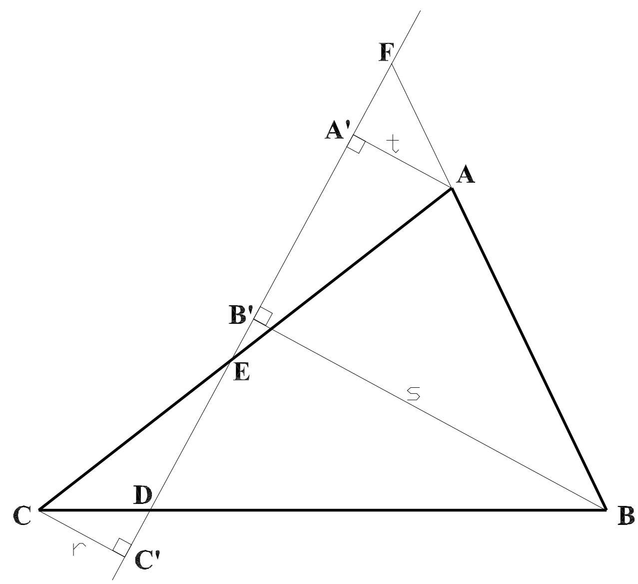

FIGURE 12. Setup for Menelaus's Theorem where the line DEF enters the triangle intersecting two sides, the positive direction is defined to be clockwise around the triangle.

Our goal in this section is to prove Menelaus's Theorem which we used to prove Simson's Theorem. As a reminder the theorem states: *Given* $\triangle ABC$*, let points* D*,* E*, and* F *lie on lines* BC*,* CA*, and* AB *respectively, and assume that none of these points is a vertex of the triangle. Then* D*,* E*, and* F *are collinear if and only if an even number of them lie on the segments* BC*,* CA*, and* AB *and*

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1.$$

Proof. First, we are going to prove that if the three points D, E, and F form a line, then the product of the ratios between lengths from vertex to point and point to vertex, as we traverse the perimeter of the triangle, is -1 using a proof by similar triangles. Our goal will be first to prove the magnitude of the product and then to address its sign. First, we drop perpendiculars from the vertices A, B, and C to the line DEF labeling them A', B', and C'. Then, by angle-angle, we have three sets of similar triangles: $\triangle AA'F \sim \triangle BB'F$, $\triangle AA'E \sim \triangle CC'E$, and $\triangle BB'D \sim \triangle CC'D$. As a consequence, when looking only at the magnitude of the sides, we have the following relationships:

$$\frac{AF}{BF} = \frac{t}{s}, \qquad \frac{BD}{CD} = \frac{s}{r}, \qquad \frac{CE}{AE} = \frac{r}{t}.$$

Multiplying together the left sides and likewise the right, results in

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{t}{s} \cdot \frac{s}{r} \cdot \frac{r}{t} = 1.$$

Thus the magnitude of the product is 1. Now to clarify the sign of the result we recall that we define as positive the path following the sides clockwise around the triangle from A to B to C.

We can therefore see that when we move around the triangle from A to F to B to D to C to E and back to A, all of the segments have positive value except for one. The same is true for any line DEF that enters the triangle. As we have assumed that P is not a vertex of the triangle and thus that the Simson Line with pole P does not cross through a vertex, the line DEF must cross either zero or two of the segments AB, BC, and CA. The case of zero intersections is depicted in Figure 13 where it can be seen that using the same definition for positive direction, three of the lengths are negative and thus the product is still negative.

Thus the product of the ratios of the length must be negative. Together with our calculation of the magnitude of the ratio we have our result that

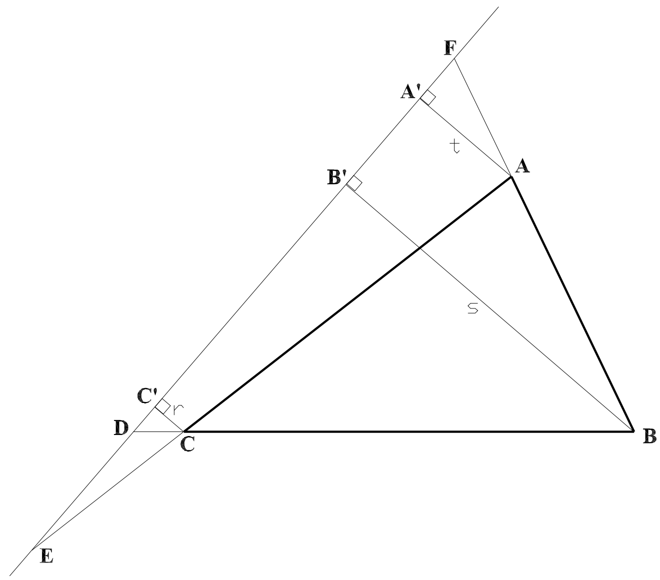

FIGURE 13. Setup for Menelaus's Theorem where the line DEF does not enter the triangle, the positive direction is defined to be clockwise around the triangle.

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1$$

as desired.

Next, we shall prove that if the product $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA}$ does indeed equal -1 then the points D, E, and F must fall on a line. Assume that $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1$, and let EF cut BC at D', then by the first part of our proof, as $D'EF$ is a line intersecting an even number of the the three sides of a triangle $\triangle ABC$ it follows that

$$\frac{AF}{FB} \cdot \frac{BD'}{D'C} \cdot \frac{CE}{EA} = -1.$$

If we set these two equations equal to one another, then $\frac{BD}{DC} = \frac{BD'}{D'C}$. On a given line segment, from one end there is a unique point making the ratio of the two resulting segments, equal to a given ratio, and therefore $D = D'$. Thus D, E, and F are collinear and our proof is complete. □

7. Proving the Converse of Simson's Theorem

To prove the converse of Simson's Theorem, we need several intermediate results, including a lemma involving parallel lines, and a discussion of the consequences of our proof by parallel lines in Section 4. The proof of the lemma is located at the end of this section. First we restate the converse of the theorem which we then prove.

Converse of Simson's Theorem *Given $\triangle ABC$, suppose that the feet of the perpendiculars from some point R to the three sides of the triangle, namely U, V, and W are collinear. Then R must lie on the circumcircle of $\triangle ABC$.*

Proof. We begin by discussing an important consequence of our proof by parallel lines. We proved in Section 4 that a Simson Line with pole P is parallel to the line QA, where Q is the other point of intersection of one of the perpendiculars with the circumcircle or the same as P if the perpendicular is tangent at that point. If therefore we let the point Q move around the circumcircle starting at the vertex A, we can note that a Simson Line can be found parallel to any line as a circuit around the circumcircle results in Simson Lines rotating through a full $180°$. Thus if as described, U, V, and W all fall on a line m, we can find a pole P on the circumcircle such that its corresponding Simson Line n is parallel to line m. If we let a, b, and c be the three sides of $\triangle ABC$, where as usual side a is opposite vertex A, then we have the situation of the following lemma.

Lemma *Suppose that lines m and n are parallel, lines b and c meet at a point A, line m meets b and c at points V and W, respectively, and line n meets b and c at points Y and Z respectively. Perpendiculars to b and c are erected at V and W, and these meet at a point R. Similarly, the perpendiculars to b and c at Y and Z meet at point P. Then P, A, and R are collinear* (Isaacs 99). Figure 14 contains an example where A falls between the two parallel lines, teh proof for A falling outside the parallel lines is similar.

Applying the Lemma three times we deduce that P, R and A are collinear as are P, R and B and P, R and C. If we assume P and R are different points, we therefore have P and R both collinear with all three points A, B, and C, but, as the vertices of a triangle, these points cannot all be collinear and our assumption that P and R are different must be false, thus P and R are the same point and R falls on the circumcircle as desired. □

7.1. **Proof of the Lemma.** An example of the lemma stated above is illustrated in Figure 14. To begin we address several trivial cases. First, if the point A happens to be the same as either P or R, then we have two points defining a line and the lemma is proven, so we can assume that A is neither point P nor point R. As such, it follows neither of the two lines m and n passes through the point A. Next, we look at what happens if either P or R falls on one of the lines b or c. Looking at R falling on b we see that if this is the case, then lines m and n must be perpendicular to line c. As a result the point P must also fall on line b and the result is proven. We can

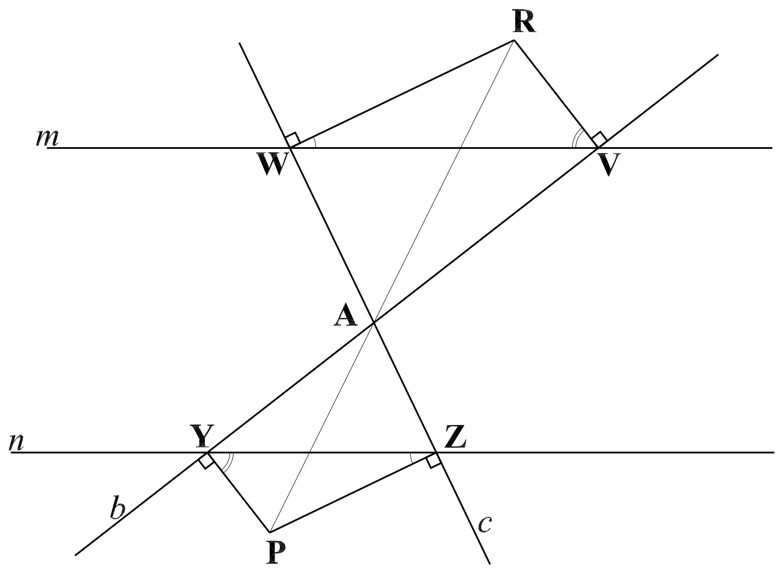

FIGURE 14. This is an example set-up of the lemma where A is located between lines m and n. The proof for the cases of A falling either above or below both lines is similar.

therefore assume that both P and R do not fall on any of the four lines described in the lemma. We note more specifically that the feet of the perpendiculars V, W, Y, and Z are all different points than A, P, and R. As such, AR is not parallel to YP or ZP. Our goal is therefore to show that the point P lies on the line AR.

Proof. Since YP is not parallel to AR, the two lines intersect at some point S. Similarly, ZP and AR intersect at a point T. We note that by construction, $YS \parallel VR$. It follows using properties of similar triangles that $AS/AQ = AY/AV$ and similarly $AT/AQ = AZ/AW$. Since $YZ \parallel VW$, we get $AY/AV = AZ/AW$ and thus that $AS/AQ = AY/AV = AZ/AW = AT/AQ$. This of course means that $AS = AT$ and therefore that S and T are the same point. As S is on YP and T is on ZP and P is the only point these two lines have in common, this means that the three points S, T, and P are the same. Specifically, this means that P in on the line AR and the three points A, R, and P are collinear as desired.

□

8. CONCLUSION

We have presented four different proofs of Simson's Theorem, each slightly different in its approach. We looked at an algebraically intensive analytic proof which required little previous experience in geometry. One of our proofs was a synthetic proof using basic principles that didn't require intermediate results. Our proof by parallel lines utilized several complicated intermediate results that have applications in other related problems. And our final proof used another very specific geometric theorem. These are not all possible methods for proving the theorem, but each one has aspects that may inspire methods of proof in other problems.

For those people interested in this type of project, there are several possibilities for further study. There are other proofs available to be studied that require a different background or set of skills. Each of these proofs can also be looked at using an obtuse triangle to notice any modifications necessary to that case. There are also other theorems in geometry that may lend themselves to this type of project where one explores various proofs. Some examples are Ceva's Theorem and its related results, Menelaus's Theorem which we looked at briefly here, as well as many other theorems of interest in the field including some interesting results in non-Euclidean geometry.

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