Convexity and The Art Gallery Theorem

Bahiyyih Parish

May 19, 2009

Abstract

Basic ideas from convex geometry in Euclidean space are developed. The finite and infinite versions of Helly's theorem are proved. Some applications of Helly's theorem are examined. Our study culminates in proof of Krasnosselsky's Art Gallery Theorem in n-dimensional space.

1 Introduction

In this paper, we examine the visibility of points in two dimensional and n-dimensional sets in Euclidean space, when the sets have one simple boundary line or boundary surface, that is, no holes. Two points are **visible** from each other in a set when the entire line segment between them is contained in the set.

Our main question is, given a simply-bounded set A in Euclidean space, what is the minumum number of points necessary in a subset $B \subseteq A$ for every point in A to be visible from at least one of the points B? In other words, what is the minimum number of points in A required to "see" the entire set A?

1.1 The Simplest Case

In the two dimensional case, our question becomes "What is the minimum number of full angle security cameras required to assure that all points in an art gallery are visible on camera?" When the art gallery can be represented as a polygonal set in $\mathbb{R}^2$, there is a fairly straightforward answer. The solution presented below has been modified from section 4.2 of Edward Burger and Michael Starbird's *The Heart of Mathematics.*

Any non-triangular polygonal set in $\mathbb{R}^2$ can be subdivided into two adjacent polygonal sets by connecting a pair of mutually visible vertices. Subdivision can continue in this manner until the entire set has been divided into adjacent, non-overlapping triangles. Each vertex of each triangle is also a vertex of the original polygon.

It is always possible to choose three colors, and color the vertices of the subdivided polygon so that the each triangle that makes up the polygon has three different colors, one at each vertex. Consider that if we select a random triangle in the polygon, and color the three vertices each a different color, then the remainder of the polygon, excluding the triangle, will be divided into up to three disjoint sub-polygons (See Figure 1).

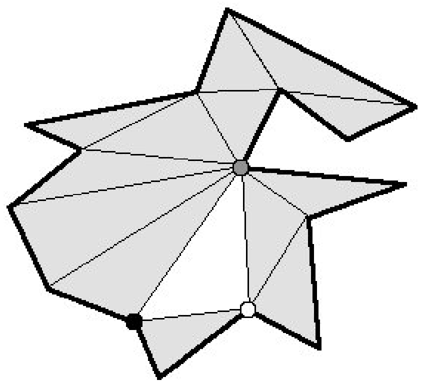


Figure 1: A triangularly subdivided polygon has been divided into three disjoint parts, in grey, by selecting a triangle from the polygon to color

These parts are disjoint because if they came in contact at any point, the polygon would have holes in it. The vertices in each of the three sub-polygons are not yet colored, but the color of the remaining vertex of each triangle adjacent to the original triangle is now determined. Color that vertex, and we have divided the sub-polygon that contains it into two smaller, disjoint, and not yet colored subsets. This process can be repeated until every vertex is colored with one of three colors, and every triangle has three different colored vertices.

If a security camera is placed in a triangle, then every point in the triangle is visible to the camera, because the triangle is convex. If there are n vertices colored with three colors, then the color that is associated with the least number of vertices must color less than or equal to $\frac{n}{3}$ vertices. If a camera is placed at each vertex of that color then the entire gallery will be visible to $\frac{n}{3}$ or less cameras. Thus any polygonal set in $\mathbb{R}^2$ can always be entirely "seen" by $\frac{n}{3}$ or fewer points.

In the n-dimensional case, the problem does not admit a basic solution. In this paper, we develop principles of convexity theory, and we eventually use Helly's Theorem, one of the most powerful results in convexity, to demonstrate a condition for when only one point is necessary to see an entire n-dimensional set. Sets that contain such a point are called **starshaped**. Partitioning sets into subsets that are all starshaped is one way of setting an upper limit on the number of points required to see an entire set.

Most of this paper is paraphrased from Webster's Convexity. Where other sources are used, reference is given.

2 Preliminaries

Before looking at convexity, Helly's Theorem, and Krasnosselsky's Theorem, we need to build some of the concepts that will be used throughout the paper.

2.1 Flats

In order to study the art gallery theorem in $\mathbb{R}^n$, we need a way to characterize sets in $\mathbb{R}^n$. One of the most basics subsets of $\mathbb{R}^n$ is a subspace. Flats in $\mathbb{R}^n$ are a generalization of subspaces.

A subset B of $\mathbb{R}^n$ is a **flat** if, for any two points in B, the set B contains the entire line defined by those two points. Thus for any two points $\mathbf{a}, \mathbf{b} \in B$, and a free variable $\lambda \in \mathbb{R}$, every point in the parametrically defined line

$$\mathbf{x} = \mathbf{b} + \lambda(\mathbf{a} - \mathbf{b}) = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$$

must be contained in B (See Figure 2). If we set $\mu = (1-\lambda)$ we can characterize a line through points $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ as the set

$$\{\lambda\mathbf{a} + \mu\mathbf{b} | \lambda + \mu = 1\}.$$

Consequently, a set $B \subseteq \mathbb{R}^n$ is a flat if it satisfies $\lambda\mathbf{a}+\mu\mathbf{b} \in B$ whenever $\mathbf{a}, \mathbf{b} \in B$ and $\lambda + \mu = 1$.

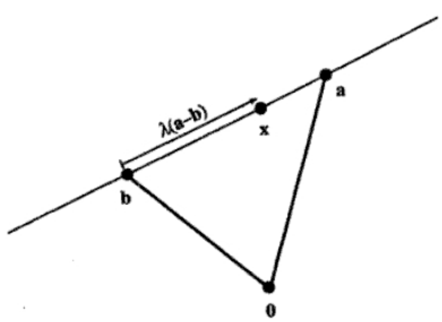


Figure 2: A line through points **a** and **b**

[1] **Lemma** A flat that contains the origin is a subspace.

Proof Let A be a flat in $\mathbb{R}^n$ which contains the origin. Suppose that **a, b** $\in A$ and $\lambda \in \mathbb{R}$. Since A is a flat and $\mathbf{a},\mathbf{0} \in A$, therefore $\lambda\mathbf{a} + (1 - \lambda)\mathbf{0} \in A$, i.e. $\lambda\mathbf{a} \in A$. Thus A is closed under scalar multiplication. Since A is a flat and $\mathbf{a}$, $\mathbf{b} \in A$, therefore $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} \in A$. But A is closed under scalar multiplication, so $2(\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}) \in A$, i.e. $\mathbf{a} + \mathbf{b} \in A$. Thus A is closed under addition, and A is a subspace of the real vector space $\mathbb{R}^n$.

Theorem Each nonempty flat in $\mathbb{R}^n$ is exactly a translated subspace of $\mathbb{R}^n$

[1]image from Webster

Proof Let F be a flat in $\mathbb{R}^n$ and $\mathbf{a}$ be some point in F. Consider the set $(F - \mathbf{a}) = \{\mathbf{f} - \mathbf{a} | \mathbf{f} \in F\}$. If a flat contains the origin, then it is a subspace. Since $(F - \mathbf{a})$ contains $\mathbf{a} - \mathbf{a} = 0$, we only need to show that $(F - \mathbf{a})$ is a flat. We start with two general points $f_1, f_2 \in F$. If $\lambda + \mu = 1$, then

$$\lambda(f_1) + \mu(f_2) \in F.$$

Subtracting $\mathbf{a}$ from the left term, we get the inclusion statement

$$\lambda(f_1) + \mu(f_2) - (\lambda + \mu)\mathbf{a} \in (F - \mathbf{a})$$

which simplifies to

$$\lambda(f_1 - \mathbf{a}) + \mu(f_2 - \mathbf{a}) \in (F - \mathbf{a}).$$

Thus $(F - \mathbf{a})$ is a flat, and every flat can be translated to form a subspace.

Next we show that a translated subspace is always a flat. Suppose S is a subspace of $\mathbb{R}^n$ and for some $\mathbf{p} \in \mathbb{R}^n$, $F = S + \mathbf{p} = \{\mathbf{s} + \mathbf{p} | \mathbf{s} \in S\}$. We can show that F is a flat. If $\mathbf{x}, \mathbf{y} \in S$ and $\lambda + \mu = 1$, then $\lambda\mathbf{x} + \mu\mathbf{y} \in S$, since S is closed under linear combinations. Adding $\mathbf{p}$ to the left term, we get

$$\lambda\mathbf{x} + \mu\mathbf{y} + \underbrace{(\lambda + \mu)}_{1}\mathbf{p} \in (S + \mathbf{p}),$$

where (S+$\mathbf{p}$)= $\{\mathbf{s} + \mathbf{p} | \mathbf{s} \in S\}$. Our expression simplifies to

$$\lambda(x + \mathbf{p}) + \mu(y + \mathbf{p}) \in (S + \mathbf{p}).$$

Thus $(S + \mathbf{p})$ is a flat, and translating any subspace results in a flat.■

2.2 Affine Combinations

Affine combinations provide some insight into the nature of flats.

A point $x \in \mathbb{R}^n$ is an **affine combination** of points $a_1, a_2, ..., a_m$ in $\mathbb{R}^n$ if there exist scalars $\lambda_1, \lambda_2, ...\lambda_m$ where $\lambda_1 + \lambda_2 + ... + \lambda_m = 1$ such that

$$x = \lambda_1 a_1 + \lambda_2 a_2 + ... + \lambda_2 a_2$$

The **affine hull** of A, written aff(A) is the set of all affine combinations of the points in set A.

It can be shown that if A is a flat, then A=aff(A). In other words, for any subset $a_1, a_2, ..., a_m$ of a flat A, the affine combination $\lambda_1 a_1 + \lambda_2 a_2 + ... + \lambda_m a_m \in A$.

A set B in $\mathbb{R}^n$ is **affinely dependent** if there exists a $b \in B$ such that $b \in$ aff$(B \setminus \{b\})$.

Another definition of affine dependence is given in the following theorem.

Theorem Let A be a set in $\mathbb{R}^n$. Then A is affinely dependent if and only if there exist some distinct points $a_1, a_2, ..., a_m$ of A and scalars $\lambda_1, \lambda_2, ...\lambda_m$, not all zero, such that $\lambda_1 a_1 + \lambda_2 a_2 + ... + \lambda_m a_m = 0$ and $\lambda_1 + \lambda_2 + ... + \lambda_m = 0$.

Proof Suppose set A is affinely dependent. Then there is an $\mathbf{a}_1 \in A$, and some $\mathbf{a}_2, \mathbf{a}_3, ..., \mathbf{a}_k \in (A \setminus \{\mathbf{a}\})$ such that $\mathbf{a}_1 = \lambda_1 \mathbf{a}_2 + \lambda_2 \mathbf{a}_3 + ... + \lambda_k \mathbf{a}_{k+1}$ for some $\lambda_1 + ... + \lambda_k = 1$. Scaling this equation by -1, but calling it $-\lambda_m$, we get

$$-\lambda_m \mathbf{a}_1 = -(\lambda_1 \mathbf{a}_2 + \lambda_2 \mathbf{a}_3 + ... + \lambda_k \mathbf{a}_{k+1})\lambda_m$$

which simplifies to

$$0 = \lambda_m \mathbf{a}_1 - (\lambda_2 \lambda_m \mathbf{a}_2 + ... + \lambda_k \lambda_m \mathbf{a}_{k+1}).$$

Separating out the constants of the $\mathbf{a}_i$, we show that the sum of the constants is zero

$$\lambda_1 - \lambda_m(\underbrace{\lambda_1 + ... + \lambda_k}_{1}) = 0.$$

Thus if A is affinely dependent, then the other conditions of the theorem are fulfilled. The proof of the converse is somewhat similar. ■

2.3 Dimension

Another concept we define in order to prove the art gallery theorem in n dimensions is dimension in $\mathbb{R}^n$. We begin by considering a theorem that will lead to a definition of affine bases. Its proof demonstrates one way to take advantage of connections to linear algebra.

Theorem 2.3 An affinely independent set in $\mathbb{R}^n$ cannot contain more than $n + 1$ points.

Proof (of contrapositive) Consider a set $A = \{a_1, \ldots a_m\}$ of m distinct points in $\mathbb{R}^n$, where $m > n + 1$. Then the system of n+1 linear homogeneous equations in m unknowns

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{m-1} \\ \lambda_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

has a nontrivial solution. Equivalently, the equations

$$\lambda_1 a_1 + ... + \lambda_m a_m = 0, \qquad \lambda_1 + ... + \lambda_m = 0$$

have a solution for which not all of $\lambda_1, \lambda_2, \ldots \lambda_m$ are zero and thus, the set $\{a_1, \ldots a_m\}$ is affinely dependent. ■

The following theorem, presented without proof, will enable us to define the dimension of a flat, and later the dimension of any set in $\mathbb{R}^n$

Theorem Every flat in $\mathbb{R}^n$ is the affine hull of some finite affinely independent subset of $\mathbb{R}^n$, called an affine basis. Moreover, the number of elements in such a subset is determined uniquely by the flat itself. ■

A flat in $\mathbb{R}^n$ which is the affine hull of some affinely independent set of $r+1$ points has **dimension** r and is called an r-flat.

Note that this definition is consistent with the definition of dimension of subspaces in linear algebra. For example, in linear algebra, a two dimensional subspace is defined by three affinely independent points: two points which are linearly independent, and the origin. We can verify the affine independence of these three points as follows. Since for any two nonzero linearly independent vectors $\mathbf{a}$ and $\mathbf{b}$, $\lambda\mathbf{a} + \mu\mathbf{b} = 0$ has no solution for any λ, μ, we know that $\lambda\mathbf{a} + \mu\mathbf{b} + \tau 0 = 0$ must also have no solution, including when $\lambda + \mu + \tau = 0$. Thus, by Theorem 2.3, these three points are affinely independent.

Using the concept of affine hulls, we can extend the definition of dimension to apply to any subset of $\mathbb{R}^n$.

The dimension of a subset $A \in \mathbb{R}^n$ is defined as the dimension of the flat $\mathrm{aff}(A)$.

2.4 Norms and Distance

In order to define open and closed sets in the next section, we need a formal definition of distance. The **norm** of a vector $\mathbf{x}$=$(x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$ is defined as

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

The **distance** between two vectors $\mathbf{x}$ and $\mathbf{y}$ is the norm of the difference vector

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})} = \sqrt{(x_1 - y_1)^2 + (x_2 + y_2)^2 + \cdots + (x_n + y_n)^2}.$$

The following Theorem from page 26 of *Convexity* will be useful in our final proof of Krasnosselsky's Theorem.

Theorem 2.4 If, for some $\alpha > 0$, $\|\mathbf{x} + \lambda\mathbf{y}\| \geq \|\mathbf{x}\|$ whenever $0 < \lambda < \alpha$, then $\mathbf{x} \cdot \mathbf{y} \geq 0$.

Proof Let $\alpha > 0$ be such that $\|\mathbf{x}\| \leq \|\mathbf{x} + \lambda\mathbf{y}\|$ whenever $0 < \lambda < \alpha$. Then, whenever $0 < \lambda < \alpha$,

$$\|\mathbf{x}\|^2 \leq \|\mathbf{x} + \lambda\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\lambda\mathbf{x} \cdot \mathbf{y} + \lambda^2\|\mathbf{y}\|^2$$

whence

$$\mathbf{x} \cdot \mathbf{y} + \frac{1}{2}\lambda\|\mathbf{y}\|^2 \geq 0.$$

Letting $\lambda \to 0_+$ in the last inequality, we deduce that $\mathbf{x} \cdot \mathbf{y} \geq 0$.

2.5 Open, Closed, and Bounded Sets

The set of points whose distance from a point $\mathbf{a} \in \mathbb{R}^n$ is less than r with $r > 0$,

$$\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\text{-}\mathbf{a}\| < r\}$$

is called an **open ball** with center $\mathbf{a}$ and radius r, and is written $B(\mathbf{a}; r)$. A **closed ball** is similarly defined as $\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\text{-}\mathbf{a}\| \leq r\}$, and is written $B[\mathbf{a}; r]$.

A point $\mathbf{b}$ of a set B in $\mathbb{R}^n$ is an **interior point** of B if it is the center of some open ball which lies in B. The set of all interior points of B is denoted by int(B).

A set B in $\mathbb{R}^n$ is called **open** if each point in the set is an interior point, i.e. if $B = \text{int}(B)$.

A point $\mathbf{b}$ of $\mathbb{R}^n$ is a **closure point** of a set B if the intersection of B and every possible open ball centered at $\mathbf{b}$ is nonempty. The set of all closure points of B is written $\text{cl}(B)$.

A set B in $\mathbb{R}^n$ is called **closed** if it contains all of its closure points.

Theorem Let A be a set in $\mathbb{R}^n$. Then $\text{cl}(A)$ =$(\text{int}(A^c))^c$ where A^c, called "A complement", indicates the set of all points in $\mathbb{R}^n$ that are not in A.

Proof If $\mathbf{x} \in \text{cl}(A)$, then each open ball with center $\mathbf{x}$ must contain a point of A, and so $\mathbf{x}$ cannot belong to $\text{int}(A^c)$, the set of all points that have an open ball not in A. Therefore $\mathbf{x}$ must belong to $(\text{int}(A^c))^c$. Conversely, if $\mathbf{x} \in (\text{int}(A^c))^c$ then each open ball centered at $\mathbf{x}$ must contain a point of A, i.e. $\mathbf{x}\in \text{cl}A$. Thus $\text{cl}A$ =$(\text{int}(A^c))^c$ ■

A set in $\mathbb{R}^n$ is called **bounded** if there exists a number $r \in \mathbb{R}^n$ such that $\|\mathbf{a}\| \leq r$ for all $\mathbf{a} \in A$.

2.6 Convergence and Closure

Convergence of infinite sequences of elements in a set in $\mathbb{R}^n$ provides another way to characterize closure. Proof that a set is closed is often written in terms of convergence of sequences contained in the set.

Definition An infinite sequence $\mathbf{x}_1, ..., \mathbf{x}_k ...$, in $\mathbb{R}^n$ is said to **converge** to a point $\mathbf{x}$ if, by making k arbitrarily large, we can make the distance between $\mathbf{x}_k$ and $\mathbf{x}$ as small as we want. In symbols, a sequence $\{\mathbf{x}_k\}$ converges to $\mathbf{x}$ if $\|\mathbf{x}_k - \mathbf{x}\| \to 0$ as $k \to \infty$.

Theorem 2.6 If A is a set in $\mathbb{R}^n$ and $\mathbf{x}\in \mathbb{R}^n$ then $\mathbf{x} \in \mathbf{cl}(A)$ if and only if there exists an infinite sequence of points of A which converges to $\mathbf{x}$.

Proof First we show that the existence of the convergent sequence implies that $\mathbf{x}\in\text{cl}A$. Suppose that $\mathbf{x}_1, ..., \mathbf{x}_k, ...$ is a sequence of points of A that converges to $\mathbf{x}\in \mathbb{R}^n$. Convergence implies that for any scalar $r > 0$, there exists some point $\mathbf{x}_k$ of the sequence such that $\|\mathbf{x}_k - \mathbf{x}\| < r$. Since $\mathbf{x}_k \in A$, the intersection of the open ball $B(\mathbf{x}$;r) and A is nonempty for every $r > 0$. Thus $\mathbf{x}$ is a closure point of A, that is $x \in \text{cl}A$.

Next we show that if $\mathbf{x} \in \text{cl}A$ then there exists an infinite sequence of points that converge to $\mathbf{x}$. If $\mathbf{x} \in \text{cl}A$ then for each positive integer k, the intersection of $B(\mathbf{x};1/k)$ and A is nonempty. Thus for each k, there exists an $\mathbf{x}_k \in A$ such that $\|\mathbf{x}_k - \mathbf{x}\| < 1/k$. Since for large enough k, we can make $\mathbf{x}_k$ arbitrarily close to $\mathbf{x}$, the sequence $\mathbf{x}_1, ..., \mathbf{x}_k ...$ converges to $\mathbf{x}$. ■

With Theorem 2.6, we can now also redefine a **closed** set A as a set for which every convergent sequence of points $\mathbf{a}_1, \mathbf{a}_2...$ with $\mathbf{a}_i \in A$, converges to a point contained in A.

2.7 The Bolzano-Weierstrasss Theorem and Compactness

Lemma Every infinite sequence in $\mathbb{R}^1$ has an infinite monotone subsequence.

Proof A term in a sequence after which all subsequent terms are smaller is called a peak element. If a sequence has an infinite number of peak elements, then an infinite subsequence made of those peak elements is strictly decreasing. If the sequence has a finite number of peak elements, then there is a last one, $\mathbf{x}_l$. Every element $\mathbf{x}_m$ after $\mathbf{x}_l$ is followed by some later element that is either greater than or equal to it since $\mathbf{x}_m$ is not a peak element. An infinite subsequence can be made from these elements that satisfies $\mathbf{x}_n \leq \mathbf{x}_q$ for $n < q$, that is, which is monotonically increasing.$\blacksquare$

The Bolzano-Weierstrass Theorem Every bounded infinite sequence of points of $\mathbb{R}^n$ contains a convergent subsequence.

Proof A standard result from calculus states that every bounded monotonic sequence is convergent. Thus every sequence in $\mathbb{R}^1$ contains a convergent subsequence.

To extend this result to sequences in $\mathbb{R}^n$, let $\mathbf{x}_1, ..., \mathbf{x}_k...$ be a bounded sequence in $\mathbb{R}^n$. Since $|x_{ki}| \leq \|\mathbf{x}_k\|$ for all $\mathbf{x}_k$, we know that the n sequences of the coordinates of the terms of $\mathbf{x}_1, ..., \mathbf{x}_k...$ must also be bounded. Since the sequence of the first coordinates, $x_{11}, ..., x_{k1}...$ is a bounded sequence of real numbers, it contains a convergent subsequence. Thus there exists a subsequence of $\mathbf{x}_1, ..., \mathbf{x}_k...$ such that the sequence of its first coordinates converges. Then by the same conditions, there exists a subsequence of that subsequence such that the sequence of second coordinates also converges. Taking subsequences of each new sequence n times, we are left with an infinite subsequence of $\mathbf{x}_1, ..., \mathbf{x}_k...$ in which all of the n coordinate sequences converge. Since $\|\mathbf{x}_k - \mathbf{x}\| \leq |x_{k1} - x_1| + ... + |x_{kn} - x_n|$ for any $\mathbf{x}$ in $\mathbb{R}^n$, we have found a convergent subsequence of $\mathbf{x}_1, ..., \mathbf{x}_k...$$\blacksquare$

Definition A subset of $\mathbb{R}^n$ is **compact** if each sequence of its points contains some subsequence that converges to a point of the subset.

Theorem In $\mathbb{R}^n$, a set is compact if and only if it is both closed and bounded.

Proof Suppose a set A in $\mathbb{R}^n$ is closed and bounded. Every infinite sequence $\mathbf{a}_1, \mathbf{a}_2, ...$ of points in A is also bounded. By Bolzano-Weierstrass, each $\mathbf{a}_1, \mathbf{a}_2, ...$ contains a convergent subsequence. By the closure of A, each of these subsequences converges to a point in A. Thus A is compact.

Suppose now that A is compact. If $\mathbf{x} \in \text{cl}A$, then by Theorem 2.6, there is an infinite sequence of points of A, $\mathbf{a}_1, \mathbf{a}_2, ...$ which converges to $\mathbf{x}$. Since limits are unique, all subsequences of this sequence also converge to $\mathbf{x}$. By the compactness of A, the sequence contains some subsequence that converges to a point of A. Thus $\mathbf{x} \in A$ and A is closed.

Now suppose that A is compact but not bounded. Then, for each positive integer k, there must exist a point $\mathbf{x}_k$ of A such that $\|\mathbf{x}_k\| \geq k$. The sequence $\mathbf{x}_1, ..., \mathbf{x}_k, ...$ of points of A contains no bounded infinite subsequence, and thus no convergent infinite subsequence, contrary to the hypothesis that A is compact. Thus A is both closed and bounded. ■

Presented without proof, the following result from Webster's *Convexity* will be useful in our final proof of Krasnosselsky's Theorem.

Theorem 2.7 In $\mathbb{R}^n$ the convex hull of an open set is open and the convex hull of a compact set is compact.

2.8 Hyperplanes

Hyperplanes are useful for dividing $\mathbb{R}^n$ into two halves, and will be instrumental in our eventual proof of Krassnosselski's theorem. Hyperplanes are, as they sound, an extension of the concept of a plane to $\mathbb{R}^n$.

A **hyperplane** is a flat in $\mathbb{R}^n$ of dimension $n-1$.

For a line L in $\mathbb{R}^2$ with equation $c_0 + c_1x + c_2y = 0$, we have $L = \{(x, y)|c_0 + c_1x + c_2y = 0\}$. The equation of a plane P in $\mathbb{R}^3$, $c_0 + c_1x + c_2y + c_3z = 0$ gives us $P = \{(x, y, z)|c_0 + c_1x + c_2y + c_3z = 0\}$. From this pattern we find that the following provides a general formula for hyperplanes H in $\mathbb{R}^n$

$$H = \{(x_1, x_2, ..., x_n)|c_0 + c_1x_1 + c_2x_2 + \ldots, c_nx_n = 0\}.$$

A **halfspace** S in $\mathbb{R}^n$ consists of all the points that lie on one side of a hyperplane. Halfspaces can be either open

$$S = \{(x_1, x_2, ..., x_n)|c_0 + c_1x_1 + c_2x_2 + \cdots + c_nx_n > 0\}$$

or closed

$$S = \{(x_1, x_2, ..., x_n)|c_0 + c_1x_1 + c_2x_2 + \cdots + c_nx_n \geq 0\}.$$

A hyperplane H **supports** a set A in $\mathbb{R}^n$ at the points $(H \cup \text{cl}A)$ if the intersection of H and $\text{cl}(A)$ is nonempty, and if A lies entirely in one of the closed halfspaces determined by H. In this case we say that H is a support hyperplane to set A.

A plane that is tangent to a ball is an example of a support hyperplane, and every hyperplane H is a support hyperplane to the open halfspaces defined by H. However, the line that is tangent to $y = \sin(x)$ at $x = 2$ is not an example of a a support hyperplane to the set of points that satisfy $y = \sin(x)$, since the set does not lie entirely on one side of the line.

3 Convexity

3.1 Convex Sets

If $\mathbf{x}$ and $\mathbf{y}$ are distinct points of $\mathbb{R}^n$ then the line through $\mathbf{x}$ and $\mathbf{y}$ can be written as $\mathbf{x} + \lambda(\mathbf{y}\textbf{-}\mathbf{x})$ where the parameter $\lambda \in \mathbb{R}$ can take on any value. Restricting λ

to between 0 and 1 results in an expression for the line segment between $\mathbf{x}$ and $\mathbf{y}$. If we set μ equal to $1-\lambda$ we find that

$$\{\mu\mathbf{x}+\lambda\mathbf{y}|\lambda,\mu\geq 0,\lambda+\mu=1\}$$

is another representation of the line segment between $\mathbf{x}$ and $\mathbf{y}$.

The set A in $\mathbb{R}^n$ is **convex** if whenever it contains two points, it also contains the line segment between them. In other words, A in $\mathbb{R}^n$ is convex if for every $\mathbf{x}$, $\mathbf{y}$ in A, the sum $\mu\mathbf{x}+\lambda\mathbf{y}$ is also in A for every $\lambda,\mu\geq 0$ for which $\lambda+\mu=1$.

To demonstrate that a set is convex, one can show that it contains the entire line segment between any two of its points. For example, we demonstrate that all open halfspaces are convex as follows. Consider two points $\mathbf{y}=(y_1,...,y_n)$ and $\mathbf{z}=(z_1,...,z_n)$ in the arbitrary open halfspace $P=\{(x_1,...,x_n)|a_1x_1+\ldots+a_nx_n>a_0;a_0,...,a_n\in\mathbb{R}\}$. By their definitions, $\mathbf{y}$ and $\mathbf{z}$ satisfy $a_1y_1+a_2y_2+...+a_ny_n>a_0$ and $a_1z_1+a_2z_2+...+a_nz_n>a_0$. Consequently, for some λ and μ as defined above,

$$\lambda(a_1y_1+a_2y_2+...+a_ny_n)+\mu(a_1z_1+a_2z_2+...+a_nz_n)>\lambda a_0+\mu a_0=a_0.$$

Thus

$$a_1(\lambda y_1)+...+a_n(\lambda y_n)+a_1(\mu z_1)+...+a_n(\mu z_n)>a_0$$

and

$$a_1(\lambda y_1+\mu z_1)+...+a_n(\lambda y_n+\mu z_n)>a_0.$$

Therefore $\lambda\mathbf{y}+\mu\mathbf{z}$ is in P for any $\mathbf{y}$, $\mathbf{z}$ in P, and P is convex.

We can show by similar means that all closed halfspaces, all balls, and all flats are convex.

Theorem The intersection of an arbitrary family of convex sets in $\mathbb{R}^n$ is convex.

Proof Let $\{A_i|i\in I\}$ where I is a set of natural numbers, be a family of convex sets in $\mathbb{R}^n$. If $\mathbf{a},\mathbf{b}\in\cap\{A_i|i\in I\}$ then $\mathbf{a},\mathbf{b}\in A_i$ for each $i\in I$. Since A_i is convex, $\lambda\mathbf{a}+\mu\mathbf{b}\in A_i$ for any $\lambda,\mu\geq 0$ with $\lambda+\mu=1$. Having $\lambda\mathbf{a}+\mu\mathbf{b}\in A_i$ for all $i\in I$ implies that $\lambda\mathbf{a}+\mu\mathbf{b}\in\cap\{A_i|i\in I\}$. Thus $\mathbf{a},\mathbf{b}\in\cap\{A_i|i\in I\}$ implies $\lambda\mathbf{a}+\mu\mathbf{b}\in\cap\{A_i|i\in I\}$ so the intersection of an arbitrary family of convex sets is convex. ■

3.2 Convex Combinations and The Convex Hull

The point $\mathbf{x}\in\mathbb{R}^n$ is a **convex combination** of points $\mathbf{a}_1,...,\mathbf{a}_n$ of set $A\in\mathbb{R}^n$ if there exist scalars $\lambda_1,...,\lambda_n$ with all λ_i greater than zero and $\lambda_1+\lambda_2+\cdots+\lambda_n=1$ such that $\mathbf{x}=\lambda_1a_1+\lambda_2a_2+...\lambda_na_n$. Analogous to the result for affine combinations of points in a flat, it can be shown that a convex set contains all possible convex combinations of its points.

The intersection of all convex sets in $\mathbb{R}^n$ that contain a set A in $\mathbb{R}^n$ is called the **convex hull** of set A, and is written conv$A$. Since the intersection of any family of convex sets in $\mathbb{R}^n$ is convex, we know that conv$A$ is always convex. Also, for any convex set B that contains A, conv$A \subseteq B$, so conv$A$ is the smallest convex set that contains A.

The following theorem provides another way to think about conv$A$.

Theorem 3.2 For a set A in $\mathbb{R}^n$, conv$A$ is the set of all convex combinations of points of A.

Proof Let B be the set of all convex combinations of points of A. Since conv$A$ is convex, we know that it contains every possible convex combination of points in conv$A$. Since $A \subseteq$ conv$A$, we know that conv$A$ must also contain every possible convex conbination of points in A, that is, $B \subseteq$ conv$A$.

Since $A \subseteq B$, we know by the definition of conv$A$ that conv$A \subseteq B$ when B is convex. Next we demonstrate that B is convex.

If $\mathbf{x}$, $\mathbf{y} \in B$, then $\mathbf{x}$ and $\mathbf{y}$ can be written as a convex combination of points in A. That is, for some points $\mathbf{a}_1, ..., \mathbf{a}_m, \mathbf{b}_1, ..., \mathbf{b}_p \in A$, with $\lambda_1, ..., \lambda_m, \mu_1, ..., \mu_p \geq 0$ and $\lambda_1 + \cdots + \lambda_m = \mu_1 + \cdots + \mu_p = 1$, we have

$$\mathbf{x} = \lambda_1\mathbf{a}_1 + \cdots \lambda_m\mathbf{a}_m \qquad \text{and} \qquad \mathbf{y} = \mu_1\mathbf{b}_1 + \cdots \mu_p\mathbf{b}_p.$$

Let $\lambda, \mu \geq 0$ and $\lambda + \mu = 1$. Then

$$\lambda\mathbf{x} + \mu\mathbf{y} = \lambda\lambda_1\mathbf{a}_1 + \cdots + \lambda\lambda_m\mathbf{a}_m + \mu\mu_1\mathbf{b}_1 + \cdots + \mu\mu_p\mathbf{b}_p.$$

We find that the scalars on the right side of this equality are all positive and sum to one.

$$\lambda\lambda_1 + \cdots \lambda\lambda_m + \mu\mu_1 + \cdots \mu\mu_p = \lambda(\lambda_1 + \cdots + \lambda_m) + \mu(\mu_1 + \cdots + \mu_p) = \lambda + \mu = 1.$$

So the point $\lambda\mathbf{x} + \mu\mathbf{y}$ can be written as a convex combination of points in A, and is therefore an element of set B. Thus set B is convex, and conv$A \subseteq B$, so B=conv$A$. ■

Theorem 3.2 implies that any point in conv$A$ can be written as a convex combination of points in A. It can be extended to show that if A is r-dimensional, any point in conv$A$ can be expressed as a convex combination of $r + 1$ or fewer points in A

3.3 Radon's Theorem

The following is essential to our eventual discussion of Krasnosselsky's theorem.

Radon's Theorem Let $\mathbf{a}_1, ..., \mathbf{a}_m \in \mathbb{R}^n$ with $m \geq n + 2$. Then the set $\{1, ..., m\}$ can be partitioned into two subsets I and J such that the intersection of conv$\{\mathbf{a}_i | i \in I\}$ and conv$\{\mathbf{a}_j | j \in J\}$ is nonempty.

Proof We only consider the non-trivial case when $\mathbf{a}_1, ..., \mathbf{a}_m$ are distinct. Since $\mathbf{a}_1, ..., \mathbf{a}_m$ contains more than $n+1$ points, it follows that the set is affinely dependent and there exist scalars $\lambda_1, ..., \lambda_m$ not all zero such that

$$\lambda_1\mathbf{a}_1 + \cdots + \lambda_m\mathbf{a}_m = 0 \qquad \text{and} \qquad \lambda_1 + \cdots + \lambda_m = 0. \tag{1}$$

Since they sum to zero, some of these λ's will be positive, and others negative. Let $I = \{i|\lambda_i \geq 0\}$ and $J = \{j|\lambda_j < 0\}$. Since $\displaystyle\sum_{i \in I} \lambda_i = \sum_{j \in J} (-\lambda_j)$, we can rearrange the first of the equations (1) as

$$\frac{\displaystyle\sum_{i \in I} \lambda_i \mathbf{a}_i}{\displaystyle\sum_{i \in I} \lambda_i} = \frac{\displaystyle\sum_{j \in J} (-\lambda_j) \mathbf{a}_j}{\displaystyle\sum_{j \in J} (-\lambda_j)} = \mathbf{x}$$

for some $\mathbf{x}$. Rewriting the expression as

$$\frac{\lambda_{i_1} \mathbf{a}_{i_1}}{\displaystyle\sum_{i \in I} \lambda_i} + \frac{\lambda_{i_2} \mathbf{a}_{i_2}}{\displaystyle\sum_{i \in I} \lambda_i} + \cdots + \frac{\lambda_{i_p} \mathbf{a}_{i_p}}{\displaystyle\sum_{i \in I} \lambda_i} = \frac{-\lambda_{j_1} \mathbf{a}_{j_1}}{\displaystyle\sum_{j \in I} (-\lambda_j)} + \frac{-\lambda_{j_2} \mathbf{a}_{j_2}}{\displaystyle\sum_{j \in I} (-\lambda_j)} + \cdots + \frac{-\lambda_{j_q} \mathbf{a}_{j_q}}{\displaystyle\sum_{j \in I} (-\lambda_j)} = \mathbf{x}$$

we find that coefficients of the a_i and the coefficients of the a_j each sum to one. Thus $\mathbf{x}$ is a convex combination of points of both $\{\mathbf{a}_i|i \in I\}$ and $\{\mathbf{a}_j|j \in J\}$, so

$$\mathbf{x} \in \text{conv}(\{\mathbf{a}_i|i \in I\} \cap \{\mathbf{a}_j|j \in J\})$$

and I and J satisfy the requirements of the theorem. ■

4 Helly's Theorem and Some Applications

In this section, we explore Helly's Theorem and some of its applications.

4.1 Helly's Theorem

Helly's Theorem Let $\mathbb{F}$ be a finite family of convex sets in $\mathbb{R}^n$ containing at least $n+1$ members. Suppose that the intersection of every $n+1$ members of $\mathbb{F}$ is nonempty. Then the intersection of all members of $\mathbb{F}$ is nonempty.

Proof We prove by induction. The assertion is trivial when $\mathbb{F}$ has $n+1$ members. Suppose that the theorem holds true for a family of m sets in $\mathbb{R}^n$ with $m \geq n+1$, where each set of $n+1$ members has a nonempty intersection. Let $A_0, ..., A_m$ be a family of $m+1$ convex sets in $\mathbb{R}^n$ for which every $n+1$ members of the family have a non-empty intersection. We want to show that the intersection of the whole family is nonempty.

By our hypothesis, for each number $i = 0, ..., m$ there exists a point $\mathbf{a}_i$ such that $\mathbf{a}_i \in A_0 \cap A_1 \cap ... \cap A_{i-1} \cap A_{i+1} \cap ... \cap A_m$. By Radon's Theorem, $\{0, ..., m\}$ can be partitioned into two sets J and K such that conv$\{\mathbf{a}_j|j \in J\}$ and conv$\{\mathbf{a}_k|k \in K\}$ have a non-empty intersection. So there exists a point $\mathbf{a}$ such that $\mathbf{a} \in \text{conv}\{\mathbf{a}_j|j \in J\} \cap \text{conv}\{\mathbf{a}_k|k \in K\}$. Thus for each $j \in J$, $\mathbf{a}_j \in \cap(A_k|k \in K)$.

Since $\cap\{\mathbf{a}_k|k \in K\}$ is convex, we have that conv$\{\mathbf{a}_j|j \in J\} \subseteq \cap(A_k|k \in K)$. Similarly, conv$\{\mathbf{a}_k|k \in K\} \subseteq \cap(A_j|j \in J)$. Thus $\mathbf{a}\in$ conv$\{\mathbf{a}_j|j \in J\} \cap$ conv$\{\mathbf{a}_j|j \in J\} \subseteq A_0 \cap ... \cap A_m$ Thus we've shown that $A_0 \cap ... \cap A_m$ is nonempty. ■

4.2 A Simple Application

Like many proofs dependent on Helly's theorem, the following proof involves creating a new family of convex sets. Each convex set corresponds to a set in the family that we'd like to prove something about. In this theorem, a **transversal** for a family of sets is a line that intersects each of the sets of that family.

Theorem If $\mathbb{F}$ is a family of parallel line segments in $\mathbb{R}^2$, and every three members of $\mathbb{F}$ have a transversal, then all of $\mathbb{F}$ has a transversal.

Proof Without loss of generality, we'll limit $\mathbb{F}$ to families of only vertical line segments S_i, since any family of parallel lines can be rotated to vertical without changing the properties of similarly rotated transversals. Each segment can be written $S_i = \{(x, y) : x = u_i, v_i \leq y \leq w_i\}$ for $i = 1, ..., r$, where $r \geq 3$ and u_i's are distinct. For each $i = 1, ..., r$ we create the convex set C_i in $\mathbb{R}^2$ composed of all the points (m, c) for which the line $y = mx + c$ has a nonempty interesection with the segment S_i.

$$C_i = \{(m, c)|v_i \leq mu_i + c \leq w_i\}$$

C_i is convex because it is the intersection of two halfspaces

$$C_i = \{(m, c)|v_i \leq mu_i + c\} \cap \{(m, c)|mu_i + c \leq w_i\}$$

and halfspaces are convex. Since every three vertical lines have a transversal, every three of the convex sets C_i must have a common point. Thus by Helly's Theorem, there exists a point (m_0, c_0) that is common to all the sets C_i, and this corresponds to the line that is a transversal for the entire family. ■

4.3 Another Nice Application

Using Helly's Theorem, we can show that if a convex set is contained in the union of a finite family of halfspaces in $\mathbb{R}^n$, then it's contained in the union of some $n + 1$ or less of the halfspaces.

Let a convex set A be contained in $H_1 \cup H_2 \cup ... \cup H_m$ where each H_i is a halfspace and $m > n$. Then $A \cap (H_1 \cup H_2 \cup \cdots \cup H_m)^c = \varnothing$. We have that

$$A \cap (H_1 \cup H_2 \cup \cdots \cup H_m)^c = A \cap (H_1^c \cap H_2^c \cap \cdots \cap H_m^c).$$

Rearranging, we find that

$$(A \cap H_1^c) \cap (A \cap H_2^c) \cap \cdots \cap (A \cap H_m^c) = \varnothing.$$

The contrapositive of Helly's Theorem states that if the intersection of an entire family of convex sets is empty, then there exists some $n + 1$ sets from the family that also have an empty intersection. Since $(A \cap H_i^c)$ is a convex set for all i's we have from Helly's Theorem that there exists a subfamily of $n + 1$ such convex sets which has an empty intersection. So for some relabeled H_i's, $(A \cap H_1^c) \cap (A \cap H_2^c) \cap ... \cap (A \cap H_{n+1}^c) = \varnothing$, and $A \cap (H_1 \cup H_2 \cup ... \cup H_{n+1})^c = \varnothing$. Thus for some $n + 1$ H_i's, the set A is contained in $(H_1 \cup H_2 \cup ... \cup H_{n+1})$. ■

4.4 Helly's Theorem for Infinite Families

The requirement that $\mathbb{F}$ contain only a finite number of sets is necessary for Helly's theorem as stated above. For example, consider the infinite family of closed parallel halfspaces

$$\{(x_1, x_2, ..., s_n) \in \mathbb{R}^n | x_1 \geq m\}$$

where $m = 1, 2, 3, ...$ Every three elements in this family have a nonempty intersection, but the intersection of the entire family is empty.

The generalization of Helly's Theorem from finite to infinite families of convex sets requires the additional constraint that all the sets be compact.

It also requires the following lemma from page 42 of *Convexity* which is presented here without proof.

Lemma Let $\{A_i | i \in I\}$ be a family of compact sets in $\mathbb{R}^n$ whose intersection is empty. Then there exists a finite subset I^* of I such that the intersection of the family $\{A_i | i \in I^*\}$ is empty.

Helly's Theorem for Infinite Families Let $\mathbb{F}$ be an infinite family of compact sets in $\mathbb{R}_n$. If every $n+1$ sets in $\mathbb{F}$ have a nonempty intersection then the intersection of all sets in $\mathbb{F}$ is nonempty.

Proof We show the contrapositive, that if the intersection of all of the sets in a family is empty, then the intersection of some $n+1$ of them is also empty. By Helly's Theorem, we know that this is true for finite families with empty intersections. The Lemma implies that if the intersection of the possibly infinite family is empty, then the family must have a finite subfamily with an empty intersection. The contrapositive of Helly's Theorem states that some $n+1$ of them also must have an empty intersection.■

5 The Art Gallery Theorem in n Dimensions

Before we can prove the general form of Krasnosselsky's Theorem, we need a definition and a lemma.

Definition A set $\mathbf{A}$ is called **starshaped** if there exists a point $\mathbf{a}_0 \in A$, called a **starcenter** such that $(1-\lambda)\mathbf{a}_0 + \lambda\mathbf{a} \in A$ for any $\mathbf{a} \in A$ and $0 \leq \lambda \leq 1$. In other words, a set is starshaped if it contains a point that is visible to every other point in the set. All convex sets are by definition starshaped.

The following lemma relates starshapedness to sets of the form

$$A_{\mathbf{x}} = \{\mathbf{a} \in A | (1-\lambda)\mathbf{x} + \lambda\mathbf{a} \in A \text{ for } 0 \leq \lambda \leq 1\}.$$

Each $A_{\mathbf{x}}$ is the set of all points in A that are visible to a point $\mathbf{x} \in A$.

Any point $\mathbf{a}$ that is a starcenter of a nonempty closed set $A \in \mathbb{R}^n$ satisfies $\mathbf{a} \in A_{\mathbf{x}}$ for all points $\mathbf{x} \in A$. In the following lemma, we strengthen this condition.

Lemma 5 The point $\mathbf{a}$ is a starcenter of a non-empty closed set A in $\mathbb{R}^n$ if and only if $\mathbf{a} \in \mathrm{conv}A_{\mathbf{x}}$ for all $\mathbf{x} \in A$.

Proof If $\mathbf{a}$ is a starcenter then $\mathbf{a}$ is visible to all points in A, so $\mathbf{a} \in A_{\mathbf{x}} \subseteq \text{conv}A_{\mathbf{x}}$ for all $\mathbf{x} \in A$. For the other direction, we show the contraposative. Suppose $\mathbf{a}$ is not a starcenter of A. We show that there is some point $\mathbf{e} \in A$ such that $\mathbf{a}$ is not in $\text{conv}A_e$.

If $\mathbf{a}$ is not a starcenter of A then there exists a $\mathbf{b} \in A$ such that the line segment between $\mathbf{a}$ and $\mathbf{b}$ is not entirely contained in A. Thus there is a point $\mathbf{c}$ on the segment that is not contained in A. Any point not in a closed set A is an interior point of A^c. Thus, since A is closed, we can construct a closed ball C centered at $\mathbf{c}$ that shares no points with A.

If we a define a scalar α

$$\alpha = \inf\{\lambda \neq 0 | A \cap (C + \lambda(\mathbf{b} - \mathbf{c})) \neq \varnothing\}$$

then the closed ball $C + \alpha(\mathbf{b} - \mathbf{c})$ is a translate of C in the direction of point $\mathbf{b}$. Denote the ball $C + \alpha(\mathbf{b} - \mathbf{c})$ by D, and its center point $\mathbf{d}$. Our definition of α ensures that A meets the boundary, but not the interior, of D. Suppose that $\mathbf{e} \in A \cap D$

Define complementary halfspaces

$$H^- = \{\mathbf{z} | (\mathbf{z} - \mathbf{e}) \cdot (\mathbf{e} - \mathbf{d}) < 0\}$$

and

$$H^+ = \{\mathbf{z} | (\mathbf{z} - \mathbf{e}) \cdot (\mathbf{e} - \mathbf{d}) \geq 0\}.$$

The hyperplane H with equation $(\mathbf{z} - \mathbf{e}) \cdot (\mathbf{e} - \mathbf{d}) = 0$ intersects with D at point $\mathbf{e}$ but D lies entirely in the closed halfspace H^-.

The definition of α ensures that the space swept out between C and D contains no points in A (see Fig 3). [2]

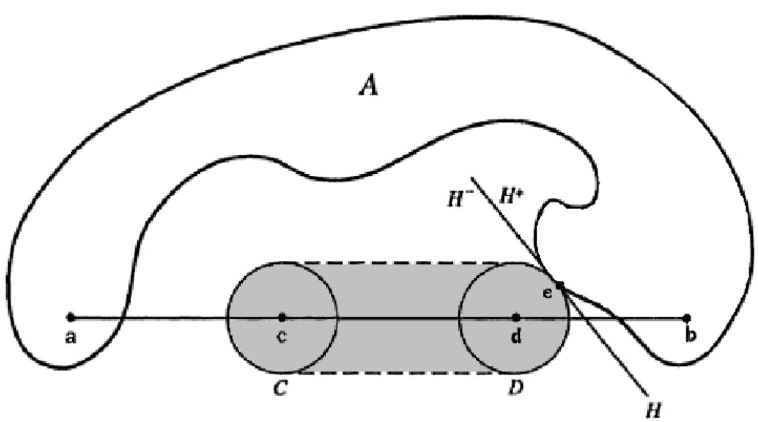


Figure 3: The convex hull of $C \cup D$ does not contain any points in A

[2]image modified from Webster

Thus for all $\theta > 0$ small enough to satisfy $\|\theta(\mathbf{b} - \mathbf{c})\| \leq \|1(\mathbf{d} - \mathbf{c})\|$, the translate of C centered at $\mathbf{d} - \theta(\mathbf{b} - \mathbf{c})$ has an empty intersection with A. Therefore

$$\|\mathbf{e} - (\mathbf{d} - \theta(\mathbf{b} - \mathbf{c}))\|^2 = \|\mathbf{e} - \mathbf{d} + \theta(\mathbf{b} - \mathbf{c})\|^2 > \|\mathbf{e} - \mathbf{d}\|^2.$$

It follows from Theorem 2.4 that

$$(\mathbf{e} - \mathbf{d}) \cdot (\mathbf{b} - \mathbf{c}) \geq 0.$$

Since $(\mathbf{b} - \mathbf{c})$ is antiparallel to $(\mathbf{a} - \mathbf{d})$, we know

$$(\mathbf{e} - \mathbf{d}) \cdot (\mathbf{a} - \mathbf{d}) \leq 0.$$

Using the definition of the norm, we have

$$(\mathbf{a} - \mathbf{e}) \cdot (\mathbf{e} - \mathbf{d}) = (\mathbf{a} - \mathbf{d} + \mathbf{d} - \mathbf{e}) \cdot (\mathbf{e} - \mathbf{d}) = (\mathbf{a} - \mathbf{d}) \cdot (\mathbf{e} - \mathbf{d}) - \|\mathbf{e} - \mathbf{d}\|^2 < 0$$

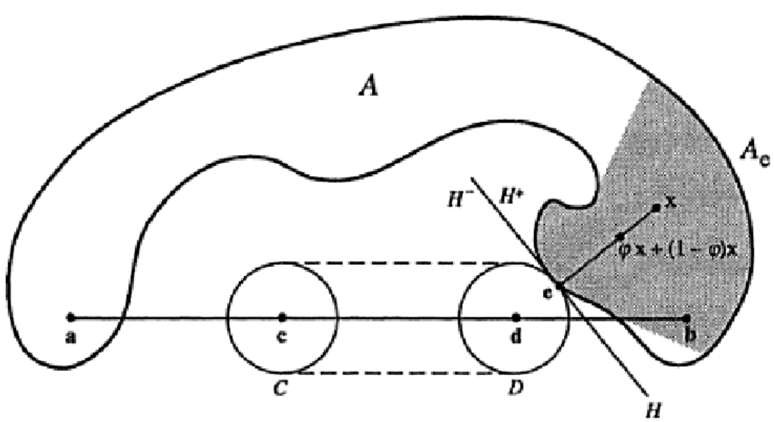
which shows that $\mathbf{a} \in H^-$.

Next we show that $A_e \subseteq H^+$. Let $\mathbf{x} \in A_e$. Since A is disjoint from the interior of D and $\varphi\mathbf{x} + (1 - \varphi)\mathbf{e} \in A$ for some $0 \leq \varphi \leq 1$,

$$\|\varphi\mathbf{x} + (1 - \varphi)\mathbf{e} - \mathbf{d}\|^2 = \|\mathbf{e} - \mathbf{d} + \varphi(\mathbf{x} - \mathbf{e}) \geq \|\mathbf{e} - \mathbf{d}\|^2.$$

See Figure 4. Using Theorem 2.4 again, we have that $(\mathbf{x} - \mathbf{e}) \cdot (\mathbf{e} - \mathbf{d}) \geq 0$, whence $\mathbf{x} \in H^+$.

Figure 4: Every point $\mathbf{x}$ in A_e is in H^+



[3]Thus A_e, is contained in H^+. ConvA_e is then also contained in H^+ since H^+ is convex. Since $\mathbf{a} \in H^-$ and H^- and H^+ have an empty intersection, we

[3]image from Webster

conclude that $\mathbf{a}$ is not contained in A_e. ■

Krasnosselsky's Theorem Let A be an infinite compact set in $\mathbb{R}^n$. Suppose that, for each $n+1$ points of A, there is some point of A from which all of these points are visible in A. Then A is starshaped.

Proof First we show that $A_{\mathbf{x}}$ is closed. Let $\mathbf{a}$ be a point in $\mathrm{cl}(A_{\mathbf{x}})$. Choose a sequence of points $\{\mathbf{a}_n\}$ in $A_{\mathbf{x}}$ that converges to $\mathbf{a}$. Since each $\mathbf{a}_n$ is in $A_{\mathbf{x}}$, we know $\{(1-\lambda)\mathbf{x}+\lambda\mathbf{a}_n\}$ is in A for each n, when $0 \leq \lambda \leq 1$. By the closure of A, if we fix λ at any allowable value, then the limit as $n \to \infty$ of the sequence $\{(1-\lambda)\mathbf{x}+\lambda\mathbf{a}_n\}$ of points in A converges to a point in A. Since

$$\lim_{n\to\infty}((1-\lambda)\mathbf{x}+\lambda\mathbf{a}_n)) = (1-\lambda)\mathbf{x}+\lambda\mathbf{a},$$

we know $(1-\lambda)\mathbf{x}+\lambda\mathbf{a}) \in A$ for all $0 \leq \lambda \leq 1$. Thus $\mathbf{a} \in A_{\mathbf{x}}$, and $A_{\mathbf{x}}$ is closed.

Since $A_{\mathbf{x}}$ is closed and is a subset of the bounded set A, we know $A_{\mathbf{x}}$ is compact. By Theorem 2.7, the set $\mathrm{conv}A_{\mathbf{x}}$ is also compact.

Let $\mathbf{x}_0, ..., \mathbf{x}_n$ be any $n+1$ points of A. By the hypothesis, there exists some point $\mathbf{y} \in A$ from which these $n+1$ points are visible in A, that is, there exists some $\mathbf{y}$ such that $\mathbf{x}_0, ..., \mathbf{x}_n \in A_{\mathbf{y}}$. Thus

$$\mathbf{y} \in A_{\mathbf{x}0} \cap ... \cap A_{\mathbf{x}n} \subseteq (\mathrm{conv}A_{\mathbf{x}0}) \cap ... \cap (\mathrm{conv}A_{\mathbf{x}n}),$$

and so each group of $n+1$ members from the infinite family $\mathbb{F} = \{\mathrm{conv}A_{\mathbf{x}} | \mathbf{x} \in A\}$ of compact convex sets in $\mathbb{R}^n$ have a common point. Therefore, by Helly's theorem for infinite families, there exists a point $\mathbf{a}$ which belongs to every member of $\mathbb{F}$. In view of the Lemma 5, $\mathbf{a}$ is a starcenter of A, and thus A is starshaped. ■

6 Acknowledgements

I'd like to thank Laura Schueller and Bob Fontenot for reading this. I'd like to give special thanks to Laura for reading it more than once, and to Bob for helping me understand Webster's book. I'd also like to thank Roger Webster for writing the book upon which this paper is primarily based. Finally, I'd like to thank puppies and dogs for always being there for me and for being so cute.

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