ISOMORPHISMS AND PERMUTATIONS OF SEMIFILTERS

COLIN MALLOY

1. Introduction

This paper deals with a set theoretic construct called a semifilter. We begin with an introduction to the basic properties of semifilters. We then explore topics such as semifilter multiplication and conjugation. After that we present our results relating isomorphisms of semifilters to permutations of an underlying set.

2. Preliminaries

2.1. **Semifilters.** We begin by letting W denote an arbitrary set. For now W may be finite or infinite. Before we can present the formal definition of semifilter, we must define the finite power set of W.

Definition 2.1.1. Let W be a set. Then the *finie power set* of W is defined to be the collection of all finite subsets of W. We denote the finite poewr set as $\mathcal{P}_f(W)$.

Note that if W is a finite set, then $\mathcal{P}_f(W)$ is the same as $\mathcal{P}(W)$, the traditional power set of W, since all subsets of W are finite.

Definition 2.1.2. Let $\mathbf{F} \subseteq \mathcal{P}_f(W)$ be a family of sets. $\mathbf{F}$ is said to be a semifilter on W provided that $S \in \mathbf{F}$ and $S \subseteq T$ where $T \in \mathcal{P}_f(W)$ imply that $T \in \mathbf{F}$.

2.2. **Examples.** We now present a few examples of simple semifilters.

Example 2.2.1. Let $W = \{1, 2, 3\}$. Then the collections

$$\mathbf{E} = \{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$$
$$\mathbf{F} = \{\{1\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$
$$\mathbf{G} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$
$$\mathbf{H} = \{\{1,2,3\}\}$$

are all semifilters on W.

Example 2.2.2. Let $W = \{1, 2, 3, 4\}$. Then consider the collection $\mathbf{F} = \{\{1,2\}, \{1,2,3\}\}$. This is not a semifilter because $\{1,2,3\} \subseteq \{1,2,3,4\}$ but $\{1,2,3,4\} \notin \mathbf{F}$.

Now we define an entire family of semifilters that will appear again later in the paper.

Definition 2.2.3. Let $n \in \mathbb{W}$. Then given a set, W, U_n is defined to be the collection of all finite subsets of W with at least n elements.

Theorem 2.2.4. The collection U_n is a semifilter on W.

Proof. Let $S \in U_n$ with $S \subseteq T$ for some $T \in \mathcal{P}_f(W)$. Since $S \in U_n$, we know $|S| \geq n$. Now since $S \subseteq T$, we have that $|T| \geq |S| \geq n$. Thus since $T \subseteq W$, we have that $T \in U_n$ as well. Therefore U_n is a semifilter on W. □

Here are a few examples of different semifilters from the U_n family.

Example 2.2.5. If $W = \{1, 2\}$, then we see that

$$U_0 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$
$$U_1 = \{\{1\}, \{2\}, \{1, 2\}\}$$
$$U_2 = \{\{1, 2\}\}.$$

Example 2.2.6. If $W = \{1, 2, 3\}$, then

$$U_2 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Before we proceed further, we need to emphasize the differences between two specific semifilters.

Example 2.2.7. Let W be any set. Note that $\mathbf{E} = \emptyset$ is a semifilter on W since $\emptyset$ vacuously fulfills the conditions to be a semifilter. We want to emphasize here that $\mathbf{E} = \emptyset$.

We now wish to contrast this with the semifilter $\mathbf{F} = \mathcal{P}_f(W)$. In this case, $\mathbf{F}$ is the only semifilter on W with the property that $\emptyset \in \mathbf{F}$. We want to emphasize here that $\mathbf{F}$ is not empty and rather that $\emptyset \in \mathbf{F}$.

2.3. **The collection of semifilters on W.** There will be times when we need to refer to the set of all semifilters on a set W. We denote this collection as $\mathcal{F}_W$. This collection has some interesting properties.

Definition 2.3.1. A set, S, is said to be *partially ordered* under binary relation $*$ provided the following three conditions are fulfilled:

1. **Reflexivity**: For all $a \in S$, $a * a$.
2. **Antisymmetry**: For all $a, b \in S$, if $a * b$ and $b * a$, then $a = b$.
3. **Transitivity**: For all $a, b, c \in S$, if $a * b$ and $b * c$, then $a * c$.

It turns out that all collections of sets are partially ordered under $\subseteq$, the set inclusion binary relation.

Theorem 2.3.2. Any collection of sets is partially ordered under set inclusion.

Proof. Let $\mathbf{X}$ be any collection of sets. **Reflexivity**: Let $\mathbf{F} \in \mathbf{X}$. As $\mathbf{F}$ is a set, it is by definition a subset of itself. Thus $\mathbf{F} \subseteq \mathbf{F}$.
Antisymmetry: Let $\mathbf{E}, \mathbf{F} \in \mathbf{X}$ such that $\mathbf{E} \subseteq \mathbf{F}$ and $\mathbf{F} \subseteq \mathbf{E}$. By the definition of set equality, we have that $\mathbf{E} = \mathbf{F}$.
Transitivity: Suppose $\mathbf{E}, \mathbf{F}, \mathbf{G} \in \mathbf{X}$ with $\mathbf{E} \subseteq \mathbf{F}$ and $\mathbf{F} \subseteq \mathbf{G}$. Let $S \in \mathbf{E}$. Since $\mathbf{E} \subseteq \mathbf{F}$, $S \in \mathbf{F}$. Now since $\mathbf{F} \subseteq \mathbf{G}$, $S \in \mathbf{G}$. Therefore $\mathbf{E} \subseteq \mathbf{G}$.

Therefore $\mathbf{X}$ is a partially ordered set. ☐

Now we observe that since $\mathcal{F}_W$, the family of all semifilters on W, is a collection of sets, it is partially ordered.

Given two semifilters on a set W, the union and intersection of those two semiftilers are semifilters on W as well.

Theorem 2.3.3. The collecton $\mathcal{F}_W$ is closed under unions and intersections.

Proof. Unions: Let W be a set and suppose $\mathbf{E}, \mathbf{F} \in \mathcal{F}_W$. Let $S \in \mathbf{E} \cup \mathbf{F}$ and let $T \in \mathcal{P}_f(W)$ such that $S \subseteq T$. Since $S \in \mathbf{E} \cup \mathbf{F}$, we have $S \in \mathbf{E}$ or $S \in \mathbf{F}$. Suppose without loss of generality that $S \in \mathbf{E}$. Since $\mathbf{E}$ is a semifilter, $T \in \mathbf{E}$. Thus $T \in \mathbf{E} \cup \mathbf{F}$. Therefore $\mathbf{E} \cup \mathbf{F}$ is a semifilter on W as well.

Intersections: Suppose $\mathbf{E}, \mathbf{F} \in \mathcal{F}_W$. Further suppose $S \in \mathbf{E} \cap \mathbf{F}$ and let $T \in \mathcal{P}_f(W)$ such that $S \subseteq T$. Since $S \in \mathbf{E} \cap \mathbf{F}$, we have that $S \in \mathbf{E}$ and $S \in \mathbf{F}$. As $\mathbf{E}$ and $\mathbf{F}$ are both semifilters on W, we have $T \in \mathbf{E}$ and $T \in \mathbf{F}$. Therefore $T \in \mathbf{E} \cap \mathbf{F}$, and $\mathbf{E} \cap \mathbf{F}$ is a semifilter on W. ☐

3. Antichains and Generating Sets

3.1. **Antichains.** Recall that earlier we made the distinction that in constructing semifilters, we use sets from the *finite* power set of W rather than the power set. If W is finite, this distinction is unimportant. If W is an infinite set like $\mathbb{Z}$, however, then the distinction is important. In order for a semifilter, $\mathbf{F}$, on an infinite set like $\mathbb{Z}$ to have certain desirable properties, we need the elements of $\mathbf{F}$ to be finite sets.

Since a given element, S, of a semifilter is a finite subset of W, we know that S has a finite number of subsets. This is very important for semifilters and why the finite power set is used instead of the regular power set. Suppose, for example, that $W = \mathbb{Z}$. Then $\mathbb{Z} \supset \{2, 4, 6, \ldots\} \supset \{4, 8, 12, \ldots\} \supset \ldots$. In this way we can construct a chain of proper subsets of $\mathbb{Z}$ that does not terminate. What we want, however, is a way to find the "minimal sets" of a semifilter. It is always possible to find such "minimal sets" if we limit the elements of a semifilter to finite subsets of W.

Definition 3.1.1. Let $\mathbf{F}$ be a semifilter on a set W. Then

$$M_{\mathbf{F}} := \{S \in \mathbf{F} \mid \text{ for all } T \in \mathbf{F}, \text{ if } T \neq S \text{ then, } T \nsubseteq S\}$$

is called the **antichain** of finite subsets of W.

Our goal is to show in some way that the antichain of a semifilter, $\mathbf{F}$, will be the collection of "minimal sets" of a semifilter and can somehow be used to describe the semifilter. First, we'll show that every set in a semifilter is either an element of the antichain or is a superset of an element in the antichain.

Lemma 3.1.2. Let $\mathbf{F}$ be a semifilter on some set W. Then for all $T \in \mathbf{F}$, there exists some $S \in M_{\mathbf{F}}$ such that $S \subseteq T$.

Proof. Let $T \in \mathbf{F}$. If there exists some $T_1 \in \mathbf{F}$ such that $T_1 \subseteq T$, then choose T_1. Now repeat this process for T_1. If there exists some $T_2 \in \mathbf{F}$ such that $T_2 \subseteq T_1$, choose T_2. Since $T \in \mathcal{P}_f(W)$, we know that $|T| < \infty$. Thus if we continue the process above, it will eventually terminate with some $T_i \in \mathbf{F}$. The set T_i will have the property that for all $S \in \mathbf{F}$, we have $S \nsubseteq T_i$ unless $S = T_i$. Thus $T_i \in M_{\mathbf{F}}$. Therefore for all $T \in \mathbf{F}$, there exists some subset, $T_i \subseteq T$, that is an element of $M_{\mathbf{F}}$. □

When refering to individual elements of the antichain, we call the sets minimal sets of $\mathbf{F}$. Now let's look at a few examples of antichains.

Example 3.1.3. Let $W = \{1, 2, 3\}$ and let

$$\begin{aligned} \mathbf{E} &= \{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\} \\ \mathbf{F} &= \{\{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\} \\ \mathbf{G} &= \{\{1,2,3\}\} \\ \mathbf{H} &= \mathcal{P}_f(W). \end{aligned}$$

Then

$$\begin{aligned} M_{\mathbf{E}} &= \{\{1\}\} \\ M_{\mathbf{F}} &= \{\{1,2\}, \{1,3\}, \{2,3\}\} \\ M_{\mathbf{G}} &= \{\{1,2,3\}\} \\ M_{\mathbf{H}} &= \{\emptyset\}. \end{aligned}$$

Observe that for any set W, if $\mathbf{F} = \mathcal{P}_f(W)$, then $M_{\mathbf{F}} = \{\emptyset\}$.

Example 3.1.4. Let W be any arbitrary set and let $\mathbf{I} = \emptyset$. In this instance, $\mathbf{I}$ is the empty semifilter. Since $\mathbf{I}$ contains no sets, $M_{\mathbf{I}}$ is empty as well. Thus $M_{\mathbf{I}} = \emptyset$. Compare this to the previous example where $M_{\mathbf{H}} = \{\emptyset\}$.

3.2. **Generating Sets.** When working with the antichain, we talked about reducing a semifilter to its minimal sets. Often it is more useful to start with a collection of sets and talk about a semifilter somehow associated with those sets. We now present a way to build a semifilter given an arbitrary list of subsets of a set W. This idea has two useful results. First, it allows us to express a semifilter in terms of its minimal sets rather than writing out the whole semifilter. Secondly, it says that given any arbitrary collection of sets, we are able to construct a semifilter containing those sets.

Definition 3.2.1. Let M be any collection of finite subsets of W and let $\mathbf{B}_M := \{T \in \mathcal{P}_f(W) \mid S \subseteq T \text{ for some } S \in M\}$. We then say that M is the **generating set** of $\mathbf{B}_M$.

According to Definition **??**, $\mathbf{B}_M$ is just a collection of sets from $\mathcal{P}_f(W)$. Now we wish to show that $\mathbf{B}_M$ is indeed a semifilter.

Theorem 3.2.2. The collection $\mathbf{B}_M$ is a semifilter on the set W.

Proof. Let M be any collection of finite subsets of W. Suppose $T \in \mathbf{B}_M$ and that $T \subseteq U$ for some $U \in \mathcal{P}_f(W)$. Since $T \in \mathbf{B}_M$, there exists some set $S \in M$ such that $S \subseteq T$. Thus as $T \subseteq U$, we see that $S \subseteq U$. Then by the definition of $\mathbf{B}_M$, we have $U \in \mathbf{B}_M$ as well. Therefore $\mathbf{B}_M$ is a semifilter on W. □

Now let's look at a few examples.

Example 3.2.3. Let $W = \{1, 2, 3\}$ and let

$$\begin{aligned} M_1 &= \{\{1\}\} \\ M_2 &= \{\{2\}, \{1,3\}\} \\ M_3 &= \{\{2,3\}, \{1,2,3\}\} \\ M_4 &= \{\emptyset\}. \end{aligned}$$

Then we get

$$\begin{aligned} \mathbf{B}_{M_1} &= \{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\} \\ \mathbf{B}_{M_2} &= \{\{2\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\} \\ \mathbf{B}_{M_3} &= \{\{2,3\}, \{1,2,3\}\} = M_3 \\ \mathbf{B}_{M_4} &= \mathcal{P}_f(W). \end{aligned}$$

Comparing Examples **??** and **??**, we see that $M_{\mathbf{E}} = M_1$ and that $\mathbf{E} = \mathbf{B}_{M_1}$. Furthermore, $M_{\mathbf{H}} = \{\emptyset\} = M_4$ and $\mathbf{H} = \mathcal{P}_f(W) = \mathbf{B}_{M_4}$. It appears that the relationship between antichains and generating sets is that the generating set for a semifilter is the antichain for that semifilter. We now formalize this.

Theorem 3.2.4. Let $\mathbf{F}$ be a semifilter on a set W. Then $\mathbf{B}_{M_{\mathbf{F}}} = \mathbf{F}$.

Proof. First we shall show that $\mathbf{F} \subseteq \mathbf{B}_{M_{\mathbf{F}}}$. Let $T \in \mathbf{F}$. By lemma **??** there exists some minimal set, S, of $\mathbf{F}$ such that $S \subseteq T$. Then by definition, $T \in \mathbf{B}_{M_{\mathbf{F}}}$. Thus $\mathbf{F} \subseteq \mathbf{B}_{M_{\mathbf{F}}}$.

Now we show that $\mathbf{B}_{M_{\mathbf{F}}} \subseteq \mathbf{F}$. Let $T \in \mathbf{B}_{M_{\mathbf{F}}}$. Then there exists some $S \in M_{\mathbf{F}}$ such that $S \subseteq T$. As $S \in M_{\mathbf{F}}$, we see that $S \in \mathbf{F}$. Now since $S \subseteq T$ and $T \in \mathcal{P}_f(W)$, we have that $T \in \mathbf{F}$. Thus $\mathbf{F} \subseteq \mathbf{B}_{M_{\mathbf{F}}}$. Therefore $\mathbf{F} = \mathbf{B}_{M_{\mathbf{F}}}$. □

We have now shown that we can represent a semifilter by its generating sets. From now on, we shall use the notation $\langle S_1, S_2, \ldots \rangle$ to mean the semifilter generated by the sets $S_1, S_2, \ldots$ where it is assumed the $S_i \in \mathcal{P}_f(W)$. It need not be the case that the generating set of a semifilter is countable. In this paper, we do not deal with that case though. Finally, note that while a set in a semifilter must be finite, a semifilter may contain infinitely many sets as well as be generated by infinitely many sets.

4. Multiplication of Semifilters

4.1. **Product Semifilters.**

Thus far we've seen some ways to generate new semifilters from subsets of W and from other semifilters. We've also presented a way to express a semifilter in terms of its minimal sets. Now we will present another concept that allows us to construct new semifilters from old ones as well as express some semifilters in terms of other ones.

Definition 4.1.1 (Product semifilter)**.** Let $\mathbf{E}$ and $\mathbf{F}$ be semifilters on a set W. Then the collection

$$\mathbf{EF} := \{U \in \mathcal{P}_f(W) \mid U = S \cup T \text{ where } S \in \mathbf{E}, T \in \mathbf{F} \text{ and } S \cap T = \emptyset\}$$

is said to be the **product semifilter** of $\mathbf{E}$ and $\mathbf{F}$. We sometimes use dot notation $(\mathbf{E} \cdot \mathbf{F})$ or $(\mathbf{E})(\mathbf{F})$ (parenthetical notation) for semifilter multiplication when visual representation becomes an issue.

What this definition says is that the product of semifilters $\mathbf{E}$ and $\mathbf{F}$ is the collection of disjoint unions of sets from $\mathbf{E}$ and $\mathbf{F}$. That is, suppose $S \in \mathbf{E}$ and $T \in \mathbf{F}$. If $S \cap T = \emptyset$, then $S \cup T \in \mathbf{EF}$.

Let's look at some examples of product semifilters before presenting any theorems regarding them.

Example 4.1.2. In this first example we will explicitly construct a product semifilter. Let's begin by letting $W = \{1, 2, 3\}, \mathbf{E} = \langle \{1\} \rangle$, and $\mathbf{F} = \langle \{2\} \rangle$. Before constructing $\mathbf{EF}$, let us first list the elements of $\mathbf{E}$ and $\mathbf{F}$ explicitly.

$$\mathbf{E} = \{\{1\},\{1,2\},\{1,3\},\{1,2,3\}\}$$
$$\mathbf{F} = \{\{2\},\{1,2\},\{2,3\},\{1,2,3\}\}$$

To construct **EF**, we begin by noticing that $\{1\}$ and $\{2\}$ are disjoint. Thus $\{1\} \cup \{2\} = \{1,2\} \in \mathbf{EF}$. Now observe that $\{1\} \in \mathbf{E}$ and $\{2,3\} \in \mathbf{F}$ are disjoint. Thus $\{1\} \cup \{2,3\} = \{1,2,3\} \in \mathbf{EF}$. Next we notice that the only remaining pair of disjoint sets is $\{1,3\} \in \mathbf{E}$ and $\{2\} \in \mathbf{F}$. Thus we consider $\{1,3\} \cup \{2\} = \{1,2,3\}$, but we already have $\{1,2,3\} \in \mathbf{EF}$. Finally we see that

$$\mathbf{EF} = \{\{1,2\},\{1,2,3\}\} = \langle\{1,2\}\rangle .$$

In this case the minimal set of **EF** is $\{1,2\}$, which is the union of the minimal sets of **E** and **F**. In general this is not true however.

Example 4.1.3. Let $W = \{1,2,3,4\}$ and let $\mathbf{E} = \langle\{1\},\{2\},\{3,4\}\rangle$ and $\mathbf{F} = \langle\{1,2\},\{3\},\{4\}\rangle$. Again let's list the elements of these semifilers explicitly.

$$\begin{aligned}\mathbf{E} = \; & \{\{1\},\{2\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}, \\ & \{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\} \\ \mathbf{F} = \; & \{\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}, \\ & \{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\}\end{aligned}$$

By comparing the two semifilters, we see that

$$\begin{aligned}\mathbf{EF} = \; & \{\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{1,2,3\},\{1,2,4\}, \\ & \{1,3,4\},\{2,3,4\},\{1,2,3,4\}\}.\end{aligned}$$

Thus $\mathbf{EF} = \langle\{1,3\},\{1,4\},\{2,3\},\{2,4\}\rangle$.

There are several things worth noting in Example **??**. First, minimal sets of the original filters need not be in the product. Observe that $\{3,4\} \in \mathbf{E}$, but $\{3,4\} \notin \mathbf{EF}$. Second, the generating set for **EF** in Example **??** is the disjoint unions of the generating sets for **E** and **F**. This turns out to be the general rule: the generating sets of a product semifilter, **EF**, are the disjoint unions of generating sets from **E** and **F**.

It turns out that multiplication of semifilters has many other desirable properties. Just like traditional multiplication, semifiter multiplication is associative, commutative, has a "zero" element, and has an identity. We prove all this now.

Theorem 4.1.4. Let $\mathbf{E}, \mathbf{F}$, and $\mathbf{G}$ be semifilters on a set W. Multiplication of semifilters satisfies the following properties:

(1) It is associative. That is, $\mathbf{E}(\mathbf{FG}) = (\mathbf{EF})\mathbf{G}$.
(2) It is commutative. That is, $\mathbf{EF} = \mathbf{FE}$.
(3) There is a "zero" semifilter. That is, there exists some semifilter $\mathbf{0}$ such that for any semifilter $\mathbf{E}$, we have $\mathbf{E0} = \mathbf{0}$.
(4) There is a multiplicative identity. That is, there exists some semifilter $\mathbf{1}$ such that for any semifilter $\mathbf{E}$, we have $\mathbf{E1} = \mathbf{E}$.

Proof. Let $\mathbf{E}, \mathbf{F}$, and $\mathbf{G}$ be semifilters on a set W.

(1). Suppose $S \in \mathbf{E}(\mathbf{FG})$. Then $S = X \cup (Y \cup Z)$ for some $X \in \mathbf{E}$, $Y \in \mathbf{F}$, and $Z \in \mathbf{G}$ such that $X \cap (Y \cup Z) = \emptyset$ and $Y \cap Z = \emptyset$. Since X is disjoint from the union of two disjoint sets, the three sets must be pairwise disjoint. Thus $(X \cup Y) \cap Z = \emptyset$. Now since the union operator is associative, we have that $X \cup (Y \cup Z) = (X \cup Y) \cup Z$. Thus $S \in (\mathbf{EF})\mathbf{G}$. Since the above process is reversible, we have that $\mathbf{E}(\mathbf{FG}) = (\mathbf{EF})\mathbf{G}$.

(2). Suppose $S \in \mathbf{EF}$. Then $S = X \cup Y$ for some disjoint $X \in \mathbf{E}$ and $Y \in \mathbf{F}$. Since the union operator is commutative, $S = X \cup Y = Y \cup X$. Thus $S \in \mathbf{FE}$ and $\mathbf{EF} \subseteq \mathbf{FE}$. Since this argument is reversible, we have that $\mathbf{FE} \subseteq \mathbf{EF}$ and thus $\mathbf{EF} = \mathbf{FE}$.

(3). The "zero" semifilter is $\emptyset$ (the empty semifilter). Consider $\mathbf{E} \cdot \emptyset$. If $S \in \mathbf{E} \cdot \emptyset$ then S must be the union of two disjoint sets, one of which is an element of $\emptyset$. This however is impossible as $|\emptyset| = 0$. Therefore $\mathbf{E} \cdot \emptyset = \emptyset$

(4). The multiplicative identity is $\mathcal{P}_f(W)$. We'll first show that $\mathbf{E} \cdot \mathcal{P}_f(W) \subseteq \mathbf{E}$. Let $S \in \mathbf{E} \cdot \mathcal{P}_f(W)$. Then $S = X \cup Y$ for some $X \in \mathbf{E}$ and $Y \in \mathcal{P}_f(W)$ such that $X \cap Y = \emptyset$. Clearly $X \subseteq X \cup Y$ and $X \cup Y \in \mathcal{P}_f(W)$. Thus by the definition of a semifilter, $X \cup Y \in \mathbf{E}$. Thus $\mathbf{E} \cdot \mathcal{P}_f(W) \subseteq \mathbf{E}$.

Now we'll show that $\mathbf{E} \subseteq \mathbf{E} \cdot \mathcal{P}_f(W)$. Let $S \in \mathbf{E}$. Then since $\emptyset \in \mathcal{P}_f(W)$ and $S \cap \emptyset = \emptyset$, we have that $S = S \cap \emptyset \in \mathbf{E} \cdot \mathcal{P}_f(W)$. Thus we have $\mathbf{E} \subseteq \mathbf{E} \cdot \mathcal{P}_f(W)$. Therefore $\mathbf{E} \cdot \mathcal{P}_f(W) = \mathbf{E}$. □

One interesting property of multiplication of semifilters is that depending on the two semifilters being multiplied, the product may have more or fewer elements than the original two semifilters.

Example 4.1.5. Recall that in Example **??**, $|\mathbf{E}| = |\mathbf{F}| = 4$. However, for that example $|\mathbf{EF}| = 2$. So in Example **??**, the product semifilter ends up being *smaller* than both the two terms of the product.

Now look back at Example **??**. In that example, $|\mathbf{E}| = |\mathbf{F}| = 7$ while $|\mathbf{EF}| = 9$. So in example **??**, the product is *larger* than both the original filters.

Example 4.1.6. Another strange property of semifilter multiplication is that any factorization of $\mathcal{P}_f(W)$ (the multiplicative identity) is trivial. That is, if $\mathbf{EF} = \mathcal{P}_f(W)$ then $\mathbf{E} = \mathbf{F} = \mathcal{P}_f(W)$.

Consider $\mathcal{P}_f(W)$. If we wanted to write $\mathcal{P}_f(W)$ as the product of two semifilters, we would have to write $\emptyset$ as the disjoint union of two elements of $\mathcal{P}_f(W)$. The only way to do this is to write $\emptyset = \emptyset \cup \emptyset$ while noting that $\emptyset \cap \emptyset = \emptyset$. Thus if $\mathcal{P}_f(W) = \mathbf{EF}$ then $\emptyset \in \mathbf{E}$ and $\emptyset \in \mathbf{F}$.

This implies that $\mathbf{E} = \mathbf{F} = \mathcal{P}_f(W)$. So the only way to factor $\mathcal{P}_f(W)$ is as $\mathcal{P}_f(W) = \mathcal{P}_f(W) \cdot \mathcal{P}_f(W)$.

The fact that $\mathcal{P}_f(W)$ (the multiplicative identity) can only be factored trivially implies that there is no such thing as "semifilter inverses" in the traditional sense that $x \cdot x^{-1} = 1$. That is, given a semifilter $\mathbf{E}$, there does not exist some $\mathbf{E}^{-1}$ with the property that $\mathbf{E} \cdot \mathbf{E}^{-1} = \mathbf{1}$ unless $\mathbf{E} = \mathcal{P}_f(W)$.

It turns out that the semifilter $\mathbf{F} = \emptyset$ has some interesting properties as well. One such property is that for most sets W, there are numerous ways to write $\mathbf{F}$ as a product.

Example 4.1.7. In order to write $\emptyset$ as a non-trivial product, the two semifilters must have the property that their minimal sets are pairwise non-disjoint.

Let $\mathbf{E}, \mathbf{F}$ be semifilters such that neither is $\emptyset$ or $\mathcal{P}_f(W)$. Then $\mathbf{EF} = \emptyset$ if only if given arbitrary minimal sets $X \in \mathbf{E}$ and $Y \in \mathbf{F}$, we have that $X \cap Y \neq \emptyset$.

If we suppose $\mathbf{EF} = \emptyset$, then given any $X \in \mathbf{E}$ and $Y \in \mathbf{F}$, it must be that $X \cap Y \neq \emptyset$. Since X and Y are arbitrary, this must be true for minimal sets as well.

Now suppose any pair of minimal sets from $\mathbf{E}$ and $\mathbf{F}$ are non-disjoint. Then as any minimal set of $\mathbf{E}$ is not disjoint from any minimal set of $\mathbf{F}$, any set in $\mathbf{E}$ will not be disjoint from any set in $\mathbf{F}$. Thus no union of sets will be in the product semifilter. Therefore $\mathbf{EF} = \emptyset$ if and only if every pair of minimal sets $X \in \mathbf{E}$ and $Y \in \mathbf{F}$ have the property that $X \cap Y \neq \emptyset$.

Example 4.1.8. Let $W = \{1, 2, 3\}$ and $\mathbf{E} = \langle\{1\}\rangle$. Then $\mathbf{E}^2 = \mathbf{EE} = \emptyset$ since for any two sets $X, Y \in \mathbf{E}$, we have $X \cap Y \neq \emptyset$. Now let $\mathbf{F} = \langle\{1, 2\}\rangle$. Note that $\mathbf{E} \neq \mathbf{F}$, but we still have that $\mathbf{EF} = \emptyset$.

Example **??** shows that when some semifilters are squared, the product is $\mathbf{F} = \emptyset$. Also, it gives an example of two non-empty semifilters whose product is empty.

Semifilter multiplication has many useful applications, but there are still many open questions regarding factorization of semifilters. When W is finite, it is a simple process to construct product semifilters. However, given any arbitrary semifilter on W, it can be extremely difficult to determine whether that semifilter is able to be factored.

5. Conjugates

5.1. **Introduction to Conjugates.** This section deals with a construction called the conjugate of a semifilter. Let us begin with the definition and then explore some examples.

Definition 5.1.1. Let $\mathbf{F}$ be a semifilter on a set W. The collection $\overline{\mathbf{F}} := \{S \in \mathcal{P}_f(W) \mid \text{such that for all } X \in \mathbf{F},\ S \cap X \neq \emptyset\}$ is called the conjugate of $\mathbf{F}$.

Example 5.1.2. Let $W = \{1, 2\}$. Then the following are all semifilters on W:

$$\begin{aligned}
\mathbf{F}_1 &= \langle \emptyset \rangle \\
\mathbf{F}_2 &= \langle \{1\}, \{2\} \rangle \\
\mathbf{F}_3 &= \langle \{1\} \rangle \\
\mathbf{F}_4 &= \langle \{2\} \rangle \\
\mathbf{F}_5 &= \langle \{1, 2\} \rangle \\
\mathbf{F}_6 &= \emptyset
\end{aligned}$$

Let's begin with $\mathbf{F}_1$. Since $\mathbf{F}_1 = \langle \emptyset \rangle$, we have that $\mathbf{F}_1 = \mathcal{P}_f(W)$. Now as $\emptyset \in \mathbf{F}_1$, if $S \in \overline{\mathbf{F}_1}$ then $S \cap \emptyset \neq \emptyset$. However, this is impossible. Therefore $\overline{\mathbf{F}_1} = \emptyset = \mathbf{F}_6$.

Now consider $\mathbf{F}_2$. Every set contained in $\overline{\mathbf{F}_2}$ must intersect with both $\{1\}$ and $\{2\}$. So we observe that $\{1, 2\} \in \mathbf{F}_2$. Next we see that no proper subset of $\{1, 2\}$ is in $\overline{\mathbf{F}_2}$ since any subset will be missing 1 or 2 and thus will not have a non-empty intersectin with $\{1\}$ or $\{2\}$. Thus $\overline{\mathbf{F}_2} = \langle \{1, 2\} \rangle = \mathbf{F}_5$.

Following the same method as above, we get the following results:

$$\begin{aligned}
\overline{\mathbf{F}_1} &= \mathcal{P}_f(W) = \mathbf{F}_6 \\
\overline{\mathbf{F}_2} &= \mathbf{F}_5 \\
\overline{\mathbf{F}_3} &= \mathbf{F}_3 \\
\overline{\mathbf{F}_4} &= \mathbf{F}_4 \\
\overline{\mathbf{F}_5} &= \mathbf{F}_2 \\
\overline{\mathbf{F}_6} &= \mathbf{F}_1 = \emptyset
\end{aligned}$$

Now turn your attention to Figures **??** and **??**. Figure **??** is the semifilter lattice for $|W| = 2$. The labels of the semifilters in the figures corresponds to the the semifilters from Example **??** We arrange the semifilters in the lattice just as we would if we thought of them as just subsets of $\mathcal{P}_f(W)$.

In Figure **??**, we took the original semifilter lattice and where we wrote $\mathbf{F}_i$ before, we've now written $\overline{\mathbf{F}_i}$. What we find is that other than being flipped over, the lattice is the same as before.

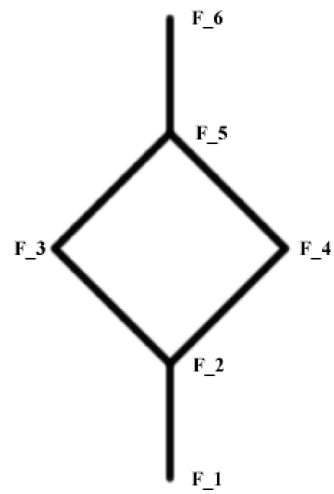

FIGURE 1. The semifilter lattice of $\mathcal{F}_W$ for $|W| = 2$.

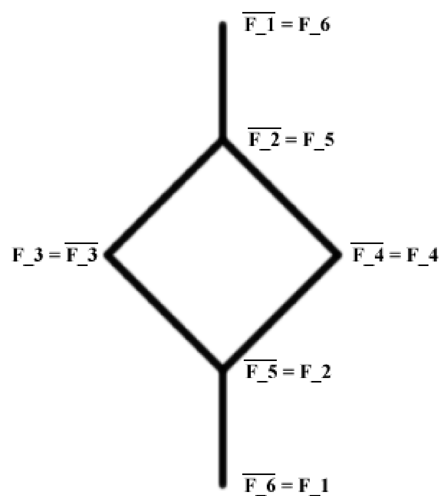

FIGURE 2. The lattice of conjugates corresponding to the lattice in Figure **??**

From looking at Example **??**, it is strongly suggested that for any semifilter $\mathbf{F}$, its conjugate, $\overline{\mathbf{F}}$, is also a semifilter. We formalize this in the following theorem.

Theorem 5.1.3. Let $\mathbf{F}$ be a semifilter on some set W. Then $\overline{\mathbf{F}}$ is also a semifilter on W.

Proof. Suppose $S \in \overline{\mathbf{F}}$ and that $S \subseteq T$ for some $T \in \mathcal{P}_f(W)$. Since $S \in \overline{\mathbf{F}}$, we have that given any $X \in \mathbf{F}$, we have $S \cap X \neq \emptyset$. Now consider $T \cap X$. Since $S \subseteq T$, we find that

$$T \cap X = (S \cap X) \cup ((T - S) \cap X) \neq \emptyset$$

since $S \cap X \neq \emptyset$. Therefore $T \in \overline{\mathbf{F}}$ and $\overline{\mathbf{F}}$ is a semifilter on W. □

Now we know for certain that $\overline{\mathbf{F}}$ is a semifilter. Conjugates have many more interesting properties that we will present in the following section.

5.2. **Results Concerning Conjugates.** Our first theorem relates semifilters whose product is the emptyset and the conjugates of those semifilters. Before we present the proof, let's look at an example.

Example 5.2.1. Let $W = \{1, 2, 3, 4\}$ and suppose $\mathbf{E} = \langle\{1, 2\}\rangle$ and $\mathbf{F} = \langle\{2, 3\}\rangle$.

Note that $\mathbf{EF} = \emptyset$ since $\{1, 2\} \cap \{2, 3\} \neq \emptyset$. Now observe:

$$\overline{\mathbf{E}} = \langle\{1\}, \{2\}\rangle$$
$$\overline{\mathbf{F}} = \langle\{2\}, \{3\}\rangle .$$

Since $\{2\} \in \overline{\mathbf{F}}$, we have that $\{1, 2\} \in \overline{\mathbf{F}}$. Thus $\mathbf{E} \subseteq \overline{\mathbf{F}}$. Likewise, since $\{2\} \in \overline{\mathbf{E}}$, we see that $\{2, 3\} \in \overline{\mathbf{E}}$ and so $\mathbf{F} \subseteq \overline{\mathbf{E}}$.

Based on this example, it seems that if $\mathbf{EF} = \emptyset$ then $\mathbf{E} \subseteq \overline{\mathbf{F}}$ and $\mathbf{F} \subseteq \overline{\mathbf{E}}$. This is true. In fact, the three statements turn out to be equivalent as we show in the following theorem.

Theorem 5.2.2. Let $\mathbf{E}$ and $\mathbf{F}$ be semifilters on a set W. Then the following statements are equivalent:

(1) $\mathbf{EF} = \emptyset$.
(2) $\mathbf{E} \subseteq \overline{\mathbf{F}}$.
(3) $\mathbf{F} \subseteq \overline{\mathbf{E}}$.

Proof. We will first show that property (1) implies properties (2) and (3). Suppose $\mathbf{EF} = \emptyset$ and let $S \in \mathbf{E}$ and $T \in \mathbf{F}$ arbitrarily. If $S \cap T = \emptyset$, then $S \cup T \in \mathbf{EF}$. However, $\mathbf{EF} = \emptyset$. Thus $S \cap T \neq \emptyset$. As both S and T are aribtrary sets of $\mathbf{E}$ and $\mathbf{F}$, $S \cap T \neq \emptyset$ implies that $S \in \overline{\mathbf{F}}$ and $T \in \overline{\mathbf{E}}$. Therefore $\mathbf{E} \subseteq \overline{\mathbf{F}}$ and $\mathbf{F} \subseteq \overline{\mathbf{E}}$.

Now we will show that properties (2) and (3) imply property (1). Suppose without loss of generality that $\mathbf{E} \subseteq \overline{\mathbf{F}}$ (the proof is analogous for $\mathbf{F} \subseteq \overline{\mathbf{E}}$). Let $S \in \mathbf{E}$ and $T \in \mathbf{F}$ be arbitrary sets. $\mathbf{E} \subseteq \overline{\mathbf{F}}$ implies that $S \cap T \neq \emptyset$. Since S and T are arbitrary, there is no pair of disjoint set from $\mathbf{E}$ and $\mathbf{F}$. Therefore $\mathbf{EF} = \emptyset$. □

Example 5.2.3. Let $W = \{1,2,3,4\}$ and let $\mathbf{E} = \langle\{1,2\}\rangle$ and $\mathbf{F} = \langle\{1\}\rangle$. Then $\overline{\mathbf{E}} = \langle\{1\},\{2\}\rangle$ and $\overline{\mathbf{F}} = \langle\{1\}\rangle$.

First notice that since $\{1\} \subseteq \{1,2\}$, we see that $\mathbf{E} \subseteq \mathbf{F}$. As $\{1\} \in \overline{\mathbf{E}}$, we see that $\overline{\mathbf{F}} \subseteq \overline{\mathbf{E}}$.

Now consider $\overline{\mathbf{G}} = \langle\{1\},\{3\}\rangle$. If $S \in \overline{\mathbf{G}}$, then $S \cap \{1\} \neq \emptyset$ and $S \cap \{3\} \neq \emptyset$. Thus for all $S \in \overline{\mathbf{G}}$, we have that $\{1,3\} \subseteq S$. Thus $\mathbf{E} \not\subseteq \mathbf{G}$ since $\{1,2\} \in \mathbf{E}$, but $\{1,3\} \not\subseteq \{1,2\}$. In this instance we observe that $\overline{\mathbf{G}} \not\subseteq \overline{\mathbf{E}}$ and that $\mathbf{E} \not\subseteq \mathbf{G}$.

We see in Example **??** that $\mathbf{E} \subseteq \mathbf{F}$ and $\overline{\mathbf{F}} \subseteq \overline{\mathbf{E}}$ while $\overline{\mathbf{G}} \not\subseteq \overline{\mathbf{E}}$ and $\mathbf{E} \not\subseteq \mathbf{G}$. We now formalize this relationship between the conjugates of semifilters $\mathbf{E}$ and $\mathbf{F}$ when we know that $\mathbf{E} \subseteq \mathbf{F}$.

Theorem 5.2.4. Let $\mathbf{E}$ and $\mathbf{F}$ be semifilters on a set W such that $\mathbf{E} \subseteq \mathbf{F}$. Then $\overline{\mathbf{F}} \subseteq \overline{\mathbf{E}}$.

Proof. Suppose that $\mathbf{E} \subseteq \mathbf{F}$ and let $X \in \overline{\mathbf{F}}$. Since $X \in \overline{\mathbf{F}}$, we have that for all $T \in \mathbf{F}$, we have that $X \cap T \neq \emptyset$. Now since $\mathbf{E} \subseteq \mathbf{F}$, for all $S \in \mathbf{E}$, we see that $X \cap S \neq \emptyset$. Thus $X \in \overline{\mathbf{E}}$. Therefore $\overline{\mathbf{F}} \subseteq \overline{\mathbf{E}}$. □

The following theorems give the relationship between a semifilter $\mathbf{F}$ an its higher order conjugates. First we show that $\mathbf{F} \subseteq \overline{\overline{\mathbf{F}}}$ and then we show that $\overline{\mathbf{F}} = \overline{\overline{\overline{\mathbf{F}}}}$.

Theorem 5.2.5. Let $\mathbf{F}$ be a semifilter on a set W. Then $\mathbf{F} \subseteq \overline{\overline{\mathbf{F}}}$.

Proof. Let $S \in \mathbf{F}$. Then for all $T \in \overline{\mathbf{F}}$, we know $T \cap S \neq \emptyset$. Therefore $S \in \overline{\overline{\mathbf{F}}}$. □

Theorem 5.2.6. Let $\mathbf{F}$ be a semifilter on a set W. Then $\overline{\mathbf{F}} = \overline{\overline{\overline{\mathbf{F}}}}$.

Proof. We begin by showing that $\overline{\mathbf{F}} \subseteq \overline{\overline{\overline{\mathbf{F}}}}$. Let $S \in \overline{\mathbf{F}}$. Then $S \cap T \neq \emptyset$ for all $T \in \overline{\overline{\mathbf{F}}}$. Thus $S \in \overline{\overline{\overline{\mathbf{F}}}}$.

We will now show that $\overline{\mathbf{F}} \supseteq \overline{\overline{\overline{\mathbf{F}}}}$. Let $S \in \overline{\overline{\overline{\mathbf{F}}}}$. Then $S \cap T \neq \emptyset$ for all $T \in \overline{\overline{\mathbf{F}}}$. Then since $\mathbf{F} \subseteq \overline{\overline{\mathbf{F}}}$, for all $U \in \mathbf{F}$ we have that $S \cap U \neq \emptyset$. Thus $S \in \overline{\mathbf{F}}$. □

Note that since $\overline{\mathbf{F}} = \overline{\overline{\overline{\mathbf{F}}}}$ there are at most three higher order conjugates of a given semifilter $\mathbf{F}$.

Theorem 5.2.7. Let $\{\mathbf{F}_i\}_{i\in I}$ be a family in $\mathcal{F}_W$. Then $\overline{\cup_{i\in I}\mathbf{F}_i} = \cap_{i\in I}\overline{\mathbf{F}_i}$.

Proof. We begin by showing that $\overline{\cup_{i\in I}\mathbf{F}_i} \subseteq \cap_{i\in I}\overline{\mathbf{F}_i}$. Let $S \in \overline{\cup_{i\in I}\mathbf{F}_i}$. Then for all $T \in \cup_{i\in I}\mathbf{F}_i$, we have $S \cap T \neq \emptyset$. Thus for each $\mathbf{F}_i$, for all

$T \in \mathbf{F}_i$, we have $S \cap T \neq \emptyset$. Thus for each i, we have $S \in \overline{\mathbf{F}_i}$. Therefore $S \in \cap_{i \in I} \overline{\mathbf{F}_i}$.

Now we show that $\overline{\cup_{i \in I} \mathbf{F}_i} \supseteq \cap_{i \in I} \overline{\mathbf{F}_i}$. Let $S \in \cap_{i \in I} \overline{\mathbf{F}_i}$. Then for each $\mathbf{F}_i$, given $T \in \mathbf{F}_i$, we have $S \cap T \neq \emptyset$. Let $U \in \cup_{i \in I} \mathbf{F}_i$. Then $U \in \mathbf{F}_i$ for some i. Then we have that $S \cap U \neq \emptyset$. As U is arbitrary, we have $S \in \overline{\cup_{i \in I} \mathbf{F}_i}$. □

6. Isomorphisms of Semifilters

We are now working towards the most important results of the paper. The first of these results is the Isomorphism-Permutation theorem. This theorem states that every isomorphism of semifilters is determined by a permutation of the base set W.

6.1. **Preliminaries.**

Before presenting the proof of the Isomorphism-Permuation Theorem, we start with some definitions and preliminary results.

All results in this section depend on W being a finite set. We implicitly assume that W is finite for all discussions in this section.

We first present a short-hand notation for ease of reading. We will often work with sets of the form $W - \{a\}$ where $a \in W$. As this notation can become quite cumbersome we will denote this set by W_a.

Next we present a way to relate an arbitrary set $X \in \mathbf{F}$ in a one-to-one correspondence with intersections of sets of the form W_a.

Lemma 6.1.1. Let $\mathbf{F}$ be a semifilter on a finite set W. Suppose $X \in \mathbf{F}$. Then there exists a unique set $\{a_1, a_2, \ldots, a_n\}$ such that $X = W_{a_1} \cap W_{a_2} \cap \ldots \cap W_{a_n}$ where for each i, we have $W_{a_i} \in \mathbf{F}$.

Proof. Let $\mathbf{F}$ be a semifilter on a finite set W and suppose $X \in \mathbf{F}$. Then $X \in \mathcal{P}_f(W)$ and thus $X \subseteq W$. Then there exists a unique set $A = \{a_1, a_2, \ldots, a_n\}$ such that $X \cup A = W$ and $X \cap A = \emptyset$.

Consider $W_{a_1} \cap W_{a_2} \cap \ldots \cap W_{a_n}$. Since for each i, we have $a_i \notin W_{a_i}$, we see $a_i \notin \cap_{i=1}^n W_{a_i}$. Furthermore, we have that $X \subseteq W_{a_i}$ since $a_i \notin X$. Thus $X \subseteq \cap_{i=1}^n W_{a_i}$. Since $A \nsubseteq \cap_{i=1}^n W_{a_i}$ and $X \subseteq \cap_{i=1}^n W_{a_i}$, it must be that $\cap_{i=1}^n W_{a_i} = X$ since $X \cup A = W$. Therefore

$$X = W_{a_1} \cap W_{a_2} \cap \ldots \cap W_{a_n}.$$

□

For the Isomorphism-Permutation Theorem, we will be working with permutations of the elements of W.

Definition 6.1.2. We say that σ is a permutation of a set, W, provided that $\sigma : W \to W$ is a bijection.

Before we proceed, however, we must extend the definition of permutation to the finite power set of W so that we can apply permutations to semifilters. We denote the set of all permutations of W by S_W. The elements of this set act upon the set W and induce a permutation on it. A semifilter, however, is a collection of sets and thus outside the domain and range of permutations of W. We now define σ' to induce the permutation σ upon semifilters.

Definition 6.1.3. Let $\sigma \in S_W$. Then σ' is defined to be the bijection $\sigma' : \mathcal{P}_f(W) \to \mathcal{P}_f(W)$ where if $\mathbf{F} \subseteq \mathcal{P}_f(W)$ then $\sigma'(\mathbf{F}) = \{\sigma(S) \mid S \in \mathbf{F}\}$.

For a given $\mathbf{F}$, $\sigma'(\mathbf{F})$ is determined by σ acting upon the underlying sets of $\mathbf{F}$.

Let us now turn our attention to an example to illustrate this point.

Example 6.1.4. Let $W = \{1, 2, 3\}$. Let $\sigma \in S_W$ such that $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$. Consider the semifilter $\mathbf{F} = \langle\{1\}\rangle$.

By applying σ, we see that

$$\begin{aligned} \sigma(\{1\}) &= \{2\} \\ \sigma(\{1,2\}) &= \{2,3\} \\ \sigma(\{1,3\}) &= \{1,2\} \\ \sigma(\{1,2,3\}) &= \{1,2,3\}. \end{aligned}$$

Thus $\sigma'(\mathbf{F}) = \{\{2\}, \{2,3\}, \{1,2\}, \{1,2,3\}\} = \langle\{2\}\rangle$.

Next we will define what we call the mass intersection of $\mathbf{F}$ for a given semifilter $\mathbf{F}$.

Definition 6.1.5. Given a semifilter $\mathbf{F}$, we define for all $S_i \in \mathbf{F}$

$$I_{\mathbf{F}} := \cap_{S_i \in \mathbf{F}} S_i.$$

Note that $I_{\mathbf{F}}$ may be empty.

Example 6.1.6. Let W be any set and let $\mathbf{F} = \mathcal{P}_f(W)$. Thus $\emptyset \in \mathbf{F}$. Then $I_{\mathbf{F}} = \emptyset$ since for some i, we have $S_i = \emptyset$.

Example 6.1.7. Now let $W = \{1, 2, 3, 4\}$ be a set and let $\mathbf{F}$ be a semifilter such that $\mathbf{F} = \langle\{1\}, \{2,3\}, \{3,4\}\rangle$. As the sets $\{1\}, \{2,3\}, \{3,4\}$ are the generating sets, we may simply look at their intersection since every other set in the semifilter will be a superset of one of these three. We find that $\{1\} \cap \{2,3\} \cap \{3,4\} = \emptyset$. Therefore $I_{\mathbf{F}} = \emptyset$.

From these two examples, we see that $I_{\mathbf{F}}$ can be empty in a variety of situations. Now let's look at some examples where $I_{\mathbf{F}}$ is non-empty.

Example 6.1.8. Let $W = \{1, 2, 3, 4\}$ and $\mathbf{F} = \langle \{1, 2\}, \{1, 3\}, \{1, 4\} \rangle$. In this situation, we see that $I_{\mathbf{F}} = \{1, 2\} \cap \{1, 3\} \cap \{1, 4\} = \{1\}$.

Now we'll present a way to characterize the elements of W that appear in sets of a semifilter $\mathbf{F}$.

Lemma 6.1.9. Given a finite set W, suppose $a \in W$ and let $\mathbf{F}$ be a non-empty semifilter on W. Then either $a \in I_{\mathbf{F}}$ or $W_a \in \mathbf{F}$.

Proof. We begin by supposing that $a \notin I_{\mathbf{F}}$. Then there exists some $S \in \mathbf{F}$ such that $a \notin S$. As $a \notin S$, we have that $S \subseteq W_a$. Then by definition of the semifilter, since $W_a \in \mathcal{P}_f(W)$, we have that $W_a \in \mathbf{F}$.

Now suppose that $W_a \notin \mathbf{F}$. Since $W_a \notin \mathbf{F}$, by the definition of the semifilter no subset of W_a is an element of $\mathbf{F}$. If $S \in \mathcal{P}_f(W)$ such that $a \notin S$, then $S \subseteq W_a$. Thus if $T \in \mathbf{F}$, then $a \in T$. As T is arbitrary, we have that $a \in I_{\mathbf{F}}$.

Now suppose that $a \in I_{\mathbf{F}}$ and $W_a \in \mathbf{F}$. We immediately find a contradiction since $a \notin W_a$ but $a \in I_{\mathbf{F}}$ implies that for all $S \in \mathbf{F}$ we have that $a \in S$. Thus it cannot be that $a \in I_{\mathbf{F}}$ and $W_a \in \mathbf{F}$. □

From Lemma **??** we see that given a semifilter, $\mathbf{F}$, and an element $a \in W$, we can characterize how a appears within the sets of $\mathbf{F}$. The lemma states that either a is in every set of $\mathbf{F}$, or there exists some set, $S \in \mathbf{F}$, such that $S \cup \{a\} = W$.

Given a semifilter, $\mathbf{F}$, if we are given the generating sets of $\mathbf{F}$, it is easy to characterize the elements of W in this way. If $X_1, \ldots, X_i$ are the generating sets of $\mathbf{F}$, then for some $a \in W$, we have that $a \in I_{\mathbf{F}}$ if and only if for each i, $a \in X_i$. Let's look at an example.

Example 6.1.10. Let $W = \{1, 2, 3, 4, 5\}$ be a set and $\mathbf{E}, \mathbf{F}$, and $\mathbf{G}$ be semifilters such that $\mathbf{E} = \langle \{1\} \rangle$, $\mathbf{F} = \langle \{1, 2, 5\}, \{1, 3, 5\}, \{1, 4, 5\} \rangle$, and $\mathbf{G} = \langle \{1, 2\}, \{1, 3\}, \{1, 4\}, \{5\} \rangle$. Then we see that

$$\begin{aligned} I_{\mathbf{E}} &= \{1\} \\ I_{\mathbf{F}} &= \{1, 5\} \\ I_{\mathbf{G}} &= \emptyset. \end{aligned}$$

6.2. **The Isomorphism-Permutation Theorem.** In this section, we present a proof of the Permutation-Isomorphism Theorem. There are three lemmas we prove beforehand that greatly simplify the final proof.

To begin we define what it means for two semifilters to be isomorphic.

Definition 6.2.1. Let $\mathbf{E}$ and $\mathbf{F}$ be semifilters on some set W. Then $\mathbf{E}$ and $\mathbf{F}$ are said to be *isomorphic* if there exists an order preserving bijection, ϕ, between them. We denote this $\mathbf{E} \cong \mathbf{F}$. That is, if $S, T \in \mathbf{E}$ with $S \subseteq T$, then $\phi(S) \subseteq \phi(T)$.

Since an isomorphism, ϕ, is a bijection, we know that $|\mathbf{E}| = |\mathbf{F}|$. Furthermore, as ϕ is order preserving, we know that if $S \in \mathbf{E}$, then $|S| = |\phi(S)|$.

Now let us turn our attention to these three lemmas.

Lemma 6.2.2. Let W be a set such that $|W| = n$. Let $\sigma \in S_W$ and for $i = 1, 2, \dots, m$ let $a_i \in W$ distinct. Then

$$\sigma\left(W_{a_1} \cap W_{a_2} \cap \ldots \cap W_{a_m}\right) = \sigma\left(W_{a_1}\right) \cap \sigma\left(W_{a_2}\right) \cap \ldots \cap \sigma\left(W_{a_m}\right).$$

Proof. Suppose that $a \in \sigma\left(W_{a_1} \cap W_{a_2} \cap \ldots \cap W_{a_m}\right)$. Then there exists some $b \in W$ such that $a = \sigma(b)$ and that for all i, we have $b \in W_{a_i}$. Since $b \in W_{a_i}$, we know that $\sigma(b) \in \sigma\left(W_{a_i}\right)$. Thus, as this is true for all i, we have that $\sigma(b) \in \sigma\left(W_{a_1}\right) \cap \ldots \cap \sigma\left(W_{a_m}\right)$. Therefore $\sigma\left(W_{a_1} \cap W_{a_2} \cap \ldots \cap W_{a_m}\right) \subseteq \sigma\left(W_{a_1}\right) \cap \sigma\left(W_{a_2}\right) \cap \ldots \cap \sigma\left(W_{a_m}\right)$.

Now suppose $a \in \sigma(W_{a_1}) \cap \ldots \cap \sigma(W_{a_m})$. Then there exists some $b \in W$ such that $a = \sigma(b)$ and that for all i, we have that $b \in W_i$. Since for all i we have that $b \in W_i$, we see that $b \in W_{a_1} \cap \ldots \cap W_{a_m}$. Thus $a = \sigma(b) \in \sigma\left(W_{a_1} \cap \ldots \cap W_{a_m}\right)$. Therefore $\sigma\left(W_{a_1} \cap W_{a_2} \cap \ldots \cap W_{a_m}\right) \supseteq \sigma\left(W_{a_1}\right) \cap \sigma\left(W_{a_2}\right) \cap \ldots \cap \sigma\left(W_{a_m}\right)$. □

The next lemma is very similar to the last one, but deals with isomorphisms instead of permutations.

Lemma 6.2.3. Let W be a set such that $|W| = n$. Let $\mathbf{E}$ and $\mathbf{F}$ be semifilters on W such that $\mathbf{E} \cong \mathbf{F}$ with isomorphism ϕ. If $X \in \mathbf{E}$ such that for some $a_1, a_2, \dots, a_m \in W$ distinct, $X = W_{a_1} \cap \cdots \cap W_{a_m}$ (by Lemma **??**), then

$$\phi\left(X\right) = \phi\left(W_{a_1} \cap W_{a_2} \cap \ldots \cap W_{a_m}\right) = \phi(W_{a_1}) \cap \phi(W_{a_2}) \cap \ldots \cap \phi(W_{a_m}).$$

Proof. First note that $|W_{a_1} \cap \ldots \cap W_{a_m}| = n - m$. As ϕ is order preserving, we also have that $|\phi(W_{a_1} \cap \ldots \cap W_{a_m})| = n - m$. Next observe that since the a_i's are distinct, we know that the W_{a_i}'s are distinct. Thus the $\phi(W_{a_i})$'s are distinct. As there are m such sets we see that $|\phi(W_{a_1}) \cap \ldots \cap \phi(W_{a_m})| = n - m$.

Now we will show that $\phi(W_{a_1} \cap \ldots \cap W_{a_m}) \subseteq \phi(W_{a_1}) \cap \ldots \cap \phi(W_{a_m})$. First note that for all i, we know that $a_i \notin W_{a_1} \cap \ldots \cap W_{a_m}$. Thus for all i, we see that $W_{a_1} \cap \ldots \cap W_{a_m} \subseteq W_{a_i}$. Now since ϕ is order preserving, for each i, we have that $\phi(W_{a_1} \cap \ldots \cap W_{a_m}) \subseteq \phi(W_{a_i})$. Since this is true for each i, we have that $\phi(W_{a_1} \cap \ldots \cap W_{a_m}) \subseteq \phi(W_{a_1}) \cap \ldots \cap \phi(W_{a_m})$.

Now since $|\phi(W_{a_1} \cap \ldots \cap W_{a_m})| = |\phi(W_{a_1}) \cap \ldots \cap \phi(W_{a_m})| = n - m$ and $\phi(W_{a_1} \cap \ldots \cap W_{a_m}) \subseteq \phi(W_{a_1}) \cap \ldots \cap \phi(W_{a_m})$ is must be that

$$\phi(W_{a_1} \cap \ldots \cap W_{a_m}) = \phi(W_{a_1}) \cap \ldots \cap \phi(W_{a_m}).$$

□

As stated, the goal of this section is to prove the Isomorphism-Permutation Theorem. This theorem states that any isomorphism of semifilters on a finite set W is determined by a permutation of W. To prove this, we will use an algorithm to construct a permutation of W depending on the structure of the semifilters and the isomorphism, ϕ. We will now present the algorithm used to construct σ while proving that it works in the proof of the theorem.

Algorithm 6.2.4. Let W be a set such that $|W| = n$. Let $\mathbf{E}$ and $\mathbf{F}$ be semifilters on W such that $\mathbf{E} \cong \mathbf{F}$ under isomorphism ϕ. For $i = 1, 2, \dots, n$ let a_i be the distinct elements of W. Now for each $a_i \in W$, do the following:

(1) if $a_i \in I_{\mathbf{E}}$ then define $\sigma(a_i) := b_i$ for some $b_i \in I_{\mathbf{F}}$ such that if $j \neq i$ then $\sigma(a_j) = \sigma(a_i)$;
(2) if $W_{a_i} \in \mathbf{E}$, then define $\sigma(a_i) := b_i$ where b_i is the unique element of W such that $b_i \notin \phi(W_{a_i})$.

Let us now note some important properties of the algorithm. First we know from Lemma **??** that for each $a \in W$, either $a \in I_{\mathbf{E}}$ or $W_a \in \mathbf{E}$, but not both. By following the algorithm, σ is defined for all of W.

Next, observe that we do indeed produce a bijection from W onto itself. Recall that since ϕ is order preserving, we know that $|I_{\mathbf{E}}| = |I_{\mathbf{F}}|$. Part (1) of the algorithm says for each $b \in I_{\mathbf{F}}$, we define σ such that there is a unique $a \in I_{\mathbf{E}}$ for which $\sigma(a) = b$. So $\sigma : I_{\mathbf{E}} \to I_{\mathbf{F}}$ is a bijection. Now recall that by Lemma **??** if $a \notin I_{\mathbf{E}}$, then $W_a \in \mathbf{E}$. Since ϕ is order preserving, $\phi(W_a) = W_b$ for some $b \in W$. Then since ϕ is bijective, defining $\sigma(a)$ to be dependent on $\phi(W_a)$ causes $\sigma : (W - I_{\mathbf{E}}) \to (W - I_{\mathbf{F}})$ to be bijective.

Example 6.2.5. Let $W = \{1,2,3,4,5\}$ and let $\mathbf{E} = \langle \{1,2,3\}, \{2,3,4\} \rangle$ and $\mathbf{F} = \langle \{1,3,4\}, \{2,3,4\} \rangle$. Assume that $\mathbf{E} \cong \mathbf{F}$ under isomorphism, $\phi : \mathbf{E} \to \mathbf{F}$, where ϕ is defined on as follows:

$$\begin{aligned}
\phi(\{1,2,3\}) &= \{2,3,4\} \\
\phi(\{2,3,4\}) &= \{1,3,4\} \\
\phi(\{1,2,3,4\}) &= \{1,2,3,4\} \\
\phi(\{1,2,3,5\}) &= \{2,3,4,5\} \\
\phi(\{2,3,4,5\}) &= \{1,3,4,5\}.
\end{aligned}$$

Note that $I_{\mathbf{E}} = \{2,3\}$ and that $I_{\mathbf{F}} = \{3,4\}$. Thus by the algorithm, we define $\sigma(2) := 4$ and $\sigma(3) := 3$. Now we see that

$$\phi(W_5) = \phi(\{1,2,3,4\}) = \{1,2,3,4\}.$$

Thus $\sigma(5) = 5$. Likewise, since $\phi(W_4) = \{2, 3, 4, 5\}$, we have $\sigma(4) = 1$. Finally, as $\phi(W_1) = \{1, 3, 4, 5\}$, we have $\sigma(1) = 2$.

By using the algorithm for this isomorphism, we have defined σ such that:

$$\begin{aligned} \sigma(1) &= 2 \\ \sigma(2) &= 4 \\ \sigma(3) &= 3 \\ \sigma(4) &= 1 \\ \sigma(5) &= 5. \end{aligned}$$

Testing σ, we see:

$$\begin{aligned} \sigma'(\{1,2,3\}) &= \{2,4,3\} \\ \sigma'(\{2,3,4\}) &= \{4,3,1\} \\ \sigma'(\{1,2,3,4\}) &= \{2,4,3,1\} \\ \sigma'(\{1,2,3,5\}) &= \{2,4,3,5\} \\ \sigma'(\{2,3,4,5\}) &= \{4,3,1,5\}. \end{aligned}$$

In this instance, the algorithm works to define a σ such that σ determines ϕ.

Let's now present one final lemma before the Isomorphism-Permutation Theorem. With this lemma, all the pieces of the puzzle will be in place and the proof is more straightforward.

Lemma 6.2.6. Let W be a set such that $|W| = n$. Now let $\mathbf{E}$ and $\mathbf{F}$ be semifilters on W such that $\mathbf{E} \cong \mathbf{F}$ under isomorphism ϕ. Now let $\sigma : W \to W$ be defined as in Algorithm **??**. If $W_a \in \mathbf{E}$ then $\sigma'(W_a) = \phi(W_a)$.

Proof. Since ϕ is an isomorphism, we know that $\phi(W_a) = W_b$ for some $b \in W$. Now by Algorithm **??** we know that $\sigma(a) = b$. Since a is the unique element of W such that $a \notin W_a$ and σ is a bijection from W onto itself, we know that $\sigma(a) = b \notin \sigma(W_a)$. Now since σ is a bijection, we know that $|\sigma(W_a)| = n - 1$. Thus b must be the only element of W such that $b \notin \sigma(W_a)$. Therefore $\sigma(W_a) = W_b = \phi(W_a)$. □

We are finally ready to present the Isomorphism-Permutation Theorem and its proof.

Theorem 6.2.7. Let W be a set such that $|W| = n$. Let $\mathbf{E}$ and $\mathbf{F}$ be semifilters on W such that $\mathbf{E} \cong \mathbf{F}$ under isomorphism ϕ. Then ϕ is determined by a permutation of W.

Proof. We claim that σ constructed according to Algorithm **??** will determine ϕ. That is, for any set $X \in \mathbf{E}$, we have $\sigma(X) = \phi(X)$.

Let $X \in \mathbf{E}$ be an aribtrary set. In accordance with Lemma **??**, there exist distinct sets $W_{a_1}, \ldots, W_{a_m}$ such that $X = W_{a_1} \cap \ldots \cap W_{a_m}$. Now observe that by Lemma **??** we have that

$$\sigma\left(W_{a_1} \cap \ldots \cap W_{a_m}\right) = \sigma(W_{a_1}) \cap \ldots \cap \sigma(W_{a_m}).$$

Now by applying Lemma **??** to each $\sigma(W_{a_i})$, we find that

$$\sigma(W_{a_1}) \cap \ldots \cap \sigma(W_{a_m}) = \phi(W_{a_a}) \cap \ldots \cap \phi(W_{a_m}).$$

Finally Lemma **??** implies that

$$\phi(W_{a_1}) \cap \ldots \cap \phi(W_{a_m}) = \phi\left(W_{a_1} \cap \ldots \cap W_{a_m}\right) = \phi(X).$$

Since X is an arbitrary set in $\mathbf{E}$, we find that ϕ is determined by σ. □

7. Permutations on Semifilters

7.1. **Permutation Groups on Semifilters.** We have just shown that all isomorphisms on semifilters on a finite set are determined by permutations. We now turn our attention to the relation $\mathbf{F} \cong \mathbf{F}$. An isomorphism from $\mathbf{F}$ onto itself is always determined by I, the identity permutation. However, depending on $\mathbf{F}$ it may also be determined by other permutations. We are interested in what permutations will determine an isomorphism from $\mathbf{F}$ onto itself. We are also interested in how the permutations act. In order for σ to determine the relation, if $X \in \mathbf{F}$, we know that $\sigma'(X) \in \mathbf{F}$. However, we also want to know when σ fixes all of the individual sets of $\mathbf{F}$. That is, when for all $X \in \mathbf{F}$, we have that $\sigma'(X) = X$.

First we show that for a given isomorphism ϕ, there may be multiple permutations that determine ϕ.

Theorem 7.1.1. Let $\mathbf{E}$ and $\mathbf{F}$ be semifilters on a set W such that $\mathbf{E} \cong \mathbf{F}$. Suppose $|I_{\mathbf{E}}| = n$. Then there are $n!$ permutations of W that determine ϕ.

Proof. Recall Algorithm **??**. In part (1) of the algorithm, elements of $I_{\mathbf{E}}$ are mapped to $I_{\mathbf{F}}$ arbitrarily. We have already shown that mapping them in this way will always result in a permutation that determines ϕ. The number of bijections between two n elements sets is $n!$. Thus there are at least $n!$ permutations that determine ϕ.

Now note that if $a \in I_{\mathbf{E}}$, then $\sigma(a) \in I_{\mathbf{F}}$. So we now need to show that for any $W_a \in \mathbf{E}$, the image of a under σ as defined by the algorithm is the only image that works.

Recall from part (2) of Algorithm **??** that for $W_a \in \mathbf{E}$, the image of a under σ is forced to be a unique element b. Choose some $c \in W$ with $c \neq b$ and consider $\sigma(a) = c$. In order for σ to determine ϕ, we need that $\sigma'(W_a) = \phi(W_a)$. However if $\sigma(a) = c$, then $\sigma'(W_a) = W_c$ while $\phi(W_a) = W_b$ and we have assumed that $b \neq c$. Thus b is the unique image of a under σ that causes σ to determine ϕ. Thus there is no way to construct new permutations that will still determine ϕ. Therefore the number of permutations that determine ϕ is $n!$. □

We now introduce two sets of permutations determined by how permutations fix a given semifilter.

Definition 7.1.2. Let $\mathbf{F}$ be a semifilter on a set W. Then we define

$$R_{\mathbf{F}} := \{\sigma \in S_W \mid \text{for all } X \in \mathbf{F}, \sigma'(X) = X\}.$$

In other words, $R_{\mathbf{F}}$ is the set of all permutations on W that fix every set of $\mathbf{F}$. The identity permutation, I, is a permutation the fixes every element of W. For all $a \in W$, we have that $I(a) = a$. Thus I fixes every set of W and so fixes every semifilter. Thus $I \in R_{\mathbf{F}}$ for all $\mathbf{F}$.

Definition 7.1.3. Let $\mathbf{F}$ be a semifilter on a set W. Then we define

$$T_{\mathbf{F}} := \{\sigma \in S_W \mid \text{for all } X \in \mathbf{F}, \sigma'(X) \in \mathbf{F}\}.$$

The set $T_{\mathbf{F}}$ is similar to, but less restrictive than $R_{\mathbf{F}}$. Instead of insisting that σ fix each set of the semifilter $\mathbf{F}$, we just need σ to fix $\mathbf{F}$. That is, $\sigma \in T_{\mathbf{F}}$ if $\sigma'(\mathbf{F}) = \mathbf{F}$.

As $R_{\mathbf{F}}$ and $T_{\mathbf{F}}$ are sets of permutations on W, both are subsets of S_W. Furthermore, we have that $R_{\mathbf{F}} \subseteq T_{\mathbf{F}}$. This is true because if $\sigma \in R_{\mathbf{F}}$ then $\sigma'(X) = X$. As $X \in \mathbf{F}$, then $\sigma \in T_{\mathbf{F}}$. Thus $R_{\mathbf{F}} \subseteq T_{\mathbf{F}}$.

Theorem 7.1.4. Let $\mathbf{F}$ be a semifilter on W. Then $R_{\mathbf{F}} = \{I\}$ if and only if $|I_{\mathbf{F}}| \leq 1$.

Proof. First let $|I_{\mathbf{F}}| = n$ and suppose $R_{\mathbf{F}} = \{I\}$. Let $\phi : \mathbf{F} \to \mathbf{F}$ be an isomorphism. Then $\phi(X) = X$ for each $X \in \mathbf{F}$. By Theorem **??**, we have that the number of permutations that determine ϕ is equal to $n!$. However $|R_{\mathbf{F}}| = 1$. So there cannot be another isomorphism that fixes every set of $\mathbf{F}$. Thus it must be that $n = 0$ or $n = 1$. Therefore $|I_{\mathbf{F}}| \leq 1$.

Now suppose $|I_{\mathbf{F}}| \leq 1$. Again consider the identity isomorphism ϕ from above. As $|I_{\mathbf{F}}| \leq 1$, the number of permutations that determine ϕ is $0!$ or $1!$ by Theorem **??**. In either case, there is only 1 permutation that determines ϕ. As the identity permutation is a permutation that always determines any ϕ, it must be that $R_{\mathbf{F}} = \{I\}$. □

Theorem 7.1.5. Let $\mathbf{F}$ be a semifilter on W. Then $T_{\mathbf{F}} = S_W$ if and only if $\mathbf{F} = U_k$ for some k.

Proof. Recall that U_k is defined to be the collection of all subsets of W with at least k elements.

Suppose $T_{\mathbf{F}} = S_W$. This means that for any set $X \in \mathbf{F}$ and any permutation $\sigma \in S_W$, we have $\sigma'(X) \in \mathbf{F}$. Since this is true for all $\sigma \in S_W$, it must be that all sets with the same cardinality as X are in $\mathbf{F}$. Let Y be a minimal set of $\mathbf{F}$ with $|Y| = k$. Then all subsets of W with cardinality k are in $\mathbf{F}$. We claim that the collection of all sets of size k are exactly the generating set of $\mathbf{F}$. There cannot be another generating set with more than k elements since it would then have a subset of size k in the semifilter and would not be a generating set. Also there cannot be a set in $\mathbf{F}$ with fewer than k elements. If this were so, all sets of the same size would also be in $\mathbf{F}$. One of those sets would be a subset of Y, but we assumed that Y was a minimal set. So $\mathbf{F}$ is generated by all subsets of W of size k. Thus $\mathbf{F} = U_k$.

Now suppose $\mathbf{F} = U_k$. For any set $X \in \mathbf{F}$ with $|X| \geq k$ every other subset of W of the same size is also in $\mathbf{F}$. Thus for any permutation, $\sigma \in S_W$, as $|X| = |\sigma'(X)|$ we have that $\sigma'(X) \in \mathbf{F}$. As every permutation in S_W perserves U_k, it must be that $T_{U_k} = S_W$. □

Theorem 7.1.6. Let W be a finite set. Then $T_{\mathbf{F}} \neq \{I\}$ for all $\mathbf{F} \in \mathcal{F}_W$ if and only if $|W| = 3$ or 4. Equivalently, there exists some $\mathbf{F} \in \mathcal{F}_W$ such that $T_{\mathbf{F}} = \{I\}$ if and only if $|W| \neq 3$ and $|W| \neq 4$.

Proof. Let W be a finite set where $|W| = n$. We begin with cases $n = 1$ and $n = 2$. Note that for this proof we label the elements of W as the integers 1 through n. These are only labels and the theorem is valid for all finite sets, no matter the labels. If a finite sets has elements with different labels, we simply relabel them and proceed with the proof.

Case 1: Suppose $W = \{1\}$. As $|W| = 1$, there is only one permutation of W – namely the identity permutation. Therefor for any semifilter $\mathbf{F}$, we have that $T_{\mathbf{F}} = \{I\}$ since $I \in T_{\mathbf{F}}$ for all $\mathbf{F}$ (on any W).

Case 2: Suppose $W = \{1, 2\}$. Consider the semifilter $\mathbf{F} = \langle\{1\}\rangle$. We endeavor to find all $\sigma \in S_W$ such that $\sigma'(X) \in \mathbf{F}$ for all $X \in \mathbf{F}$. As $\{1\} \in \mathbf{F}$ is the only singleton set in $\mathbf{F}$, for any $\sigma \in T_{\mathbf{F}}$ we must have that $\sigma(1) = 1$. Now that $\sigma(1)$ is defined, since σ is a permutation it must be that $\sigma(2) = 2$. Thus $\sigma = I$ and therefore $T_{\mathbf{F}} = \{I\}$.

We now present the cases where $n \geq 5$ and save the cases where $n = 3, 4$ for later.

Case 3: Suppose that W is a finite set such that $|W| = 2k + 1$ for some integer $k \geq 2$. Then $W = \{1, 2, \ldots, 2k + 1\}$. We now describe an algorithm to construct a semifilter, $\mathbf{F}$, for which $T_{\mathbf{F}} = \{I\}$.

First, arrange the elements of W into an array in the following way:

$$\begin{array}{l|l|l|l|l} 1 & 2 & \dots & k & k+1 \\ \hline k+2 & k+3 & \dots & 2k+1 & \end{array}.$$

Note that there are $k+1$ entries in the first row and k entries in the second row. Define $\mathbf{F}$ in the following way:

(1) Let $\{1, 2, \dots, k+1\}$ be a minimal set of $\mathbf{F}$.
(2) Suppose p is an entry in the ith column of the top row and that q is an entry in the jth column of the bottom row. Then if $j \geq i$, make $\{p, q\}$ a minimal set of $\mathbf{F}$.

Thus $\{p, q\}$ is a minimal set of $\mathbf{F}$ only if q is directly below p or to the right of p.

To illustrate this construction, we present an example before finishing this case. Let $W = \{1, 2, 3, 4, 5, 6, 7\}$. Then our array is:

$$\begin{array}{c|c|c|c} 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \end{array}.$$

Following the above rules, we construct

$$\mathbf{F} = \langle \{1, 2, 3, 4\}, \{1, 5\}, \{1, 6\}, \{1, 7\}, \{2, 6\}, \{2, 7\}, \{3, 7\} \rangle .$$

In this instance, if $\sigma \in T_{\mathbf{F}}$, then $\sigma'(\{1, 2, 3, 4\}) = \{1, 2, 3, 4\}$ since $\{1, 2, 3, 4\}$ is the only four-element minimal set of $\mathbf{F}$. Notice that if σ is a bijection, then $a \in W$ is an element of an m-element minimal set if and only if $\sigma(a)$ is an element of an m-element minimal set. We now observe that $\sigma(1) = 1$ since the element 1 is the only element of $\{1, 2, 3, 4\}$ also in exactly three other minimal sets. Likewise $\sigma(2) = 2$ since 2 is the only element of $\{1, 2, 3, 4\}$ in exactly two other minimal sets. For similar reasons $\sigma(3) = 3$ and $\sigma(4) = 4$. Now we observe $\sigma(5) = 5$. This is because 5 is the unique element of W contained in a single two-element minimal set and not in the $k+1$-element minimal set. Similary $\sigma(6) = 6$ as it is the unique element of W contained in two two-element minimal sets and not in the $k+1$-element minimal set. Finally $\sigma(7) = 7$. Thus for the $\mathbf{F}$ we constructed, $T_{\mathbf{F}} = \{I\}$.

Now let's extend this argument to the general case. Suppose that we've followed the rules from above and have constructed

$$\mathbf{F} = \langle \{1, 2, \dots, k+1\}, \{p_1, q_1\}, \{p_2, q_2\}, \dots, \{p_\ell, q_\ell\} \rangle .$$

In this case, $\{1, 2, \dots, k+1\}$ is the only minimal set with more than two elements. If $\sigma \in T_{\mathbf{F}}$ then $\sigma'(\{1, 2, \dots, k+1\}) = \{1, 2, \dots, k+1\}$. The following arguments repeatedly make use of the array we constructed and the second rule for constructing $\mathbf{F}$. We observe that the element 1 is in k two-element minimal sets of $\mathbf{F}$, the element 2 is in $k-1$, 3 is in $k-2$, and so on. The pattern clearly is that for $1 \leq i \leq k+1$, the element i is in $i-1$ two-element minimal sets of $\mathbf{F}$. Thus for all $a \in \{1, 2, \ldots, k+1\}$, we must have that $\sigma(a) = a$.

As $\{1, 2, \ldots, k+1\}$ is the only minimal set of $\mathbf{F}$ with more than two elements, if $i \geq k+1$ then i is in contained in only two-element minimal sets.

Now consider $k+2$. There is only one entry in the array that is directly above or to the left. Thus $k+2$ is contained in one two-element minimal set. There is only one number above and to the left of $k+3$. Thus $k+3$ is contained in two two-element minimal sets. Likewise, $k+4$ is contained in three two-element minimal sets. This pattern continues through $2k+1$. Since for all $a \neq b$ contained in $\{k+2, k+3, \ldots, 2k+1\}$, the elements a and b will be contained in a different number of minimal sets, it must be that $\sigma(a) = a$ for each $a \in \{k+2, k+3, \ldots, 2k+1\}$.

Recall that σ is an arbitrary permutation with the property that $\sigma'(X) \in \mathbf{F}$. We've now shown that for all $a \in W$, $\sigma(a) = a$. Therefore $T_{\mathbf{F}} = \{I\}$.

Case 4: Suppose that W is a finite set such that $|W| = 2k$ for some integer $k \geq 3$. Then $W = \{1, 2, \ldots, 2k\}$ and $W_{2k} = \{1, 2, \ldots, 2k-1\}$. Construct $\mathbf{E}$, a semifilter on W_{2k}, as you would in case 3. Now take the minimal sets of $\mathbf{E}$ and construct $\mathbf{F}$, a semifilter on W, by letting the minimal sets of $\mathbf{E}$ be the minimal sets of $\mathbf{F}$. As $W_{2k} \subseteq W$, the minimal sets of $\mathbf{E}$ will be subsets of W and thus can be minimal sets of $\mathbf{F}$.

Let $\sigma \in T_{\mathbf{F}}$. First, since $2k$ is the only element of W that is not contained in a minimal set. Thus $\sigma(2k) = 2k$. Now for all $a \in \{1, 2, \ldots, 2k-1\}$, the same arguments from case 3 apply. Thus we have that $\sigma(a) = a$ for all $a \in W$. Therefore $\sigma = I$ and $T_{\mathbf{F}} = \{I\}$.

We now show that if $n = 3$ or 4, there does not exist a semifilter on W such that $T_{\mathbf{F}} = \{I\}$. Unfortunately, the only way we have found to do this is to check all possible cases (that is, this is a proof by exhaustion).

Before breaking into cases, let us make an aside. Claim: If $\mathbf{E}$ and $\mathbf{F}$ are semifilters on a finite set W, such that $\mathbf{E} \cong \mathbf{F}$ and $T_{\mathbf{F}} \neq \{I\}$, then $T_{\mathbf{F}} \neq \{I\}$. Let σ be a permuation that determines ϕ. Now suppose $X \in \mathbf{E}$ and $Y \in \mathbf{F}$ such that $\sigma(X) = Y$. Further suppose that $\sigma_1 \in T_{\mathbf{E}}$. Since $\sigma_1 \in T_{\mathbf{E}}$, we have that $\sigma_1(X) \in \mathbf{E}$. Since $\sigma_1(X) \in \mathbf{E}$ and $\mathbf{E} \cong \mathbf{F}$, we have that $\sigma(\sigma_1(X)) \in \mathbf{F}$. Now as X and Y are arbitrary, we have that $\sigma \circ \sigma_1 \in T_{\mathbf{F}}$. Since permutations are bijections, they have unique inverses. Thus if there exists $\sigma_2 \in T_{\mathbf{E}}$ such that $\sigma_1 \neq \sigma_2$, then $\sigma \circ \sigma_1 \neq \sigma \circ \sigma_2$. Therefore if $T_{\mathbf{E}} \neq \{I\}$, then $T_{\mathbf{F}} \neq \{I\}$.

Case 5: Suppose that $W = \{1, 2, 3\}$.

Consider $\emptyset$, the empty semifilter. $T_\emptyset = S_W$ since the conditions are vacuously fulfilled.

Consider semifilters of the form $\mathbf{F} = \langle\{1\}\rangle$. In this case 2 and 3 may be permuted as neither appear in a generating set of $\mathbf{F}$. Thus $T_\mathbf{F} \neq \{I\}$ for all semifilters of the form $\mathbf{F} = \langle\{1\}\rangle$.

Now let $\mathbf{F} = \langle\{1\}, \{2\}\rangle$. In this instance, as $\{1\}$ and $\{2\}$ are the only generating sets of $\mathbf{F}$, the elements 1 and 2 may be permuted. Thus $T_\mathbf{F} \neq \{I\}$.

Suppose $\mathbf{F} = \langle\{1\}, \{2\}, \{3\}\rangle$. In this case, $T_\mathbf{F} = S_W$ since any permutation of the elements of W will determine $\mathbf{F} \cong \mathbf{F}$. Therefore $T_\mathbf{F} \neq \{I\}$.

Let $\mathbf{F} = \langle\{1, 2\}\rangle$. As $\{1, 2\}$ is the only generating set, 1 and 2 may be permuted. Thus $T_\mathbf{F} \neq \{I\}$.

Consider semifilters of the form $\mathbf{F} = \langle\{1, 2\}, \{3\}\rangle$. We observe that we can permute 1 and 2 without affecting their relationships with 3. Thus $T_\mathbf{F} \neq \{I\}$.

Now let $\mathbf{F} = \langle\{1, 2\}, \{2, 3\}\rangle$. As $I_\mathbf{F} = \{2\}$, we must leave 2 alone. However, if we permute 1 and 3, then we still preserve the semifilter. Thus $T_\mathbf{F} \neq \{I\}$.

Now suppose $\mathbf{F} = \langle\{1, 2\}, \{1, 3\}, \{2, 3\}\rangle$. Then the numbers 1 and 2 may be permuted without affecting $\mathbf{F}$. Therefore $T_\mathbf{F} \neq \{I\}$.

Consider $\mathbf{F} = \langle\{1, 2, 3\}\rangle$. For any permutation, $\sigma \in S_W$, we have that $\sigma(\{1, 2, 3\}) = \{1, 2, 3\}$. Therefore $T_\mathbf{F} \neq \{I\}$.

Now suppose $\mathbf{F} = \langle\emptyset\rangle$. Then $\mathbf{F} = \mathcal{P}_f(W)$ and thus for any permutation, $\sigma \in S_W$, we have that $\sigma'(\mathbf{F}) = \mathbf{F}$. Therefore $T_\mathbf{F} \neq \{I\}$.

These ten cases represent all semifilters on W up to isomorphism. Thus we have shown that if $n = 3$, then for all semifilters on W, $T_\mathbf{F} \neq \{I\}$.

The case of $n = 4$ is shown in a similar way (with many more cases).

Case 6: Now suppose that $|W| = 4$. We'll let $W = \{1, 2, 3, 4\}$ act as a representative for all four element sets. For this case, we again show that for each semifilter, $\mathbf{F}$, on W there exists a non-trivial permutation of W that determines the isomorphism $\mathbf{F} \cong \mathbf{F}$. Up to isomorphism, the semifilters for which we must show this are:

$\mathbf{F}_1 = \langle \emptyset \rangle$
$\mathbf{F}_2 = \langle \{1\} \rangle$
$\mathbf{F}_3 = \langle \{1\}, \{2\} \rangle$
$\mathbf{F}_4 = \langle \{1\}, \{2\}, \{3\} \rangle$
$\mathbf{F}_5 = \langle \{1\}, \{2\}, \{3\}, \{4\} \rangle$
$\mathbf{F}_6 = \langle \{1, 2\} \rangle$
$\mathbf{F}_7 = \langle \{1, 2\}, \{3, 4\} \rangle$
$\mathbf{F}_8 = \langle \{1, 2\}, \{1, 3\} \rangle$
$\mathbf{F}_9 = \langle \{1, 2\}, \{1, 3\}, \{1, 4\} \rangle$
$\mathbf{F}_{10} = \langle \{1, 2\}, \{1, 3\}, \{2, 3\} \rangle$
$\mathbf{F}_{11} = \langle \{1, 2\}, \{2, 3\}, \{3, 4\} \rangle$
$\mathbf{F}_{12} = \langle \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\} \rangle$
$\mathbf{F}_{13} = \langle \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\} \rangle$
$\mathbf{F}_{14} = \langle \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\} \rangle$
$\mathbf{F}_{15} = \langle \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\} \rangle$
$\mathbf{F}_{16} = \langle \{1, 2\}, \{3\} \rangle$
$\mathbf{F}_{17} = \langle \{1, 2\}, \{3\}, \{4\} \rangle$
$\mathbf{F}_{18} = \langle \{1, 2\}, \{1, 3\}, \{4\} \rangle$
$\mathbf{F}_{19} = \langle \{1, 2, 3\} \rangle$
$\mathbf{F}_{20} = \langle \{1, 2, 3\}, \{1, 2, 4\} \rangle$
$\mathbf{F}_{21} = \langle \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\} \rangle$
$\mathbf{F}_{22} = \langle \{1, 2, 3\}, \{4\} \rangle$
$\mathbf{F}_{23} = \langle \{1, 2, 3\}, \{1, 4\} \rangle$
$\mathbf{F}_{24} = \langle \{1, 2, 3\}, \{1, 4\}, \{2, 4\} \rangle$
$\mathbf{F}_{25} = \langle \{1, 2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\} \rangle$
$\mathbf{F}_{26} = \langle \{1, 2, 3, 4\} \rangle$
$\mathbf{F}_{27} = \emptyset$

We begin the process of finding non-trivial permutations for these 28 semifilters by first observing that if all four elements of W do not appear in any of the minimal sets of a semifilter, we may adapt the permutation used for the case where $n = 3$ on the semifilter with "similar" minimal sets. We've chosen our 28 representatives so that if three or fewer elements appear in the minimal sets, then 4 is definitely one of the elements that does not appear. In this case, if 4 does not appear then we can start with $\sigma^* \in S_{\{1,2,3\}}$ and define $\sigma \in S_W$ in the following way:

$$\begin{aligned}
\sigma(1) &:= \sigma^*(1) \\
\sigma(2) &:= \sigma^*(2) \\
\sigma(3) &:= \sigma^*(3) \\
\sigma(4) &:= 4.
\end{aligned}$$

Then σ will affect the minimal sets of the semifilter on W in the same way that σ^* affects the minimal sets of the similar semifilter on $\{1,2,3\}$.

Take, for example, $\mathbf{F}_3 = \langle\{1\},\{2\}\rangle$. In case 5, for the semifilter with the same minimal sets, we showed showed that permuting 1 and 2 while leaving 3 alone will induce the isomorphism $\mathbf{F} \cong \mathbf{F}$. Thus if we define σ in the way given above, we get

$$\begin{aligned} \sigma(1) &= 2 \\ \sigma(2) &= 1 \\ \sigma(3) &= 3 \\ \sigma(4) &= 4 \end{aligned}$$

which is a permutation that induces $\mathbf{F}_3 \cong \mathbf{F}_3$. From this, we see that the following semifilters fall under this argument:

$$\mathbf{F}_1,\ \mathbf{F}_2,\ \mathbf{F}_3,\ \mathbf{F}_4,\ F_6,\ \mathbf{F}_8,\ \mathbf{F}_{10},\ \mathbf{F}_{16},\ \mathbf{F}_{19},\ F_{27}.$$

We now embark upon our journey of checking the remaining semfilter from the list.

Let $\sigma_1 \in S_W$ be defined as follows:

$$\begin{aligned} \sigma_1(1) &= 2 \\ \sigma_1(2) &= 1 \\ \sigma_1(3) &= 3 \\ \sigma_1(4) &= 4. \end{aligned}$$

Then σ_1 fixes the following semifilters:

$$\mathbf{F}_5,\ \mathbf{F}_7,\ \mathbf{F}_{14},\ \mathbf{F}_{15},\ \mathbf{F}_{17},\ \mathbf{F}_{20},\ \mathbf{F}_{22},\ \mathbf{F}_{24},\ \mathbf{F}_{25},\ \mathbf{F}_{26}.$$

Thus for each semifilter, $\mathbf{F}$, from this list, $T_{\mathbf{F}} \neq \{I\}$.

Now define $\sigma_2 \in S_W$ in the following way:

$$\begin{aligned} \sigma_2(1) &= 1 \\ \sigma_2(2) &= 2 \\ \sigma_2(3) &= 4 \\ \sigma_2(4) &= 3. \end{aligned}$$

Then we see that σ_2 will in the isomorphism $\mathbf{F} \cong \mathbf{F}$ for the following semifilters:

$$\mathbf{F}_9,\ \mathbf{F}_{21}.$$

Then for each semifilter in the list, $T_{\mathbf{F}} \neq \{I\}$.

Now let $\sigma_3 \in S_W$ be defined in this way:

$$\begin{aligned}\sigma_3(1) &= 4\\ \sigma_3(2) &= 3\\ \sigma_3(3) &= 2\\ \sigma_3(4) &= 1.\end{aligned}$$

Then $\sigma_3(\mathbf{F}) = \mathbf{F}$ for the following semifilters:

$$\mathbf{F}_{11},\ \mathbf{F}_{12}.$$

Thus for these two semifilter, $T_{\mathbf{F}} \neq \{I\}$.

Let $\sigma_4 \in S_W$ be as follows:

$$\begin{aligned}\sigma_4(1) &= 1\\ \sigma_4(2) &= 3\\ \sigma_4(3) &= 2\\ \sigma_4(4) &= 4.\end{aligned}$$

Then σ_4 fixes the following semifilters:

$$\mathbf{F}_{13},\ \mathbf{F}_{18},\ \mathbf{F}_{23}.$$

Thus for these last three semifilters, $T_{\mathbf{F}} \neq \{I\}$.

With the above four permutations we have shown that all semifilters on W that do not inherit a non-trivial $T_{\mathbf{F}}$ from a semifilter on $\{1, 2, 3\}$ also have associated non-trivial $T_{\mathbf{F}}$'s. Therefore all semifilters on a four element set have an associated non-trivial $T_{\mathbf{F}}$. This concludes the proof. □

7.2. **Automorphisms of $\mathcal{F}_W$.** Recall that $\mathcal{F}_W$ is the collection of all semifilters on the set W. Earlier when we covered isomorphisms of semifilters, we looked at the image of a single semifilter under a permutation.

We now concern ourselves with how permutations affect every single semifilter in $\mathcal{F}_W$. If we apply a permutation of the elements of W to $\mathcal{F}_W$, will this preserve the structure of $\mathcal{F}_W$? Before we can proceed in answering that question, we must present some definitions.

Definition 7.2.1. Let W be a set and let $\sigma \in S_W$. Then σ'' is defined to be the bijection $\sigma'' : \mathcal{F}_W \to \mathcal{F}_W$ where for each $\mathbf{F} \in \mathcal{F}_W$ we define $\sigma''(\mathbf{F}) := \sigma'(\mathbf{F})$.

Theorem **??** states that every permutation of W determines an isomorphism of semifilters. Thus defining σ'' in terms of σ', we know that the image of a semifilter under σ'' is a semifilter.

Definition 7.2.2. A function $\Phi : \mathcal{F}_W \to \mathcal{F}_W$ is said to be an *automorphism* of $\mathcal{F}_W$ if it fulfills the following:

(1) Φ is a bijection,
(2) for all $\mathbf{E}, \mathbf{F} \in \mathcal{F}_W$, we have $\Phi(\mathbf{EF}) = \Phi(\mathbf{E})\Phi(\mathbf{F})$,
(3) for all $\mathbf{G} \in \mathcal{F}_W$, we have $\Phi\left(\overline{\mathbf{G}}\right) = \overline{\Phi(\mathbf{G})}$,
(4) and if $\mathbf{E} \subseteq \mathbf{F}$ then $\Phi(\mathbf{E}) \subseteq \Phi(\mathbf{F})$.

This definition of an automorphism of $\mathcal{F}_W$ implicitly relies on the definition of an isomorphism between $\mathcal{F}_{W_1}$ and $\mathcal{F}_{W_2}$. However, for an isomorphism to exist, it must be that $|W_1| = |W_2|$. In the finite case, this relationship makes an isomorphism uninteresting since $\mathcal{F}_{W_1}$ and $\mathcal{F}_{W_2}$ will behave exactly the same by default except that the elements of W_1 and W_2 may have different labels. The automorphism case, however, is interesting since it turns out that every automorphism on $\mathcal{F}_W$, where W is finite, is determined by a unique permutation of W.

Before formalizing that fact in a theorem, however, we present a lemma.

Lemma 7.2.3. Let W be a finite set such that $|W| = n$ and suppose that $\mathbf{F}$ is a semifilter on W. Then $\mathbf{F} = \langle\{a\}\rangle$ if and only if $\mathbf{F}^2 = \emptyset$ and there exist non-trivial semifilters $\mathbf{F}_1, \ldots, \mathbf{F}_{n-1}$ such that $\mathbf{F} \cdot \mathbf{F}_1 \cdot \ldots \cdot \mathbf{F}_{n-1} = \{W\}$.

Proof. Suppose that for some $a \in W$, we have that $\mathbf{F} = \langle\{a\}\rangle$. Since $I_{\mathbf{F}} \neq \emptyset$, no two sets of $\mathbf{F}$ are disjoint. Therefore $\mathbf{F}^2 = \emptyset$. Now let $a_1, \ldots, a_{n-1}$ be distinct elements of W_a. Then

$$\mathbf{F} \cdot \prod_{i=1}^{n-1} \langle\{a_i\}\rangle = \{\{a, a_1, a_2, \ldots, a_{n-1}\}\} = \{W\}.$$

Now suppose that $\mathbf{F}$ is a semifilter on W such that $\mathbf{F}^2 = \emptyset$ and there exist non-trivial semifilters $\mathbf{F}_1, \ldots, \mathbf{F}_{n-1}$ such that $\mathbf{F} \cdot \mathbf{F}_1 \cdot \ldots \cdot \mathbf{F}_{n-1} = \{W\}$. As $\mathbf{F} \cdot \mathbf{F}_1 \cdot \ldots \cdot \mathbf{F}_{n-1} \neq \emptyset$ and $\mathbf{F} \cdot \mathbf{F}_1 \cdot \ldots \cdot \mathbf{F}_{n-1} \neq \mathcal{P}_f(W)$, we have that $\mathbf{F} \neq \emptyset$ and $\mathbf{F} \neq \mathcal{P}_f(W)$. Now since $\mathbf{F}^2 = \emptyset$, it must be that given $S, T \in \mathbf{F}$ distinct, $S \cap T \neq \emptyset$. Thus $I_{\mathbf{F}} \neq \emptyset$.

Aside: Suppose $\mathbf{G}_1, \ldots, \mathbf{G}_n$ are non-trivial semifilters on W such that $\prod_{i=1}^{n} \mathbf{G}_i = \{W\}$. Then there must exist pairwise disjoint sets $X_1, \ldots, X_n$ where $X_i \in \mathbf{G}_i$ and $\cup_{i=1}^{n} X_i = W$. Since the X_i's are disjoint, non-trivial, there are n of them and their union is W, it must be that for each i, $|X_i| = 1$. As the $X_i's$ are disjoint and all of cardinality 1, we have that for each i, $X_i = \{a_i\}$ where $a_i \in W$ and for $i \neq j$, we have $a_i \neq a_j$. Thus for each i, there exists a distinct $a_i \in W$ such that $\{a_i\} \in \mathbf{G}_i$.

Thus by the above argument, since $\mathbf{F}, \mathbf{F}_1, \ldots, \mathbf{F}_{n-1}$ are all non-trivial, there exists some $a \in W$ such that $\{a\} \in \mathbf{F}$. As $|I_{\mathbf{F}}| \neq \emptyset$ and $\{a\} \in \mathbf{F}$, we have that $I_{\mathbf{F}} = \{a\}$. Therefore $\mathbf{F} = \langle \{a\} \rangle$. □

We now present the main theorem.

Theorem 7.2.4. Let W be a finite set. Suppose $\Phi : \mathcal{F}_W \to \mathcal{F}_W$ is an automorphism. Then there exists a unique $\sigma \in S_W$ that determines Φ. That is, σ determines Φ if and only if for all $\mathbf{F} \in \mathcal{F}_W$, $\Phi(\mathbf{F}) = \sigma''(\mathbf{F})$.

Proof. Let W be a finite set such that $|W| = n$. Suppose $\Phi : \mathcal{F}_W \to \mathcal{F}_W$ is an automorphism. Let us consider all principal semifilters generated by a singleton set. Since Φ preserves semifilter multiplication, by Lemma **??**, for each $a \in W$, there exists a unique $b \in W$ such that $\Phi(\langle \{a\} \rangle) = \langle \{b\} \rangle$. In this way Φ will determine a unique $\sigma \in S_W$ defined by $\sigma(a) = b$ when $\Phi(\langle \{a\} \rangle) = \langle \{b\} \rangle$.

Note that since $\Phi\left(\overline{\mathbf{G}}\right) = \overline{\Phi(\mathbf{G})}$, Φ preserves semifilter lattice relations. Thus, as the principal semifilters generated by a singleton set are "sub-semifilters" of all semiftilers in $\mathcal{F}_W$ (except $\mathbf{F} = \{W\}$ and $\emptyset$), the images of these principal, singleton-generated semifilters must determine the rest of the lattice relations. □

8. Conclusion

In this paper we presented an introduction to semifilters and some results relating semifilters and permutations. There are still many conjectures and questions left to explore.

Question 8.0.5. Given a set W such that $|W| = n \in \mathbb{N}$, what is $|\mathcal{F}_W|$?

Question 8.0.6. Let $\mathbf{E}$ be a semifilter on a set W. Are there simple conditions $\mathbf{E}$ must fulfill in order for $\mathbf{E}$ to have a non-trivial factorization, $\mathbf{E} = \mathbf{FG}$?

There are many conjectures left relating to Theorem **??**.

Conjecture 8.0.7. Suppose $\Phi : \mathcal{F}_W \to \mathcal{F}_W$ is a bijection such that if $\mathbf{E}, \mathbf{F} \in \mathcal{F}_W$, then $\Phi(\mathbf{EF}) = \Phi(\mathbf{E})\Phi(\mathbf{F})$. Φ maintains the semifilter lattice relations of $\mathcal{F}_W$. That is, if $\mathbf{E} \subseteq \mathbf{F}$, then $\Phi(\mathbf{E}) \subseteq \Phi(\mathbf{F})$.

Conjecture 8.0.8. Suppose $\Phi : \mathcal{F}_W \to \mathcal{F}_W$ is a bijection such that if $\mathbf{E}, \mathbf{F} \in \mathcal{F}_W$, then $\Phi(\mathbf{EF}) = \Phi(\mathbf{E})\Phi(\mathbf{F})$. Then for all $\mathbf{G} \in \mathcal{F}_W$, we have $\Phi\left(\overline{\mathbf{G}}\right) = \overline{\Phi(\mathbf{G})}$.

Conjecture 8.0.9. Let W be a finite set and let $\sigma \in S_W$. Then $\Phi : \mathcal{F}_W \to \mathcal{F}_W$ defined by $\Phi(\mathbf{F}) = \sigma''(\mathbf{F})$ is an automorphism of $\mathcal{F}_W$.

References

[1] Keef, Patrick W., Balof, and Barry, A. *Invariants on primary Abelian groups and a problem of Nunke* 2008: submitted to Elsevier.
[2] Keef, Patrick W. *Personal Interview* 2008.