CHAOS AND DYNAMICS

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Abstract. In this paper we will study chaos through the dynamics of the quadratic family of functions. We begin with an introduction to basic dynamical notions, including orbit analysis and periodicity. Our goal is to isolate chaos within the specific example of the quadratic functions. From here, we will form a proper definition of chaos using symbolic dynamics. This material is largely a review of *A First Course in Chaotic Dynamical Systems* by Robert L. Devaney.

In everyday life, we often sit in front of the television wondering to ourselves: "Does this meteorologist know anything about the upcoming weather conditions?" or "Why didn't I foresee this dramatic change in the stock market?" In fact, these two questions have been studied extensively in the field of mathematics as chaotic dynamical systems. Though we can never be certain of tomorrow's actual temperature or humidity, we have learned that the expertise of fellow mathematicians can productively influence whether or not we take an umbrella to work. The words "chaos" and "randomness" are often used interchangeably, but we will soon see that mathematically they are very different notions. In order to grasp the complicated concept of chaos, it is best to narrow down the focus by studying a very simple dynamical system. This paper will attempt to trace out the existence of chaos within a very simple dynamical system: the quadratic family of functions described as $Q_c x = x^2 + c$ for various values of c. Our ultimate goal is in fact to arrive at a clear yet concise definition of chaos.

A person versed in the basic concepts of calculus should be able to follow along with this material. It is primarily based on the textbook *A First Course in Chaotic Dynamical Systems* by Robert L. Devaney, a former professor of mathematics at Boston University and prominent figure in dynamics in the twentieth century. We will begin by building a foundation of basic definitions and concepts related to the functions and systems we will study. We will present a detailed equation for Q_c and then use certain members of this quadratic family to verify the simple properties of dynamical systems. We will then perform various processes on different Q_c equations to demonstrate their dynamical characteristics. In expanding on these

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characteristics we eventually hope to witness the surfacing of chaotic dynamical behavior. Finally, once chaos has been properly identified, we will set out to isolate the chaotic behavior. We define chaos as follows:

Chaos: A dynamical system F is *chaotic* if:
i. The set of periodic points in F is dense.
ii. F is transitive.
iii. F depends sensitively on initial conditions.

We cannot yet understand this definition because we do not know the meaning of many of the terms within it. Our final goal is to fully appreciate this simple definition by decoding each and every aspect. Ideally, our understanding of dynamics within the quadratic family will allow us to just graze over the basic notion of chaos.

The field of dynamics has been much improved upon over the last few decades. This fact can be attributed to the rapidly expanding industry of computer technology. Since much of the study of dynamics is based on complicated graphs and diagrams, it is a wonder that the pioneers of dynamics were able to discover what they did. Next we have the French mathematician Henri Poincare, who, in his 1890 paper on "stable and unstable manifolds," brushed upon our modern-day notion of chaos. It was not until 1930 that chaos was again addressed this time by Julia Gaston. His detailed work on what is now called the *Julia Set* could not fully be appreciated until computers in the 1980s were able to graphically demonstrate his findings. It is in fact argued that computer generated figures of the Julia Set are of the most beautiful and most complicated mathematical diagrams in existence. Our last significant contributors to dynamics are Americans Steven Smale and E. N Lorenze who, in the 1960s, came up with the concepts of *symbolic dynamics* and *sensitive dependence on initial conditions*, respectively. These two concepts will play a large role in our study of chaos. To study a topic so fresh in its discovery gives us the added benefit of using current computer technology. It is also possible that we will witness many more innovations in dynamics within our lifetimes.

This paper will be organized as follows. In Section 1, we will discover the basic properties of the quadratic family of functions Q_c. It is necessary to start out with a solid background in the quadratic family as it will be our main environment for illustrating chaos. Section 2 will focus on the process of iteration and the orbits formed from iterating points within a system. In Section 3 we will learn to analyze the orbits of a system graphically. Section 4 will address the characteristics of certain points within an orbit, namely fixed and periodic points. In Section 5 we will translate our graphical analyses into more complex bifurcation diagrams. We will then skip to a few more theoretical aspects of dynamics: the Cantor Middle-Thirds Set in Section 6 and symbolic dynamics for Section 7. Lastly, for Section 8, we will address chaos.

1. **The Quadratic Family of Functions**

To fully understand the process of dynamics it is often helpful to turn to specific examples to observe dynamical trends. For this reason we will be working our way through the study of dynamics with one simple and constant tool: the family Q_c of quadratic functions. This family of functions will act as an instrument in tracing out the steps necessary to arrive at chaos. First, we must begin with a discussion and graphical description of this family.

The *quadratic functions* are continuous functions that have the form $Q_c(x) = x^2 + c$, where c is a real-valued constant. This constant c is called a *parameter* and its value within an equation distinguishes one quadratic function from others in the family. As we will later see, the value of c will ultimately determine the dynamics of the function. For each c-value there exists a completely new and unique system of dynamics.

Now let us observe what happens to the graph of Q_c when the value of c is varied. When $c = 0$, we have the equation of a parabola in its simplest form, with its vertex located at the origin. As c increases to a positive value, the graph of the function shifts directly upward along the y-axis. The vertex in this case is located at the point $(0, c)$. A negative value for c results in a parabola with its vertex located below the x-axis at point $(0, c)$. These properties of quadratic functions may seem obvious and elementary to students of math, but they will support a strong foundation for understanding dynamics within the quadratic family. We will soon discover how to make use of these properties with new dynamical procedures.

2. **Orbit Analysis**

Now it is time to approach the notion of dynamics. For the sake of simplicity, we will begin with a dynamical description of the one-variable quadratic function. The essential process involved in dynamics is that of composing a function with itself: a process called *iteration*. We will be utilizing this process of iteration when we find orbits of the function $Q_c(x)$ at different values of x.

2.1. **Iteration.**

As calculus students, we perhaps unknowingly employed a few simple iterative processes. Finding roots of a function using Newton's Method was one of these occasions. Iteration literally means the repetition of a process over and over. To iterate a quadratic function we simply evaluate the function over and over, using the output of the last application as the input for the next. For a function $F(x)$, we denote the second iterate of F at x as $F^2(x)$, the third iterate as $F^3(x)$, and the n^{th} iterate as $F^n(x)$. If $F(x) = x^2 + 1$, then

$$F^2(x) = F(F(x)) = (x^2 + 1)^2 + 1$$

and

$$F^3(x) = F(F(F(x))) = ((x^2 + 1)^2 + 1)^2 + 1.$$

This process is also sometimes called function composition. We will now see how iteration relates to orbits of a function.

2.2. **Orbits.** Let us consider the function $Q_1(x) = x^2 + 1$ at the point $x_0 = 0$. The first four iterations of Q_1 at 0 occur as follows:

$$x_0 = 0$$

$$x_1 = F(x_0) = F(0) = 0^2 + 1 = 1$$

$$x_2 = F(x_1) = F(1) = 1^2 + 1 = 2$$

$$x_3 = F(x_2) = F(2) = 2^2 + 1 = 5$$

$$x_4 = F(x_4) = F(5) = 5^2 + 1 = 26.$$

The first four iterations of $Q_1(x)$ form the first entries of the *orbit* of F at 0. An orbit of a function at a point x_0 is described as:

Definition 2.1 (Orbit)**.** *If we have function $F(x)$ and some initial point $x_0 \in \mathbf{R}$, then the orbit of x_0 under F is the sequence of points*

$$x_0, x_1 = F(x_0), x_2 = F^2(x_0), ..., F^n(x_0), \ldots.$$

The point x_0 is called the seed of the orbit.

We can rewrite the iterations of $Q_1(x) = x^2 + 1$ at point $x = 0$ in the form of an orbit sequence: $0, 1, 2, 5, 26, \ldots$. The orbit of Q_1 at 0 increases with every iteration and will eventually converge to ∞. To better understand this phenomenon, we will investigate different outcomes for different types of orbits.

2.3. **Types of Orbits.** Within a dynamical system, there are many types of orbits. One important species of orbit is initiated by a *fixed point*.

Definition 2.2 (Fixed Point)**.** *A fixed point is a point x_0 that satisfies the equation $F^n(x_0) = x_0$ for all $n \in \mathbf{Z}^+$.*

The orbit of a fixed point is the constant sequence $x_0, x_0, x_0, \ldots$. We can find the fixed points for any function F by simply solving the equation $F(x) = x$. For example, we set $Q_{-2}(x) = x^2 - 2$ equal to x and solve the equation for x to find the fixed points of Q_{-2}.

$$x = x^2 - 2$$

$$0 = x^2 - x - 2 = (x+1)(x-2)$$

The roots of this equation, and thus the fixed points, are $x = -1$ and $x = 2$. We can verify each of these values by inserting them back into Q_{-2} and finding that they do indeed yield constant orbit sequences. Another useful method for finding fixed points involves graphing $y = Q_c(x)$ and $y = x$ on the same axes and evaluating their intersections. As one might expect, each function $Q_c(x)$ with $c > \frac{1}{4}$ has no fixed points. This is due to the fact that for these functions, the equation $Q_c(x) = x$ yields no real solutions.

Now that we have discussed fixed points, it is time to approach a different type of orbit. Just as there are some points that trigger constant orbits, there are some points that orbit through finite cycles of values and then return to their original values. These points are called periodic points and are defined as:

Definition 2.3 (Periodic Point)**.** *The point x_0 is periodic if $F^n(x_0) = x_0$ for some $n \in \mathbf{Z}^+$. The smallest n is the prime period of the orbit.*

Let us consider the orbit at 0 for function $Q_{-1}(x) = x^2 - 1$. Computing the first few compositions, we get:

$$x_0 = 0$$

$$x_1 = 0^2 - 1 = -1$$

$$x_2 = (-1)^2 - 1 = 0$$

$$x_3 = 0^2 - 1 = -1$$

It follows naturally that the orbit at 0 of F is $0, -1, 0, -1, 0, \ldots$. In different terms, 0 falls on a prime period 2 and the points -1 and 0 form a 2-cycle. We can also have seeds of orbits that are not periodic, but that converge to periodic sequences. These points, called eventually periodic, are defined as follows:

Definition 2.4 (Eventually Periodic Point). *A point x_0 is eventually periodic if it is not itself periodic, but has point x_n in its orbit that is periodic.*

There are also eventually fixed points, which can be described with a similar definition, replacing the word "periodic" with "fixed." An example of an eventually periodic point is $x_0 = 1$, iterated in the function Q_{-1}. Since $1^2 - 1 = 0$ and 0 is a periodic point for Q_{-1}, it follows that the orbit for $x_0 = 1$ is $1, 0, -1, 0, -1, \ldots$.

For each dynamical system, there may be a few fixed points, periodic points, and eventually fixed and periodic points. For the most part, however, these significant points are overwhelmed by non-fixed, non-periodic seeds that, under iteration, yield very complex orbits. We will be investigating periodic and fixed points in depth in the coming sections, and we will also investigate the less-obvious orbits.

2.4. **Other Orbits.** While a function may have a few fixed and periodic points, it has infinitely many non-fixed and non-periodic points. For instance, if we have a computer randomly select an initial seed value on the interval $(-2, 2)$ and iterate it through the function $Q_{-2}(x) = x^2 - 2$, the first few terms might look like:

$$x_0 = -1.93890429188385$$

$$x_1 = 1.75934985308563$$

$$x_2 = 1.09531190555244$$

$$x_3 = -0.80029182955509$$

This sequence does not appear to have any clear pattern. Indeed, the first 100 terms of this sequence do not paint a picture of a clear pattern. The ninety-eighth through one hundredth terms of this sequence are:

$$x_{98} = F^{98}(x_0) = -1.99690845692414$$

$$x_{99} = F^{99}(x_0) = 1.98764338533516$$

$$x_{100} = F^{100}(x_0) = 0.00319363837844$$

The distribution of the iterations is scattered and appears to be random. A similar occurrence can be observed over several orbits of the function $Q_{-2}(x) = x^2 - 2$. Figure 1 is a histogram of the orbit of $x_0 = 0.1$ under $Q_{-2}(x) = x^2 - 2$. To create this frequency graph, we split up the first 2,000 iterations of 0.1 over small subintervals of $(-2, 2)$. From this histogram, we can appreciate that the distribution of the orbit does not run evenly over the interval of $(-2, 2)$. It is clear that the orbit favors points along the edges of the interval. If the orbit was truly distributed randomly, we would see no such patterns. This occurrence suggests the presence of chaos within the orbit. As our understanding of chaotic dynamical systems matures, we

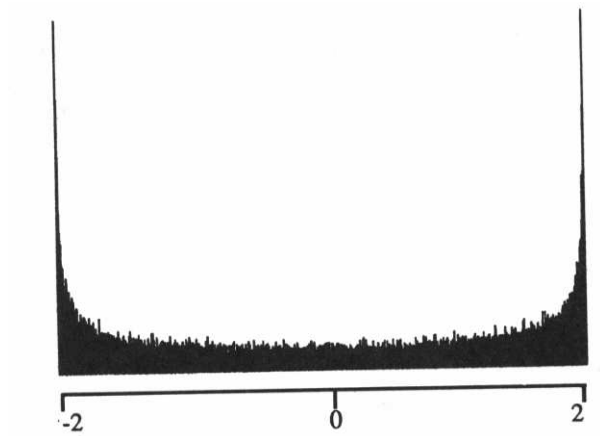

FIGURE 1. Histogram of the orbit of 0.1 under $x^2 - 2$.

will learn of better ways to evaluate orbits.

2.5. **Graphical Analysis.** We now know how to evaluate orbits of a function at specific points, but in order to examine the trends of orbits along intervals of x we must employ a new system. As mentioned previously, it is possible to find fixed points by graphing a function and finding its intersections with the line $y = x$. We can in fact gather much more information about orbits then just fixed points from the graph of a function. The technique of graphical analysis allows us to do this. Graphical Analysis is a geometric procedure used to determine the behavior of orbits under the function F from the graph of F.

To graphically analyze the orbits of a function $F(x)$, we begin by graphing the function $y = F(x)$ alongside the graph of $y = x$. To find the orbit of a point x_0, we locate the point (x_0, x_0) on the line $y = x$ and draw a vertical line to meet the function F at point $(x_0, F(x_0))$ (see Figure 2). Next, we draw a horizontal line from the function F to point $(F(x_0), F(x_0))$ on $y = x$. We then draw a vertical line back to F, which will meet at point $(F(x_0), F^2(x_0))$. The x value of this point, being $F(x_0)$, is the next point on our orbit. To further extend our orbit sequence, we simply continue this procedure: we draw vertical lines from the diagonal $y = x$ to the function F and then draw horizontal lines back to $y = x$. These orbit analysis graphs are also commonly known as "cobweb" or "staircase" diagrams.

For some functions, we can create complete orbit analyses using this technique. For example, let us consider the graphical analysis for $Q_0(x) = x^2$. Looking at Figure 2, we can see that there are only a few directions in which an orbit of x^2 can converge/diverge. The direction an orbit tends to depends on the value of the seed

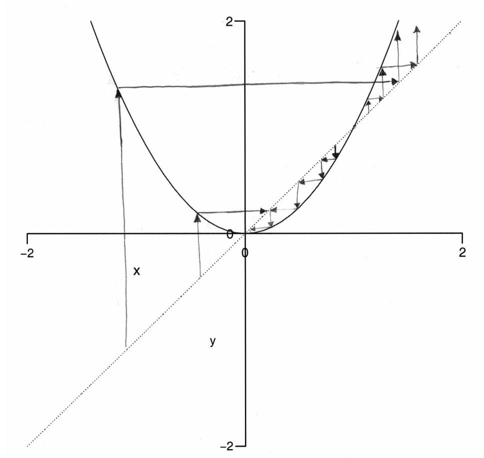

FIGURE 2. Orbit Analysis of x^2.

x_0. It can be easily determined by our methods that the fixed points of Q_0 are 0 and 1. If a seed of iteration begins at either of these points then the orbit will stay put. If $|x_0| < 1$ then the orbit of x_0 tends to 0, as pictured in Figure 2. On the other hand, if $|x_0| > 1$, the orbit of x_0 tends to ∞. Since we have accounted for all possible orbits under the function, we can say that we have performed a complete *orbit analysis*. This may not be possible for all functions, and it is certainly not possible for real applications of dynamical systems, but orbit analysis is a good tool to describe dynamical activity within the quadratic family of functions.

3. **Attracting, Repelling, and Neutral Points**

We have thus far discovered the existence of fixed points within functions of the quadratic family. The dynamics of orbits initiated at fixed points follow a constant repeating pattern. On the other hand, orbits initiated at points close to fixed points may be very complex. We will now examine such orbits through the process graphical analysis.

3.1. **The Fixed Point Theorem.** As mentioned in Section 2.3, there are some functions within the quadratic family with no fixed points. Therefore, we must commence with a theorem that will help us to determine the existence of fixed points under a function. This theorem, commonly called the *Fixed Point Theorem*, follows quite effortlessly from the *Intermediate Value Theorem* we know from early calculus. As a reminder, here is a statement of the IVT:

Theorem 3.1 (Intermediate Value Theorem). *Suppose F is continuous on the closed interval $[a, b]$. If N is any number between $F(a)$ and $F(b)$ then there must be some value c within the interval $[a, b]$ such that $F(c) = N$.*

Now we shall describe the Fixed Value Theorem and proceed with a simple proof that relies on results of IVT.

Theorem 3.2 (Fixed Value Theorem). *Suppose that F is continuous on the closed interval $[a, b]$ and that $F : [a, b] \to [a, b]$. It follows that F must have a fixed point on this interval $[a, b]$.*

Proof. Let us first consider the case where $F(a) = a$. Point $x = a$ is by definition a fixed point, so we have reached our desired consequence. We know that $F(a)$ cannot be valued at a number less than a because a is the lower bound of the interval. Now let us consider the remaining case where $F(a) > a$. This inequality can be rewritten as $F(a) - a > 0$. Combined with the first case we have $F(a) - a \geq 0$. Similarly, b is the upper bound of the interval, so it follows that $F(b) \leq b$, or $F(b) - b \leq 0$.

Since F is continuous, the intermediate value theorem says that there must be some c in $[a, b]$ such that $F(c) - c = 0$. So there must be a fixed point c on the interval $[a, b]$. □

This theorem proves the existence of fixed points on particular intervals of interest, but it does not give us a method to find the exact locations of fixed points. It does, however, give us more understanding into the fixed points we see on a graph of a function against the line $y = x$. We already have a method of finding fixed points, but solving the equation $F(x) = x$ will not be easy for all cases.

3.2. **Attracting and Repelling Fixed Points.** Referring back to Figure 2, we can see the graphical analysis of $Q_0(x) = x^2$. Recall that the orbits within the interval $(0, 1)$ all gravitated toward the fixed point $x_0 = 0$. In contrast, the orbits larger than the fixed point $x_0 = 1$ tended to infinity. The reason for this difference is that 0 and 1 act as different types of fixed points for x^2. The fixed point $x_0 = 0$ is called an *attracting fixed point* because the orbits near to it on either side converge to it. On the other hand, $x_0 = 1$ is a *repelling fixed point* because all the orbits close by to it diverge away from the point $x = 1$. Now we will learn to use calculus to distinguish between attracting and repelling points.

As it turns out, the classification of a fixed point depends on the slope of the tangent line of the function at the point. As we know from calculus, the slope of a tangent line of a function at a certain point is found by evaluating the derivative of the function at that point. The following is an important definition about orbit behavior around fixed points:

Definition 3.1. *Suppose x_0 is a fixed point of a differentiable function F.*
If $|F'(x_0)| < 1$, then the point x_0 is an attracting fixed point.
If $|F'(x_0)| > 1$, then the point x_0 is a repelling fixed point.
If $|F'(x_0)| = 1$, then the point x_0 is a neutral fixed point.

The truth of these statements can easily be verified through graphical analysis. If the slope of a tangent line at a fixed point x_0 is steep in comparison to the diagonal $y = x$, then the orbits around this point will necessarily escape to infinity. On the other hand, if the slope at x_0 is slight then the nearby orbit will converge to x_0. As apparent from Definition 3.1, negative slopes also yield the same tendencies.

Now let us finish off this section with an application of Definition 3.1.

Example. For the function $Q_2(x) = x^2 - 2$, find all fixed points and classify them as attracting, repelling, or neutral.
Answer. First we find the fixed points by setting $Q_2(x)$ equal to x.

$$x = x^2 - 2 \ \Rightarrow \ 0 = x^2 - x - 2 = (x+1)(x-2)$$

The fixed points of Q_2 therefore occur at $x_0 = -1$ and $x_0 = 2$. Now we take the derivative of Q_2 and evaluate it at the fixed points.

$$Q_2'(x) = 2x$$

Since $Q_2'(0) = 2(-1) = -2$ and $|-2| > 1$, the point $x = 0$ must be a repelling fixed point. Since $Q_2'(2) = 4 > 1$, the point $x_0 = 2$ is also a repelling fixed point.
To verify this answer, we can simply perform graphical analysis on the function and observe the behaviors of the orbits around the fixed points.

3.3. **The Attracting and Repelling Fixed Point Theorems.** Geometrically, it is easy to see why Definition 3.1 is valid, but in order to mathematically prove them we must revisit the *Mean Value Theorem* from calculus. As a reminder, here is a statement of the MVT:

Theorem 3.3 (Mean Value)**.** *Let F be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). There must therefore be a number c in (a, b) such that:*

$$F'(c) = \frac{F(b) - F(a)}{b - a}.$$

First, we will state both the *Attracting Fixed Point Theorem* and the *Repelling Fixed Point Theorem* and then we will go through a proof of the latter (the proof of the former follows similarly).

Theorem 3.4 (Attracting Fixed Point)**.** *Suppose x_0 is an attracting fixed point for $F(x)$. Then there exists an interval I containing the point x_0 that satisfies the following condition: if $x \in I$, then $F^n(x) \in I$ for all n. In other words, as n approaches infinity, $F^n(x)$ approaches x_0.*

Theorem 3.5 (Repelling Fixed Point)**.** *Suppose x_0 is a repelling fixed point for $F(x)$. Then there exists an interval I containing the point x_0 that satisfies the following condition: if $x \in I$ and $x \neq x_0$, then there is an $n > 0$ such that $F^n(x) \notin I$.*

Proof. Since $|F'(x_0)| > 1$, there is some $\lambda > 0$ such that $|F'(x_0)| > \lambda > 1$. Let $\delta > 0$ be chosen so that $|F'(x)| > \lambda$ with x in the interval $I = [x_0 - \delta, x_0 + \delta]$. Choose any point $p \in I$. By Theorem 3.3, we have:

$$\frac{|F(p) - F(x_0)|}{|p - x_0|} > \lambda \text{ , or} \qquad |F(p) - F(x_0)| > \lambda |p - x_0|$$

Since x_0 is a fixed point, we have $|F(p) - x_0| > \lambda |p - x_0|$. This mathematical statement tells us that the distance from $F(p)$ to x_0 is much greater than the distance from p to x_0. The point $F(p)$ must therefore fall outside the interval I. □

Now we have officially verified the existence of attracting and repelling fixed points, but the behavior of orbits around neutral fixed points is not quite as simple. We will investigate the behavior of orbits around neutral fixed points at a later time.

3.4. **Periodic Points.** Just as with fixed points, periodic points can be classified as attracting, repelling, or neutral. If we want to classify a periodic point x_0 that lies on an n-cycle, we must concentrate on the slope of the tangent of the n^{th} iterate of the function at x_0. In simpler terms, a periodic point of period n is attracting if it is an attracting fixed point for F^n. Likewise, a periodic point of period n is repelling if it is an repelling fixed point for F^n.

Let us derive a formula for the derivative of the n^{th} iterate. It would be very time consuming to compute F^n for a large n, let alone to evaluate its derivative at each periodic point on the cycle. Luckily, we have the Chain Rule from calculus to aid us in this calculation. Notice that:

$$(F^2)'(x_0) = F'(F(x_0)) \cdot F'(x_0) = F'(x_1) \cdot F'(x_0).$$

When we find $(F^3)'(x_0)$ using this method, we can recognize a certain pattern.

$$(F^3)'(x_0) = F'(F^2(x_0)) \cdot (F^2)'(x_0) = F'(x_2) \cdot F'(x_1) \cdot F'(x_0)$$

The derivative of F^n at x_0 is clearly the product of the derivatives of F at each point in the n period orbit. We can create a formula to state this in the general case.

Theorem 3.6 (Chain Rule Along a Cycle)**.** *Suppose $x_0, x_1, \ldots, x_{n-1}$ all lie on a cycle of period n. Then*

$$(F^n)'(x_0) = F'(x_{n-1}) \cdot \ldots \cdot F'(x_1) \cdot F'(x_0)$$

Since the derivative of F^n at the point x_0 is just the product of the derivatives of F at all points on the orbit, it does not matter which point on the cycle we start with. For example, $(F^n)'(x_{n-1}) = F'(x_{n-1}) \cdot \ldots \cdot F'(x_1) \cdot F'(x_0) = (F^n)'(x_0)$. This fact is a basic corollary of Theorem 3.6.

Now we will put our newfound knowledge of periodic points to good use and work through an example of a periodic orbit.

Example. Zero lies on a periodic orbit of $Q_{-1}(x) = x^2 - 1$. Classify this orbit as attracting, repelling, or neutral.

Answer. First we must figure out what cycle the point $x_0 = 0$ falls on. We simply plug $x = 0$ into $Q_{-1}(x)$ and begin to take iterations: $0, Q_{-1}(0) = 0^2 - 1 = -1, Q_{-1}(-1) = (-1)^2 - 1 = 0, -1, 0, \ldots$. The points 0 and -1 lie on a period 2 cycle, so in order to determine the behavior of the orbits near the periodic point $x_0 = 0$ we must evaluate $|(Q_{-1}^2)'(0)|$. To do this, we must differentiate $Q_{-1}(x)$.

$$Q'_{-1}(x) = 2x$$

Now we have all the information necessary to use the Chain Rule Along a Cycle.

$$|(Q_{-1}^2)'(0)| = |Q'_{-1}(0) \cdot Q'_{-1}(-1)| = |0 \cdot (-2)| = 0$$

Since $|(Q_{-1}^2)'(0)| = 0 < 1$, we know that the cycle is attracting.

3.5. **Neutral Fixed Point Theorem.** When the behavior of orbits starting near a fixed point cannot be identified as strictly attracted or repelled a further analysis can often be performed to describe the nature of the orbits. This incident occurs specifically around a neutral fixed point x_0 where $F'(x_0) = 1$. For example, we can graphically analyze the function $F(x) = x + x^2$ to find that the orbits with seeds slightly to the right of the neutral fixed point $x_0 = 0$ are *weakly attracted* to it. Their convergence rate to x_0 is much slower than normally observed. On the contrary, the orbits with seeds slightly to the left of this fixed point are *weakly repelled* from it. They slowly diverge from the neutral fixed point. In this case, the role of concavity of the function at x_0 is effecting the behavior of the surrounding orbits. This is an indication that differentiation can help us to understand the effects of neutral points. When we combine the following types of neutral points with their respective behaviors we form the following:

Theorem 3.7 (Neutral Fixed Point)**.** *Suppose the function $F(x)$ has a neutral fixed point at x_0 with $F'(x_0) = 1$.*

i. *If $F''(x_0) > 0$, then x_0 is weekly attracting from the left and weakly repelling from the right.*
ii. *If $F''(x_0) < 0$, then x_0 is weekly repelling from the left and weakly attracting from the right.*

Suppose that F has a neutral fixed point at x_0 with $F'(x_0) = 1$ and $F''(x_0) = 0$.

iii. *If $F'''(x_0) > 0$, then x_0 is weakly repelling.*
iv. *If $F'''(x_0) < 0$, then x_0 is weakly attracting.*

If our results after finding the third derivative prove to be inconclusive, we can extend this theorem to the n^{th} derivative as necessary. Therefore, it is possible to find even the slightest attracting and repelling forces at a neutral fixed point by examining higher derivatives. As with the Attracting and Repelling Fixed Point Theorems, we can apply this theorem to periodic points.

This concludes our introductory work with fixed and periodic points, and now we will advance to the study of bifurcations within quadratic functions.

4. **Bifurcations**

It is quite clear that the quadratic functions $Q_c(x) = x^2 + c$ have very similar graphical forms to one another. The graphs in fact vary only in their intercepts with the y-axis. As it turns out, a slight difference in y-intercept may result in a significant difference in dynamical behavior. The study of the process of bifurcation will help us to understand why this occurs.

4.1. **Fixed Points of the Quadratic Map.** With the usual technique, we can solve $0 = x^2 + c - x$ to find the fixed points of $Q_c(x)$. Applying the quadratic formula, we find the two roots of the equation:

$$r_1 = \frac{1}{2}\left(1 + \sqrt{1-4c}\right)$$

$$r_2 = \frac{1}{2}\left(1 - \sqrt{1-4c}\right)$$

These roots represent real-valued fixed points if and only if $c \leq \frac{1}{4}$. Recall that when $c > \frac{1}{4}$, our equation yields no real-valued fixed points. In this case, all orbits under the function tend to infinity. When $c = \frac{1}{4}$, we get $r_1 = \frac{1}{2}(1+0) = \frac{1}{2}(1-0) = r_2$.

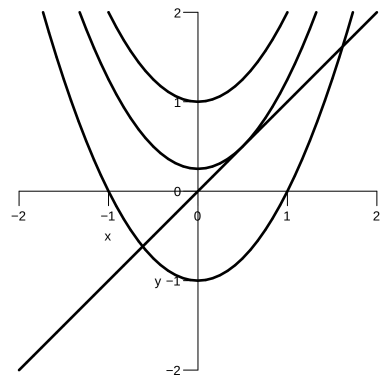

FIGURE 3. Graphs of $x^2 + 1$ (top), $x^2 + \frac{1}{4}$ (middle), and $x^2 - 1$ (bottom).

Here we have found just one real fixed point at $(\frac{1}{2}, \frac{1}{2})$. When $c < \frac{1}{4}$, we have two distinct real fixed points r_1 and r_2 with $r_1 > r_2$. In Figure 3, the graphs of the functions $Q_1(x) = x^2 + 1$, $Q_{\frac{1}{4}}(x) = x^2 + \frac{1}{4}$, and $Q_{-1}(x) = x^2 - 1$ are lined up against each other next to the diagonal line $y = x$. As expected, the function $x^2 + 1$ has no visible fixed points, while $x^2 + \frac{1}{4}$ has one and $x^2 - 1$ has two.

4.2. **The Saddle-Node Bifurcation.** As the value of c decreases from $\frac{1}{4}$ to a smaller positive value, we witness an increase of real fixed points for $Q_c(x)$. This change can be described as a splitting or *bifurcation* of fixed points at $c = \frac{1}{4}$. This particular bifurcation is called a *saddle-node bifurcation*. In fact, the definition of a saddle-node bifurcation is just a generic representation of the aforementioned occurrence.

Definition 4.1 (Saddle-Node Bifurcation)**.** *Let $F_\lambda(x)$ be a family of one-parameter functions. A saddle-node bifurcation occurs at λ_0 if the following three conditions are satisfied:*

i. *$F_\lambda(x)$ has no fixed points on an interval I for values of λ slightly less (more) than λ_0.*
ii. *$F_\lambda(x)$ has exactly one fixed point on I for $\lambda = \lambda_0$.*
iii. *$F_\lambda(x)$ has two fixed points on I for values of λ slightly more (less) than λ_0.*

When we replace $F_\lambda(x)$ with $Q_c(x)$ and λ_0 with $c = \frac{1}{4}$ in Definition 4.1, our prior observations are perfectly captured. The saddle-node bifurcation is also known as a blue sky bifurcation. Along the interval $c > \frac{1}{4}$ there are absolutely no fixed points for $Q_c(x)$. Then, suddenly, when c brushes past $\frac{1}{4}$, a fixed point appears to surface out of the "blue sky." This is a common observation for bifurcations.

As we can see in Figure 3, a saddle-node bifurcation occurs at a point where the line $y = x$ acts as a tangent to the graph of the function at that point. As expected, the diagonal $y = x$ is tangent to the function $Q_{\frac{1}{4}}(x)$ at the point $(\frac{1}{2}, \frac{1}{2})$. In this case, it is also important that $Q''_{\frac{1}{4}}(x) \neq 0$ at the point. A certain amount of concavity is required for $Q_{\frac{1}{4}}(x)$ at $(\frac{1}{2}, \frac{1}{2})$ so that no point near to it re-crosses the line $y = x$. The presence of another fixed point so near to first would certainly interfere with the destined orbit.

4.3. **The Period-Two Bifurcation.** When we delve deeper into the properties of fixed points of the quadratic functions, we find that there exist different, more subtle types of bifurcations. We begin by determining when the fixed points r_1 and r_2 are attracting, repelling, and neutral. Taking the derivative of $Q_c(x)$, we get $Q'_c(x) = 2x$. Evaluating this derivative at the point r_1 gives us

$$Q'_c(r_1) = \left(1 + \sqrt{1 - 4c}\right).$$

When $c = \frac{1}{4}$, $Q'_c(r_1) = 1$. Thus, the point r_1 is a neutral fixed point. When the value of c falls below $\frac{1}{4}$, $Q'_c(r_1) > 1$ and r_1 transforms into a repelling fixed point. Evaluating the derivative at r_2 gives us

$$Q'_c(r_2) = \left(1 - \sqrt{1 - 4c}\right).$$

Since the value $\sqrt{1 - 4c}$ is subtracted from 1 in this case, $Q'_c(x) \leq 1$ for all c. However, we must do a little more work to determine the behavior of the fixed point r_2. To find all the c-values for which r_2 is attracting, we must solve $|Q'_c(r_2)| < 1$ for c:

$$-1 < Q'_c(r_2) < 1$$

$$-1 < \left(1 - \sqrt{1 - 4c}\right) < 1$$

$$0 < \sqrt{1 - 4c} < 2 \;\Rightarrow\; 0 < 1 - 4c < 4 \;\Rightarrow\; -\frac{3}{4} < c < \frac{1}{4}$$

Therefore, the fixed point r_2 is attracting on the open c interval $(-\frac{3}{4}, \frac{1}{4})$. Now let us evaluate $|Q'_{-1}(r_2)|$:

$$|Q'_{-1}(r_2)| = |1 - \sqrt{1 - 4(-1)}| = |1 - \sqrt{5}| \approx 1.24 > 1.$$

It is clear that when $c < -\frac{3}{4}$, the point r_2 is repelling. But r_1 is also repelling for $c < -\frac{3}{4}$. In cases such as this, how can r_1 and r_2 both be repelling fixed points? This occurrence seems to contradict certain laws of physics. The answer: there exists a period 2-cycle (see Section 2.3) somewhere between these fixed points that neutralizes their conflicting repelling behavior. In order to find this periodic point, we must consider $0 = Q_c^2(x) - x = x^4 + 2cx^2 + c^2 + c - x$. Solving this equation will give us our original fixed points as well as our period 2 points. A bit of algebraic manipulation leaves us with the following periodic points:

$$q_1 = \frac{1}{2}\left(-1 + \sqrt{-4c-3}\right)$$

$$q_2 = \frac{1}{2}\left(-1 - \sqrt{-4c-3}\right).$$

The points q_1 and q_2 will be real if and only if $c \leq -\frac{3}{4}$. We have now stumbled across a new type of bifurcation; the *period doubling bifurcation.* We can apply our procedure from Section 3.2 to determine if q_1 and q_2 are attracting or repelling points. A summary of our results follows:
For the family $Q_c(x) = x^2 + c$:

i. If $-\frac{3}{4} < c < \frac{1}{4}$, then $Q_c(x)$ has a repelling fixed point at r_1, an attracting fixed point at r_2, and no period 2 cycles.
ii. If $c = -\frac{3}{4}$, then $Q_c(x)$ has a neutral fixed point where $r_2 = q_1 = q_2$ and has no period 2 cycles.
iii. If $-\frac{5}{4} < c < -\frac{3}{4}$, then r_1 and r_2 are repelling fixed points and $Q_c(x)$ has a period 2 cycle.

An n-cycle may undergo a period-doubling bifurcation itself. For instance, the above period 2 cycle eventually gives birth to a 4-cycle. This 4-cycle gives birth to an 8-cycle, and so on. Interestingly enough, a period-doubling bifurcation occurs when the graph of $F_\lambda(x)$ is perpendicular to $y = x$. Just as with the saddle-node bifurcation, calculus can be used to find period-doubling cycles. In this case we would look for $F'_\lambda(p_{\lambda_0}) = -1$, or we would take advantage of the chain rule along a cycle (Theorem 3.6) to find $(F^2_\lambda)'(p_{\lambda_0}) = 1$. Therefore, the graph of the second iterate of F_λ is tangent to $y = x$ when a bifurcation occurs.

Now we have carefully examined the behavior of fixed and periodic points for $Q_c(x) = x^2 + c$ at the significant intervals of c-values. This procedure could easily be repeated for a different family of functions. Of course, depending on the family, the results will most likely yield different behaviors and bifurcations.

4.4. **Bifurcation Diagram.** To better understand the nature of bifurcations, we will begin to look at *bifurcation diagrams* as well as orbit analysis graphs. A bifurcation diagram does not specifically display the graph of the function in question. It actually gives us a device to look at a whole family of functions on one diagram. It ultimately graphs the quantity of fixed points against different values of c or λ. To create a bifurcation diagram, we plot c values on the x-axis and x values (for fixed points) on the y-axis. See Figure 4 for an example of a bifurcation diagram. A bifurcation diagram allows us to easily locate intervals of c where the number and behavior of fixed points are changing. In the next section we will examine an advanced bifurcation diagram of $Q_c(x) = x^2 + c$ and attempt to locate the bifurcations we are now familiar with.

4.5. **Transition to Chaos.** We will now use computer technology to observe the dynamics of $Q_c(x) = x^2 + c$ over different values of c. Using the MatLab program, we first divide the interval $-2 \leq c \leq \frac{1}{4}$ into 500 evenly distributed c values. Recall that all interesting dynamics for the family Q_c occur on this interval. Next, we will plug in each of these c values into $Q_c(x)$ and use the program to classify the ultimate behavior of the orbit of 0 for each chosen c. We would ultimately like to create a bifurcation diagram of our data. We follow these rules:

i. If the orbit of 0 under Q_{c_i} is attracted to a fixed point p_1, then we plot (c_i, p_1) on our diagram.
ii. If the orbit of 0 under Q_{c_j} is attracted to a 2-cycle, q_1 and q_2, then we plot (c_j, q_1) and (c_j, q_2).
iii. In general: if the orbit of 0 under Q_{c_k} is attracted to an n-cycle, then we plot the appropriate n points vertically under $c = c_k$.
iv. If the orbit of 0 under Q_{c_l} shows no clear pattern, then we fill in the x-interval under which the orbit covers. This will appear as a vertical line on the diagram.

In order to create a bifurcation diagram we will plot the c-axis horizontally with an integer corresponding to each chosen c value (1 through 500) and the x-axis vertically with $-2 \leq c \leq 2$. For the specific MatLab commands used to create this diagram, see Appendix A. Our resulting diagram is pictured in Figure 4. It is important to keep in mind that repelling fixed points will not appear on our diagram. We know that the first bifurcation, the saddle-node bifurcation, occurs at $c = \frac{1}{4}$. We can only see the attracting half of the split of fixed points. If we were to zoom in on Figure 4, we would find that the second bifurcation, the period-doubling bifurcation, occurs at about the 280^{th} index for c, which appropriately corresponds to a c value of $-\frac{3}{4}$. Following this process, we find a period 4-cycle surfacing at about the 168^{th} c value for which $c = -\frac{5}{4}$. This number also looks familiar from our prior calculations for critical c values. As c approaches -2, the bifurcation diagram gets more difficult to read. What is happening here? Most likely, our answer is chaos. In fact, there is a lot about Figure 4 we cannot yet comprehend. As another example: why does the initial seed $x_0 = 0$ yield such interesting and complex orbits? We hope to answer these questions in sections to come.

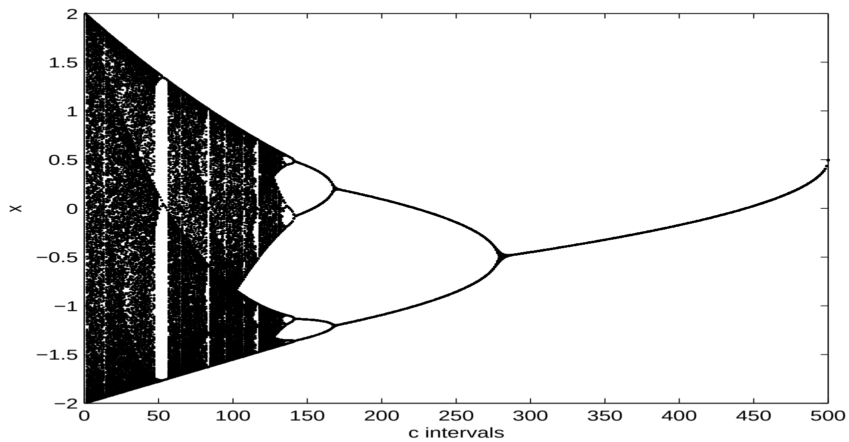

FIGURE 4. Bifurcation diagram for $Q_c(x) = x^2 + C$ with orbits initiated at point $x = 0$

5. Cantor Sets

We now have a comprehensive understanding of the dynamics of the quadratic functions for c values greater than $-\frac{5}{4}$. Unfortunately, when the value of c decreases below -2, the dynamical behavior of a function Q_c is far more difficult to follow. In this section we will approach our first basic fractal: the Cantor middle-thirds set. This fractal will act as a basis for understanding the complex behavior of orbits of the quadratic family $Q_c(x) = x^2 + c$ for values of $c < -2$. We will soon discover that the ideas of the Cantor set are closely related to these orbits.

5.1. **The Cantor Middle-Thirds Set.** The qualities and characteristics of the Cantor middle-thirds set will help us to understand the behavior of orbits in certain quadratic cases. For now, we will focus on the classic model of a Cantor set: the Cantor middle-thirds set. This set is classified as a basic fractal. Unfortunately, we will not have time here for a proper discussion of fractals. Refer to Devaney's chapter on fractals, Chapter 14, for an extensive discussion. For our purposes, it will suffice to stick to a brief description of the Cantor set.

To create the traditional Cantor set, we begin with the closed interval $[0, 1]$ and remove from this set the middle third of the interval. The middle section we remove is precisely the open interval $(\frac{1}{3}, \frac{2}{3})$, so we are left with the set of closed intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Our next progression is the same as the first; we remove the middle thirds from our remaining two sections. This leaves us with the set of four closed intervals $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$, and $[\frac{8}{9}, 1]$. We continue to repeat this process, always removing from the set the middle third section of each remaining interval. This process is pictured in Figure 5. The set left over after the process is repeated an infinite number of times is the complete Cantor middle-thirds set, denoted here as K. It is a non-empty set, since at least the endpoints of each interval from each step can be accounted for (the open intervals being removed each time do not affect the endpoints). It is also true that K is completely disconnected: each interval is interrupted by a smaller absent interval, so no definite intervals are present. These two facts suggest that K is an uncountable set, but we are not quite ready to prove it. To build up the proper foundation for the proof, we must first examine ternary expansions of the real numbers.

5.2. **Ternary Expansion.** As we might recall from calculus, the geometric series $\sum_{i=0}^{\infty} a^i$ converges absolutely if $|a| < 1$. If the series converges then

$$\sum_{i=k}^{\infty} a^i = \frac{a^k}{1-a}.$$

For example:

$$\sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^i = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$$

and

$$\sum_{i=1}^{\infty} \frac{2}{3^i} = 2\sum_{i=1}^{\infty} \left(\frac{1}{3}\right)^i = 2\left(\frac{1}{2}\right) = 1.$$

Now let us consider the ternary expansion $x = 0.022022\overline{022}$. The familiar base-10 expansion of a number has digits 0-9 that occupy each decimal place, whereas a base-three, or ternary, expansion holds only the numbers 0, 1, and 2 in each decimal place. For instance, 0.1 is the ternary expansion for $1^{-3} = \frac{1}{3}$ and 0.002 is the ternary expansion for $2 \cdot 3^{-3} = \frac{2}{27}$. By the rules of ternary expansion, we can write x as

$$x = \frac{0}{3^1} + \frac{2}{3^2} + \frac{2}{3^3} + \frac{0}{3^4} + \ldots.$$

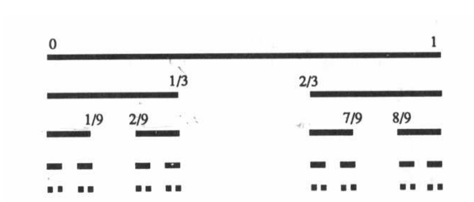

FIGURE 5. The Cantor Middle-Thirds Set: a visual representation of the first five steps (image taken from Devaney p. 76)

We can recombine these elements of x to form two geometric series

$$x = \frac{2}{3^2} + \frac{2}{3^5} + \ldots + \frac{2}{3^3} + \frac{2}{3^6} + \ldots = \frac{2}{9}\sum_{i=0}^{\infty}\left(\frac{1}{27}\right)^i + \frac{2}{27}\sum_{i=0}^{\infty}\left(\frac{1}{27}\right)^i = \frac{4}{13}$$

We have found that the ternary expansion $0.022022\overline{022}$ is equal to the fraction $\frac{4}{13}$. Now let us use the same method to check the base-ten value for $0.0\overline{2}$:

$$\sum_{i=2}^{\infty}\frac{2}{3^i} = \frac{2}{9}\sum_{i=0}^{\infty}\left(\frac{1}{3}\right)^i = \frac{1}{3}.$$

Since 0.1 is also a ternary expansion for $\frac{1}{3}$, it is possible for a number to have two different ternary expansions. For the same reason, two different ternary expansions can equal each other.

As another example, consider the ternary expansions $0.22\overline{00}$ and $0.21\overline{22}$. They are equivalent to each other and to the fraction $\frac{8}{9}$. That two different ternary expansions can equal each other is a problem that also surfaces in base-ten and binary expansions. As it turns out, the numbers that yield two ternary expansions are of a certain type. These numbers are rational numbers that can be written as $\frac{k}{3^i}$ where k is an integer that satisfies $0 \leq k \leq 3^i$. Clearly, $\frac{8}{9}$ falls under this category.

Unfortunately, it is not as easy to convert a base-ten number into its ternary equivalent. If the digits of the ternary number $0.t_1t_2t_3\ldots$ fall under an accessible pattern, we can use summations and other manipulations to decipher the number's base-ten value. Otherwise, we are left to estimate its value. Fortunately, the system of ternary expansion allows for an effortless method of mapping ternary numbers to the real interval $I = [0, 1]$. If x has a ternary expansion of the form $0.t_1t_2t_3\ldots$, then the value of the digit t_1 determines which third of the interval I x is located in. If $t_1 = 0$, then $x \in [0, \frac{1}{3}]$, if $t_1 = 1$, then $x \in [\frac{1}{3}, \frac{2}{3}]$, and $x \in [\frac{2}{3}, 1]$ if $t_1 = 2$. Once this has been determined, the same approach can be applied to t_2 to figure out which third of the third x is located in. We can follow this procedure up to the last significant digit t_n to find the exact location for x. An easy way to practice this procedure is to follow along with a diagram of the Cantor middle-thirds set (see Figure 5). At each step, we determine whether x is in the left thirds portion of the interval (if $t_n = 0$), the middle thirds section (if $t_n = 1$), or the right-hand third of the interval (if $t_n = 2$). But if $t_n = 1$, then x is not in the Cantor middle-thirds set. Therefore, each real number x in $[0, 1]$ that has a ternary expansion void of 1's belongs to the Cantor set. In other words, the Cantor set is a set of all $x \in I$ that satisfy this condition: x can be written as a ternary expansion composed of only 0's and 2's.

Thus, if we have a ternary expansion of 0's and 2's, we simply pass down Figure 5 taking the appropriate "left" or "right" path to eventually locate the number within I. If we choose the ternary expansion 0.220, for instance, and we follow the directions "right, right, left," we land on the point $\frac{8}{9}$ (see Figure 5). This is as expected, since we have already determined that the base-ten value for 0.22 is indeed $\frac{8}{9}$.

Now that we are able to comfortably translate between base-ten and ternary numbers, we may proceed with a proof the uncountability of the Cantor set.

5.3. **Uncountability of the Cantor Set.** To prove that the Cantor middle-thirds set is uncountable, we must go through one more intermediate step. This step involves the set of binary numbers, which is itself an uncountable set. If we take each member x of the Cantor set in its ternary form and change every "2" to a "1", we have created the set of binary numbers. In other words, the elements of the Cantor set are in a one-to-one correspondence with the full set of binary numbers. Since the set of binary numbers is also in a one-to-one correspondence with the set of real numbers, which is uncountable, it follows that the Cantor set is uncountable. This property of the Cantor middle-thirds set will surface again in the sections to come.

5.4. **The Quadratic Family.** Now we will apply the concepts behind the Cantor middle-thirds set to dynamics we are closely acquainted with. Up to this point we have learned the basic dynamical infrastructure of $Q_c(x) = x^2 + c$ for $-\frac{5}{4} \leq c < \infty$. We will now attempt to do this for $Q_c(x)$ with $c < -2$, but our work will not be quite as simple as before. We will come across very complex networks of orbits for each value of $c < -2$, but our newfound knowledge of Cantor sets will help us along the way.

In Figure 6, we see the function $Q_{-2.5}(x) = x^2 - 2.5$ enclosed (or almost enclosed) in a box. This box, centered at the origin, has vertices's approximately at the points $[-2.158, 2.158]$, $[2.158, 2.158]$, $[2.158, -2.158]$, and $[-2.158, -2.158]$. The number $x \approx 2.158$ has a significance we can appreciate; it is the value of the fixed point r_1 for $x^2 - 2.5$. As we have come to experience, all the interesting orbit activity for a quadratic function Q_c occurs between the two fixed points. Since $|r_1| > |r_2|$ for all possible c, this activity will certainly be captured within the interval $I = [-r_1, r_1]$, or in our case, $I = [-2.158, 2.158]$. As expected, the function $Q_{-2.5}(x)$ intersects the line $f(x) = x$ at a corner of the box (in the first quadrant of the graph).

From Figure 6, note that the lowest section of the function is poking out of the bottom of the box. Let us call the open-ended portion of I corresponding to this lower part of the graph A_1 and recognize that orbits fed from each $x \in A_1$

immediately leave I. In fact, x values within the interval $(-0.585, 0.585)$ yield orbits that leave the interval I upon their first iteration and escape to infinity. In addition to this set $A_1 = \{x : -0.585 < x < 0.585\}$, there are other sets of $x \in I$ who's orbits eventually escape to infinity through this hole in I. We know the fate of all orbits that leave I, so this leaves us to determine the fate of orbits that never escape from I. The set of x values that yield orbits completely contained in I will be denoted as Λ and will be defined as follows:

$$\Lambda = \{x : x \in I \wedge Q_c^n(x) \in I \ \forall \ n \in \mathbb{Z}^+\}.$$

The most effective way to determine the full set of Λ is by instead determining the complete complement of Λ. Naturally, what is left behind in I will make up Λ. We will call this set $\neg\Lambda$. As we have discovered, A_1 is part of this set. There are indeed many more subsets that contribute to $\neg\Lambda$. For example, let us consider the pair of open intervals that form the set A_2. If an x_0 is chosen from one of these intervals, it follows from Figure 6 that $Q_{-2.5}(x_0)$ will be in A_1 (see path 1). Therefore, the orbit of x_0 escapes from I after 2 iterations. Similarly, if x_0 is chosen from A_3, then x_0 will escape I after 3 iterations. Following a process of "backward graphical analysis," we find that, if x_0 is chosen from A_n, then x_0 will escape from I after n iterations. If we take the union of A_1, A_2, ..., A_∞, we will have the completed set of $\neg\Lambda$, as desired. In general, we will define $\neg\Lambda$ as:

$$\neg\Lambda = \{x : x \in I \wedge x \notin \Lambda\} = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_\infty$$

From the first three subsets of $\neg\Lambda$, we can see a clear pattern in the formation of the set $\neg\Lambda$. Now, if we remove subsets A_1, A_2, and A_3 from the interval I, we are left with a basic foundation for Λ. The pattern of formation for Λ is just as evident as for $\neg\Lambda$: it is a pattern eerily similar to that of the Cantor middle-thirds set. Since A_n is an open set, the endpoints of A_1, A_2, and A_3 will be left behind to remain in Λ (a similar property to that of the original Cantor set). The orbit path 2 in Figure 6 shows an orbit starting from an endpoint of one of the A_3 intervals. As we can see, this orbit is eventually fixed at r_1. Therefore, the set Λ is clearly closed and non-empty. The set Λ also contains no intervals for reasons very similar to those for the Cantor middle-thirds set. For all practical purposes, Λ is a Cantor set in its own right. In the chapters to come, we will expand on the properties of this Cantor set and on many others.

6. **Symbolic Dynamics**

We have gradually worked to secure a strong background in the properties and behaviors of dynamical systems. Now we are almost ready to describe the chaos

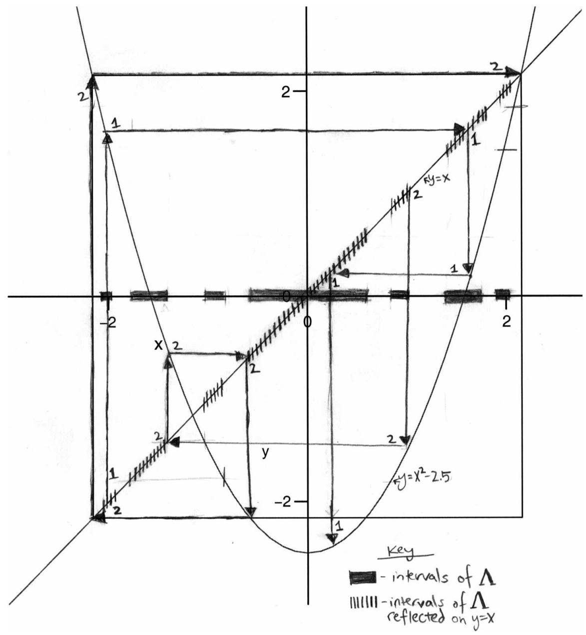

FIGURE 6. The function $x^2 - 2.5$ with the intervals A_1, A_2, and A_3

we have witnessed within the quadratic family. The only thing holding us back at this point is our lack of proper symbolic terminology. In this section we will work our way through a basic tutorial on symbolic dynamics. This will give us a more convenient way to describe the dynamics we see. Symbolic dynamics will help us to translate our familiar system of quadratic dynamics into one more readily accommodating to the theories behind chaos.

6.1. **Sequence Space.** To better understand the concept of chaos, we will first approach a chaotic model with dynamics even simpler than that of the quadratic family. We must define the "space" under which the our chaotic dynamical model will exist. This so-called space will not consist of intervals of real numbers. It will be a *sequence space*, consisting of combinations of a certain type of sequence. We will call this sequence space Σ, and define it as the following set:

Definition 6.1 (Sequence Space)**.** *The set* $\Sigma = \{(s_0 s_1 s_2 \ldots) : s_j = 0 \text{ or } 1\}$ *is the sequence space on two symbols.*

Thus, the elements of Σ are sequences of made up of zeros and ones. The entire set Σ is comprised of all possible combinations of ones and zeros (in sequence form). Clearly, sequences such as 01101001 and $1100\overline{1100}$ are distinct elements of the set. We will soon see how this space provides a foundation for chaotic behavior to occur.

Naturally, it will prove useful to measure distance between two points in our sequence space. We cannot use the traditional method for finding the distance between two sequences. Therefore, let us designate a new definition.

Definition 6.2 (Distance in Σ)**.** *Let $s = (s_0 s_1 s_2 \ldots)$ and $t = (t_0 t_1 t_2 \ldots)$ be points in Σ. The distance between s and t is*

$$d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}.$$

For example, suppose $s = (1111\overline{1111})$ and $t = (0000\overline{0000})$. The distance between these points should be as large as Σ permits since $|s_i - t_i| = 1$ for every $0 \leq i \leq \infty$. Calculating the distance, we get:

$$d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = 2.$$

Therefore, two points in the sequence space can be as far apart as 2 units. To determine the distance between points that are "close" together in the sequence space, we use the Proximity Theorem (featured in Devaney, p. 102).

Theorem 6.1 (The Proximity Theorem)**.** *Let $s, t \in \Sigma$. If $s \neq t$ but $s_i = t_i$ for $i = 0, 1, \ldots, n$, then the distance between s and t is $d[s,t] \leq \frac{1}{2^n}$.*

These new terms and concepts about sequence space will help us to express chaos in precise terms.

6.2. **The Shift Map.** Now that we have a created a space of our model, we must determine how the model will map from Σ to Σ. In order to consider the dynamics of the model, we must examine the behavior of the orbits initiating from different sequences in Σ. How shall we iterate sequences in Σ to form new sequences within the space? The answer is simple: we will iterate by removing the first term of the sequence and shift all the other terms one place to the left. This process will be made possible by the *shift map*, denoted as σ.

Definition 6.3 (Shift Map)**.** *The shift map $\sigma : \Sigma \to \Sigma$ is given by*

$$\sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 s_3 \ldots).$$

For example, $\sigma(011011\ldots) = 110110\ldots$ and $\sigma(01111\ldots) = 1111\ldots$. We can continue this process without effort, merely dropping the first term of each iteration to find the next. To find the n^{th} iteration of the sequence, we follow the following formula:

$$\sigma^n(s_0 s_1 s_2 \ldots) = (s_n s_{n+1} s_{n+2} \ldots).$$

Interestingly, if we are given a repeating sequence $s = (s_0 s_1 \dots s_{n-1} \overline{s_0 s_1 \dots s_{n-1}})$, we can iterate n times to return to the original sequence. In other words, s is a n-cycle periodic point and $\sigma^n(s) = s$. As we can see, the shift map provides for a very easy way to find periodic points within the sequence space Σ. This fact will come in handy in our discussion of chaos.

6.3. **Itinerary.** We will finish off our discussion of symbolic dynamics new terminology that specifically relates to the quadratic family of functions $Q_c = x^2 + c$ where $c < -2$. We know that for the case of $c < -2$, all the interesting dynamics occur on the interval $I = [-r_1, r_1]$ where $r_1 = \frac{1}{2}\left(1 + \sqrt{1 - 4c}\right)$. Recall from Figure 6 that, within the subinterval A_1, all orbits leave I after the first iteration. Naturally, the set λ of points whose orbits never leave I must be contained in $I - A_1$. The set $I - A_1$ is split up into 2 closed intervals, one on each side of the origin. Let us call I_0 the interval to the right of the origin and I_1 the interval to the left. Any $x \in \lambda$ will therefore travel between I_0 and I_1 (or stay within one interval) under each iteration. This fact allows us to create a distinct *itinerary* for each point $x \in \lambda$.

Definition 6.4 (Itinerary)**.** *Let $x \in \lambda$. The itinerary of x is the infinite sequence $S(x) = (s_0 s_1 s_2 \dots)$ where $s_j = 0$ if $Q_c^j(x) \in I_0$ and $s_j = 1$ if $Q_c^j(x) \in I_1$.*

The itinerary of x is an infinite sequence of 0s and 1s. For an example, consider the case where x is the attracting fixed point r_1. The itinerary of x would be $S(x) = (1111\dots)$ since x would remain in its place in I_1 for all iterations. Interestingly, each periodic point $x_p \in \lambda$ has a repeating sequence. As one might infer, this concept is the basis for the creation sequence space Σ. The mapping of real numbers within λ to their respective itineraries will come into play later under the theories of conjugacy.

Now we have built up a vocabulary of new terms about symbolic dynamics. Finally, we can proceed with a definition of chaos!

7. Chaos!

Now that we have the all the necessary materials in our toolbox, we can finally approach the complex concept of chaos. Chaos, as it is seen in the dynamical systems around us, can be described in many different ways. For the time being, we will stick to Devaney's definition of chaos, as featured in Chapter 10 of *A First Course in Chaotic Dynamical Systems*. Devaney's definition engages three very new and abstract topological ideas. These ideas are denseness, transitivity, and sensitivity, all of which pertain to orbits within dynamical systems. First, let us consider the description of a *dense set*:

Definition 7.1 (Dense Set)**.** *Suppose Y is a subset of set X and let $\epsilon > 0$. The subset Y is dense in X if, for each point $x \in X$, there exists a point $y \in Y$ within ϵ of x.*

In other words, for each point in X there is a point in Y that is arbitrarily close. As an example, the set of rational numbers is dense in the set of real numbers. In addition, the open interval $(0, 1)$ is dense in the closed interval $[0, 1]$. Obviously, there are also many cases where subspaces are not dense in a set. For instance, the integers are not dense in the reals. Let us choose an element $x \in \mathbb{R}$, say 4.7, and also designate $\epsilon = 0.1$. Clearly, there is no integer within the interval $[4.6, 4.8]$. The size of Y seems to play a part in its role as dense subset, but we often find that a dense subset is quite small in comparison to the original set. The the subset of rationals within the reals is a good example of this, since the set of rationals is countable and the set of real numbers is uncountable.

Upon investigation of the sequence space Σ, it can be shown that the subset of Σ that consists of all the periodic points in Σ is a dense subset. To prove this statement, we must show that there is a periodic point within ϵ of any point $s = (s_0 s_1 s_2 \ldots)$ in Σ. For the proof that the subset of all periodic points in Σ is dense, we are interested in the closeness of points. The Proximity Theorem (featured in Devaney, p. 102) will help us determine the distance between "close" points.

Now we may return to our proof. Let us choose a positive integer n so that $\frac{1}{2^n} < \epsilon$. Now we will look for a periodic point within $\frac{1}{2^n}$ of s. Let $t_n = (s_0 s_1 \ldots s_n \overline{s_0 s_1 \ldots s_n})$. Since the first $n + 1$ terms of s and t_n are the same, we can apply the Proximity Theorem to find:

$$d[s, t_n] \leq \frac{1}{2^n} < \epsilon.$$

The sequence t_n is repeating, so it is clearly a $n + 1$ periodic point under the shift map σ. Therefore, since s and ϵ were chosen arbitrarily, we have proved that there is a periodic point arbitrarily close to every point s in Σ.

Now that we have explicitly demonstrated the characteristics of a dense subset, we will move on to approach another important topic of chaos: transitivity. Transitivity is closely related to the concept of dense subsets. First, we will define transitivity as it relates to chaos and than we will provide an appropriate application.

Definition 7.2 (Transitivity)**.** *A dynamical system is transitive if, for each two points x and y in the system and for each $\epsilon > 0$, there exists an orbit that comes within ϵ of both points.*

A dynamical system with a *dense orbit* is transitive. This follows from the fact that a dense orbit must come arbitrarily close to all points in the system. We will propose the idea of a dense orbit through an example in the Σ domain under σ. In this case, we are hoping to find an orbit that comes arbitrarily close to every point in Σ. As we can recall, such an orbit would be formed by a point s in Σ and by its respective iterations under the shift map σ.

Consider the sequence $s' = (0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ 010\ 011\ \ldots)$ where s' is made up of all combinations of 0s and 1s of length n for $n = 0, 1, 2, \ldots, \infty$. As we can see, the first two terms are 0 and 1. These are the two possibilities for $n = 1$. These terms are followed by the four combinations for $n = 2$ and then proceeded by 2^n terms of length n for each n.

As before, choose an arbitrary $s = (s_0 s_1 s_2 \ldots)$ in Σ and an $\epsilon > 0$. Also, select n so that $\frac{1}{2^n} < \epsilon$. Somewhere along the sequence s', there is a sequence strand of length $n+1$ that is composed of the digits $s_0 s_1 s_2 \ldots s_n$ that correspond with the sequence s. If s_0 is the k^{th} term of s', we apply the shift map k times to s'. Therefore, the first $n+1$ terms of $\sigma^k(s')$ are equal to the first $n+1$ terms of s. Again putting to use the Proximity Theorem, we have:

$$d[\sigma^k(s'), s] \leq \frac{1}{2^n} < \epsilon.$$

Since s and ϵ were arbitrary choices, we have proved that the point s' yields an orbit that comes within ϵ of every point in Σ. Thus, the orbit of s' is dense and the sequence space Σ under σ is transitive. This concludes our discussion of transitivity.

A third notion that surfaces in Devaney's definition of chaos is that of *sensitive dependence on initial conditions*. Here is the definition for sensitivity:

Definition 7.3 (Sensitive Dependence on Initial Conditions)**.** *A dynamical system F depends sensitively on initial conditions if there exists a $\beta > 0$ such that for any $x \in F$ and any $\epsilon > 0$ there is a $y \in F$ within ϵ of x and a k such that $d[F^k(x), F^k(y)] \geq \beta$.*

If a system has sensitive dependence on initial conditions, then we can always find a y within ϵ of x whose orbit eventually differs from that of x by at least β. It should be noted that the orbit of y need only to be separated from the orbit of x for the k^{th} iteration.

Now we will attempt to show that the shift map depends sensitively on initial conditions. Let $\beta = 1$. For any $s \in \Sigma$ and $\epsilon > 0$, we again select an n so that $\frac{1}{2^n} < \epsilon$. Suppose there exists a $t \in \Sigma$ such that $t \neq s$ but $d[s, t] < \frac{1}{2^n}$. From the Proximity Theorem, $s_i = t_i$ for $i = 0, 1, \ldots, n$. However, there must be a $k > n$ such that $s_k \neq t_k$. Now apply k iterations to both s and t under the shift map.

The initial terms of the sequences will be different and

$$d[\sigma^k(s), \sigma^k(t)] \geq \frac{|s_k - t_k|}{2^0} + \sum_{i=1}^{\infty} \frac{0}{2^i} = 1.$$

Therefore, we have proved sensitivity for this shift and for every other shift in Σ since s was chosen arbitrarily. For each $s \in \Sigma$ all other points in Σ have orbits that eventually separate from the orbit of s by at least 1 unit.

Now that we have officially discussed the concepts behind denseness, transitivity, and sensitivity, we can proceed with our definition of chaos:

Definition 7.4 (Chaos)**.** *A dynamical system F is chaotic if:*

i. *The set of periodic points in F is dense.*
ii. *F is transitive.*
iii. *F depends sensitively on initial conditions.*

Since we have shown that the shift map σ satisfies these three conditions under the sequence space Σ, we have proved that the dynamical system $\sigma : \Sigma \to \Sigma$ is chaotic. Next we will apply this concept of chaos to the quadratic family.

8. Conjugacies and the Chaotic Nature of Q_c

8.1. **Homeomorphisms.** As we have recently discovered, the itinerary map S relates λ, the set of orbits in Q_c for $c < -2$ that never leave I, to the sequence space Σ. We would like to take this relationship a step farther. First, we hope to show that λ and Σ are indeed identical spaces. In other words, we are checking to see if λ and Σ are *homeomorphic* to each other.

Definition 8.1 (Homeomorphism)**.** *A homeomorphism is a continuous function that is one-to-one and onto. It must also have a continuous inverse.*

Homeomorphic sets map points between each other in a one-to-one correspondence. Due their continuous natures, they also map nearby points in one set to nearby points in the other.

If $c < -\frac{1}{4}(5 + 2\sqrt{5})$, then $S : \lambda \to \Sigma$ is a homeomorphism. In order to prove that the two sets λ and Σ are homeomorphic, we must be sure that they satisfy all four conditions of homeomorphism. We will not solve such an involved proof, but we will certainly consider a few more ideas that extend from homeomorphisms.

8.2. **Conjugacy.** We will now look at a concept that extends directly from homeomorphism. This concept is called *conjugacy.*

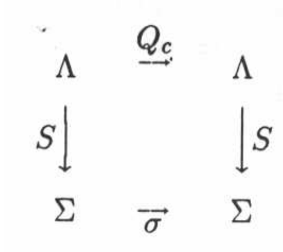

FIGURE 7. Commutative Diagram involving λ, S, and Σ (image from Devaney, page 107).

Definition 8.2 (Conjugacy)**.** *The functions* $F : X \to X$ *and* $G : Y \to Y$ *are conjugate if there is a homeomorphism* $h : X \to Y$ *such that* $h(F(x)) = G(h(x))$ *for all* $x \in X$*.*

Conjugacies play an important role in locating chaos within a system because their existence ultimately demonstrates the exact equivalence between the dynamics of two systems. If we can find that Q_c on λ is conjugate to σ on Σ, then we can conclude that their dynamics are equivalent. Since we were able to prove that shift map σ over the sequence space Σ demonstrates chaotic behavior, we can easily prove the same for a system with congruent dynamics.

For the moment, let us return to our discussion of the itinerary map S to consider an applicable example of conjugacy.

Example. Prove the following statement: if $x \in \lambda$, then $S(Q_c(x)) = \sigma(S(x))$.

Proof. First recall that if $S(x) = (s_0 s_1 s_2 \ldots)$, then $\sigma(S(x)) = (s_1 s_2 s_3 \ldots)$. Since $x \in \lambda$, it must have an itinerary $(s_0 s_1 s_2 \ldots)$ and therefore

$$x \in I_{s_0}$$

$$Q_c(x) \in I_{s_1}$$

$$Q_c^2(x) \in I_{s_2}$$

$$Q_c^3(x) \in I_{s_3}.$$

Since I_{s_j} is either I_0 or I_1, it follows that $S(Q_c(x)) = (s_1 s_2 s_3 \ldots)$ is the same sequence of 0s and 1s. Thus, we find that $S(Q_c(x)) = \sigma(S(x))$, as desired. □

To see a *commutative diagram* of this property, see Figure 7. From the diagram, we can begin with λ in the upper left-hand corner and follow either of two different paths to end up in at the same point in Σ (in the lower right-hand corner). Such commutative diagrams will be used from now on to test for conjugacy. Under

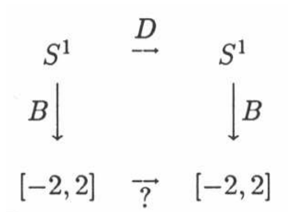

Figure 8. Commutative Diagram involving S^1, B, and D (image from Devaney, page 125).

this example, the map S is clearly a conjugacy and the functions σ and Q_c for $c < -\frac{1}{4}(5 + 2\sqrt{5})$ are conjugate.

As with the case of S, there are homeomorphisms that provide us with more than just the usual information about continuity and correspondence. In this particular case, the map S has equipped us with the ability to make a giant step toward finding chaos within the quadratic family Q_c. We have shown that the system Q_c on λ is dynamically equivalent to the shift map σ on Σ. The remaining steps of the proof follow naturally and will be omitted from our discussion.

8.3. **The Doubling Map.** Now that we have officially verified that the quadratic family of functions Q_c exhibits chaotic behavior for certain values of c, it is only natural to wonder what other simple functions are chaotic. Let us consider another example of a chaotic dynamical system. Our example will involve the unit circle S^1 in the plane, described by

$$S^1 = (x, y) \in \{\mathbb{R}^2 : x^2 + y^2 = 1\}.$$

A point in S^1 is given in its polar angle θ form, in radians. It is important to note that θ is measured modulo 2π.

Now that we have a space, we must complete the system by defining a function to act on the space. Let $D : S^1 \to S^1$ be defined as $D(\theta) = 2\theta$. To iterate this function, we simply multiply any angle θ by 2. Once the value of the iteration exceeds 2π, we must subtract $n \cdot 2\pi$ for the appropriate integer n.

To prove that D on S^1 is chaotic, we will look for the presence of a conjugacy. Let us consider a map B that first projects a point θ on the unit circle straight down to the x-axis and then doubles its distance from the origin. Such a function $B : S^1 \to [-2, 2]$ could be defined as $B(\theta) = 2\cos(\theta)$. This mapping is pictured in Figure 8. We would like to complete this diagram by finding the missing map from $[-2, 2]$ to $[-2, 2]$. We know that

$$B(D(\theta)) = B(2\theta) = 2\cos(2\theta)$$

and we are looking for a map such that

$$2\cos(\theta) \to 2\cos(\theta).$$

With a little trigonometric manipulation, we find that

$$2\cos(2\theta) = 2\left(2\cos^2(\theta) - 1\right) = \left(2\cos^2(\theta)\right)^2 - 2.$$

Therefore, the function that acts as a conjugacy in this case is $Q_{-2} = x^2 - 2$. This happens to be a familiar function. Though the value of c is not less than $-\frac{1}{4}(5 + 2\sqrt{5})$, we have seen from previous experiments and discussions that the dynamics of Q_{-2} do suggest chaotic activity. For sake of time and energy conservation, let us assume (correctly) that Q_{-2} is a chaotic dynamical system. It then follows easily that the function D on S^1 is also a chaotic dynamical system.

9. **Related Topics**

Over the course of this paper, we have been working to verify one ultimate result: that the quadratic family of functions $Q_c(x) = x^2+c$ exhibits chaotic behavior under certain constraints on c. With the introduction to bifurcation theory in Section 4, our suspicions of chaos were first aroused. In order to formally arrive at our desired conclusion, we had to familiarize ourselves with a great deal of new terminology. We began with the study of orbits and bifurcations and then moved on to more complex mathematical notions such as the Cantor set and symbolic dynamics. We have now completed our goal, but a certain cloud still hangs over our heads. What does acknowledging the presence of chaos do for us? Where do we go from here?

The fact of the matter is, if we can find chaos in a dynamical system as simple and primitive as the quadratic family, we can certainly find chaos in more complex systems. We have barely scratched the surface of chaos. For example, both sine and cosine functions have chaotic characteristics under certain parameters. As do many functions within the complex plane. Quadratic functions of the form $z^2 + c$ where z and c are complex numbers have been studied extensively in the field of chaos theory. The chaotic behavior of these functions and many other complex functions can be visually captured by their Julia set representations. Julia sets allow us to observe the outcome of a functions orbits within a plane, and the diagrams they create are as beautiful as they are complex. The study of Julia sets is an interesting extension of chaos.

As mentioned previously, there are systems we utilize in our everyday lives that demonstrate chaos. The weather predictions we tune in to in the morning are based on exploration into a massive chaotic system. The stock market acts as a multi-faceted dynamical system that is subject to chaos. There are also living creatures that express behaviors we can model using our dynamical techniques. As an example, there is a certain type of firefly in the American Midwest that will inadvertently synchronize his flash to an appropriate stimulus. This synchronization can be measured and modeled by a relatively simple dynamical system. For further information on this particular experiment, see *Nonlinear Dynamics and Chaos* by Steven H. Strogatz [2]. There are clearly many directions one can take to indulge their interests in chaotic dynamical systems.

Appendix A

MatLab commands for creating a bifurcation diagram of $Q_c(x)$ for $-2 \leq c \leq \frac{1}{4}$:

```
%Script file to produce the bifurcation diagram for
%  F(x) = x^2 + c using the orbit of 0, and values
%  of c:  -2 <= c <= 0.25

c=linspace(-2,0.25,500); %This is 500 points for c
x=zeros(1,500); %An array of initial conditions
fprintf('Going into the initial orbit\ n');

for i=1:100
  y=x.^2+c;
  x=y;
end
fprintf('Going into new orbit\ n');
%Store results:
A=zeros(400,500); %allocates memory
for i:400
  y=x.^2+c;
  A(i,:)=y;
  x=y;
end

plot(A','k.','Markersize',4);
```

References

[1] Devaney, Robert L: *A First Course in Chaotic Dynamical Systems.* Cambridge, MA: Perseus Books Publishing (1988).

[2] Strogatz, Steven H: *Nonlinear Dynamics and Chaos*. Reading, MA: Addison-Wesley Publishing Company (1994).