FUNCTIONS OF BOUNDED VARIATION

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ABSTRACT. In this paper we explore functions of bounded variation. We discuss properties of functions of bounded variation and consider three related topics. The related topics are absolute continuity, arc length, and the Riemann-Stieltjes integral.

1. Introduction

In this paper we discuss functions of bounded variation and three related topics. We begin by defining the variation of a function and what it means for a function to be of bounded variation. We then develop some properties of functions of bounded variation. We consider algebraic properties as well as more abstract properties such as realizing that every function of bounded variation can be written as the difference of two increasing functions.

After we have discussed some of the properties of functions of bounded variation, we consider three related topics. We begin with absolute continuity. We show that all absolutely continuous functions are of bounded variation, however, not all continuous functions of bounded variation are absolutely continuous. The Cantor Ternary function provides a counter example. The second related topic we consider is arc length. Here we show that a curve has a finite length if and only if it is of bounded variation. The third related topic that we examine is Riemann-Stieltjes integration. We examine the definition of the Riemann-Stieltjes integral and see when functions of bounded variation are Riemann-Stieltjes integrable.

2. Functions of Bounded Variation

Before we can define functions of bounded variation, we must lay some ground work. We begin with a discussion of upper bounds and then define partition.

2.1. Definitions.

Definition 2.1. *Let S be a non-empty set of real numbers.*

(1) *The set S is* bounded above *if there is a number M such that $M \geq x$ for all $x \in S$. The number M is called an* upper bound *of S.*
(2) *The set S is* bounded below *if there exists a number m such that $m \leq x$ for all $x \in S$. The number m is called a* lower bound *of S.*
(3) *The set S is* bounded *if it is bounded above and below. Equivalently S is bounded if there exists a number r such that $|x| \leq r$ for all $x \in S$. The number r is called a* bound *for S.*

Definition 2.2. *Let S be a non-empty set of real numbers.*

(1) *Suppose that S is bounded above. A number β is the* supremum *of S if β is an upper bound of S and there is no number less than β that is an upper bound of S. We write $\beta = \sup S$.*
(2) *Suppose that S is bounded below. A number α is the* infimum *of S if α is a lower bound of S and there is no number greater than α that is a lower bound of S. We write $\alpha = \inf S$.*

We can restate Definition 2.2 (1) in equivalent terms as follows.

Theorem 2.1. *Let S be a non-empty set of real numbers that is bounded above, and let b be an upper bound of S. Then the following are equivalent.*
(1) $b = \sup S$
(2) *For all $\epsilon > 0$ there exists an $x \in S$ such that $|b - x| < \epsilon$.*
(3) *For all $\epsilon > 0$ there exists $x \in S$ such that $x \in (b - \epsilon, b]$*

A proof of Theorem 2.1 can be found in Schumacher's text [3]. We often refer to $\sup S$ as the least upper bound of S and to $\inf S$ as the greatest lower bound of S.

The following axiom is often useful for determining the existence of the least upper bound of a set.

Axiom 2.1. *Every non-empty set of real numbers that is bounded above has a least upper bound.*

Definition 2.3. *A* partition *of an interval $[a, b]$ is a set of points $\{x_0, x_1, ..., x_n\}$ such that $a = x_0 < x_1 < x_2 \cdots < x_n = b$.*

With these definitions in hand we can define the variation of a function.

Definition 2.4. *Let $f : [a, b] \to \mathbb{R}$ be a function and let $[c, d]$ be any closed subinterval of $[a, b]$. If the set*

$$S = \left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| : \{x_i : 1 \leq i \leq n\} \text{ is a partition of } [c, d] \right\}$$

is bounded then the variation *of f on $[c, d]$ is defined to be $V(f, [c, d]) = \sup S$. If S is unbounded then the variation of f is said to be ∞. A function f is of bounded variation on $[c, d]$ if $V(f, [c, d])$ is finite.*

2.2. **Examples.** We now examine a couple of examples of functions of bounded variation, and one example of a function that is not of bounded variation.

Example 2.1. *If f is constant on $[a, b]$ then f is of bounded variation on $[a, b]$.*

Consider the constant function $f(x) = c$ on $[a, b]$. Notice that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

is zero for every partition of $[a, b]$. Thus $V(f, [a, b])$ is zero.

Another example of a function of bounded variation is a monotone function on $[a, b]$.

Theorem 2.2. *If f is increasing on $[a, b]$, then f is of bounded variation on $[a, b]$ and $V(f, [a, b]) = f(b) - f(a)$.*

Proof. Let $\{x_i : 1 \leq i \leq n\}$ be a partition of $[a, b]$. Consider

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big) = f(b) - f(a).$$

Because of the telescoping nature of this sum, it is the same for every partition of $[a, b]$. Thus we see that $V(f, [a, b]) = f(b) - f(a) < \infty$. Thus f is of bounded variation on $[a, b]$.

□

Similarly, if f is decreasing on $[a, b]$ then $V(f, [a, b]) = f(a) - f(b)$.

For the next example we first recall a theorem involving rational and irrational numbers.

Theorem 2.3. *Between any two distinct real numbers there is a rational number and an irrational number.*

We will not prove this here, but a proof is provided in Gordon's text [1].

Example 2.2. *The function f defined by*

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

is not of bounded variation on any interval.

Proof. Let $n \in \mathbb{Z}$ and $n > 0$. Let $[a, b]$ be a closed interval in $\mathbb{R}$. We construct a partition $P = \{x_0, x_1, ..., x_{n+2}\}$ of $[a, b]$ such that $V(f, [a, b]) \geq \sum_{i=1}^{n+2} |f(x_i) - f(x_{i-1})| > n$ as follows.

Recall that by definition $x_0 = a$. By Theorem 2.3 we know that between any two real numbers there is a rational number and an irrational number. Take x_1 to be an irrational number between a and b. Then take x_2 to be a rational number between x_1 and b. Continue like this, taking x_{2i+1} to be an irrational number between x_{2i} and b, and x_{2i} to be a rational number between x_{2i-1} and b. Finally $x_{n+2} = b$. Thus we have created a partition that begins with a and then alternates between rational and irrational numbers, until it finally ends with b. Now consider the sum $\sum_{i=1}^{n+2} |f(x_i) - f(x_{i-1})|$, which we know is at most the variation of f on $[a, b]$. Thus

$$\begin{aligned} V(f, [a, b]) &\geq \sum_{i=1}^{n+2} |f(x_i) - f(x_{i-1})| \\ &\geq \sum_{i=2}^{n+1} |f(x_i) - f(x_{i-1})| \\ &= |f(x_2) - f(x_1)| + \cdots + |f(x_{n+1}) - f(x_n)| \\ &= |1 - 0| + |0 - 1| + \cdots + |1 - 0| \\ &= 1 + 1 + 1 + \cdots + 1 \\ &= n. \end{aligned}$$

Thus $V(f, [a, b])$ is arbitrarily large, and so $V(f, [a, b]) = \infty$. □

We have now examined a couple of examples of functions of bounded variation, and one example of a function not of bounded variation. This should give us some insight into the behavior of functions of bounded variation and motivate the exploration of some algebraic properties of these functions, which we do in Section 2.3.

2.3. **Algebraic Properties of Functions of Bounded Variation.**

In this section we consider some of the properties of functions of bounded variation. Properties that are listed but not proven have proofs in Gordon's text [1].

Theorem 2.4. *Let f and g be functions of bounded variation on $[a, b]$ and let k be a constant. Then*

(1) *f is bounded on $[a, b]$;*
(2) *f is of bounded variation on every closed subinterval of $[a, b]$;*
(3) *kf is of bounded variation on $[a, b]$;*
(4) *$f + g$ and $f - g$ are of bounded variation on $[a, b]$;*
(5) *fg is of bounded variation on $[a, b]$;*
(6) *if $1/g$ is bounded on $[a, b]$, then f/g is of bounded variation on $[a, b]$.*

We begin with a useful lemma.

Lemma 2.1. *Let $f\colon [a, b] \to \mathbb{R}$ be a function, let $\{x_i : 0 \le i \le n\}$ be a partition of $[a, b]$, and let $\{y_i : 0 \le i \le m\}$ be a partition of $[a, b]$ such that $\{x_i : 0 \le i \le n\} \subseteq \{y_i : 0 \le i \le m\}$. Then*

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le \sum_{i=1}^{m} |f(y_i) - f(y_{i-1})|.$$

Proof. We begin by showing that adding one point to the partition $\{x_i : 0 \le i \le n\}$ gives the desired result, and then appeal to induction.

Let $\{x_i : 0 \le i \le n\}$ and $\{y_i : 0 \le i \le m\}$ be partitions as in the statement of the lemma. Suppose $y \in \{y_i : 0 \le i \le m\}$. If $y = x_j$ for some j then the sum does not change. Thus we will suppose that $y \ne x_j$ for all j. In this case y falls between two points x_{k-1} and x_k in $\{x_i : 0 \le i \le n\}$ for some k. We take the sum $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$ and write it out as follows:

$$\sum_{i=1}^{k-1} |f(x_i) - f(x_{i-1})| + |f(x_k) - f(x_{k-1})| + \sum_{i=k+1}^{n} |f(x_i) - f(x_{i-1})|.$$

We focus on $|f(x_k) - f(x_{k-1})|$. We know that

$$\begin{aligned} |f(x_k) - f(x_{k-1})| &= |f(x_k) - f(x_{k-1}) + f(y) - f(y)| \\ &\le |f(x_k) - f(y_i)| + |f(y_i) - f(x_{k-1})| \end{aligned}$$

by the triangle inequality. We relabel the partition with the extra point as $\{x_i : 0 \le i \le n+1\}$. Thus, since all the addends are positive, we can write

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le \sum_{i=1}^{n+1} |f(x_i) - f(x_{i-1})|.$$

Because there are at most a finite number of the y_i the desired results follows by induction. □

Lemma 2.1 tells us that if we add points to a partition the sum of $|f(x_i)-f(x_{i-1})|$ either does not change, or increases. This is useful when we are trying to prove things about the supremum of such sums, as we need to when looking at functions of bounded variation.

Now we prove parts of Theorem 2.4.

Proof. To prove (2) we begin by assuming that f is of bounded variation on $[a,b]$. Thus $V(f,[a,b]) = \sup\{\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|\} = r$ where r is a real number. Let $[c,d]$ be a closed subinterval of $[a,b]$ and $\{x_i : 1 \leq i \leq n\}$ be a partition of $[c,d]$. Then extend this partition to $[a,b]$ by adding the points a and b, and relabeling. So $\{x_i : 0 \leq i \leq n+2\}$ is a partition of $[a,b]$ such that $x_1 = c, x_{n+1} = d$. Then

$$\begin{aligned}\sum_{i=2}^{n+1} |f(x_i) - f(x_{i-1})| &\leq |f(x_1) - f(a)| + \sum_{i=2}^{n+1} |f(x_i) - f(x_{i-1})| + |f(b) - f(x_n)| \\ &\leq r\end{aligned}$$

Because the original partition of $[c,d]$ was arbitrary we can conclude that $r \geq V(f,[c,d])$.

To prove (3) we begin by letting $\{x_i : 1 \leq i \leq n\}$ be a partition of $[a,b]$. Consider

$$\begin{aligned}\sum_{i=1}^{n} |kf(x_i) - kf(x_{i-1})| &= |k| \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\ &\leq |k|V(f,[a,b]).\end{aligned}$$

Because the partition was arbitrary kf is of bounded variation. Further, we can observe that $V(kf,[a,b]) = |k|V(f,[a,b])$.

To prove (4) we begin again by letting $\{x_i : 1 \leq i \leq n\}$ be a partition of $[a,b]$. By repeated use of the triangle inequality we write

$$\begin{aligned}&\sum_{i=1}^{n} |f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})| \\ &\leq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| \\ &\leq V(f,[a,b]) + V(g,[a,b]).\end{aligned}$$

Notice that $V(f,[a,b])+V(g,[a,b])$ is finite and the partition we chose was arbitrary. Thus by the least upper bound axiom (Axiom 2.1) $f+g$ is of bounded variation.

To prove that $f-g$ is of bounded variation on $[a,b]$ we simply note that $f-g = f+(-1)g$. Since $(-1)g$ is of bounded variation on $[a,b]$ by Theorem 2.4(3) we know from what we have just shown that $f-g$ is of bounded variation on $[a,b]$.

A proof of (5) is provided in Gordon's text [1].

To prove (6) we assume that f and g are of bounded variation on $[a,b]$ and that $\frac{1}{g}$ is bounded on $[a,b]$. Thus we know that there exists a number M such that for all $x \in [a,b]$, $|1/g(x)| \leq M$. By Theorem 2.4(5) it suffices to show that $1/g$ is of bounded variation on $[a,b]$. Thus we begin by taking $\{x_i : 0 \leq i \leq n\}$ as an arbitrary partition of $[a,b]$ and consider the usual sum,

$$\begin{aligned}\sum_{i=1}^{n}\left|\frac{1}{g(x_i)}-\frac{1}{g(x_{i-1})}\right| &= \sum_{i=1}^{n}\left|\frac{g(x_{i-1})-g(x_i)}{g(x_i)g(x_{i-1})}\right| \\ &\le M^2\sum_{i=1}^{n}|g(x_i)-g(x_{i-1})| \\ &\le M^2\cdot V(g,[a,b]).\end{aligned}$$

Because the partition was arbitrary, we see that the sum is bounded above by $M^2\cdot V(g,[a,b])$ and so by the least upper bound axiom (Axiom 2.1) $1/g$ is of bounded variation. □

Theorem 2.5. *Let $f\colon [a,b]\to\mathbb{R}$ be a function and let $c\in(a,b)$. If f is of bounded variation on $[a,c]$ and $[c,b]$, then f is of bounded variation on $[a,b]$ and $V(f,[a,b])=V(f,[a,c])+V(f,[c,b])$*

A proof of Theorem 2.5 is provided in Gordon's text [1].

2.4. **Examples.** We now consider some functions that are not of bounded variation on a particular interval. These examples are helpful for gaining insight into what kinds of functions we might expect not to be of bounded variation, and for examining the mechanics of showing that a particular function is not of bounded variation.

Example 2.3. *If $f\colon [a,b]\to\mathbb{R}$ is of bounded variation on every closed subinterval of (a,b) it may yet fail to be of bounded variation on $[a,b]$.*

We can see this by a counterexample. Consider the following function,

$$f(x)=\begin{cases}\frac{1}{1-x}, & \text{when } x\neq 1;\\ 0, & \text{when } x=1.\end{cases}$$

This function is increasing on $(0,1)$ and so on every closed subinterval of $(0,1)$ it is of bounded variation by Theorem 2.2. However, because it has a vertical asymptote at $x=1$ we can make the sum $\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|$ as large as we like by choosing partition points close to 1. Thus $V(f,[0,1])=\infty$ and f is not of bounded variation on $[0,1]$.

The following example is especially interesting because it shows that a continuous function need not be of bounded variation.

Example 2.4. *The function f defined by*

$$f(x)=\begin{cases}\sqrt[3]{x}\sin(\pi/x), & \textit{for } x\neq 0;\\ 0, & \textit{if } x=0;\end{cases}$$

is not of bounded variation on $[0,1]$.

We begin by making a partition of $[0,1]$. We assume, without loss of generality, that the number of partition points is even, so n is even. We make our partition as follows: $x_n=1$, $x_{(n-(2k+1))}=1/(k+3)$, $x_{(n-2k)}=2/(2k+3)$, for $k=1,2,...$ and $x_0=0$.

Then

$$\begin{aligned} V(f,[0,1] &\geq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\ &\geq \sum_{i=1}^{n/2} |f(x_{2i+1}) - f(x_{2i})|. \end{aligned}$$

That is to say that we remove every other interval from the sum. Over the remaining intervals, our function has some convenient properties. The intervals we are now considering have the form $[1/(k+3), 2/(2k+3)]$. Also notice that $|f(2/(2k+3))| = \sqrt[3]{2/(2k+3)}$ and $f(1/(k+3)) = 0$. Because $\sqrt[3]{x}$ is an increasing function, $\sin(\pi/x)$ is the only component affecting when f is increasing or decreasing. Knowing how the sine function behaves we can see that f is monotone on each interval of the form $[1/(k+3), 2/((2k+3)]$. Thus

$$\begin{aligned} V(f,[0,1] &\geq \sum_{i=1}^{n/2} |f(x_{2i}) - f(x_{2i-1})| \\ &= \sum_{k=1}^{n} |f(1/(k+3))) - f(2/(2k+3))| \\ &= \sum_{k=1}^{n} \left| \sqrt[3]{\frac{2}{(2k+3)}} \right| \\ &= \sum_{k=1}^{n} \frac{\sqrt[3]{2}}{\sqrt[3]{(2k+3)}}. \end{aligned}$$

Notice that each term is smaller than the last, since the denominator is getting larger as k increases. Thus

$$\begin{aligned} \sum_{k=1}^{n} \frac{\sqrt[3]{2}}{\sqrt[3]{2k+3}} &\geq \sum_{k=1}^{n} \frac{1}{\sqrt[3]{k+3}} \\ &= \sum_{k=2}^{n+1} \frac{1}{\sqrt[3]{k}}. \end{aligned}$$

This is a p-series, with $p = 1/3$, so we know that it diverges. This means that by choosing n large enough we can make the sum

$$\sum_{k=1}^{n} \frac{\sqrt[3]{2}}{\sqrt[3]{(2k+3)}}$$

as large as we like and thus $V(f,[0,1]) = \infty$. Thus, though the function is continuous it is not of bounded variation on $[0,1]$.

2.5. **Functions of Bounded Variation as a Difference of Two Increasing Functions.** In this section we examine the fact that a function of bounded variation can be written as the difference of two increasing functions. Later in the section we refine this property, showing that a function of bounded variation can be written as the difference of two strictly increasing functions.

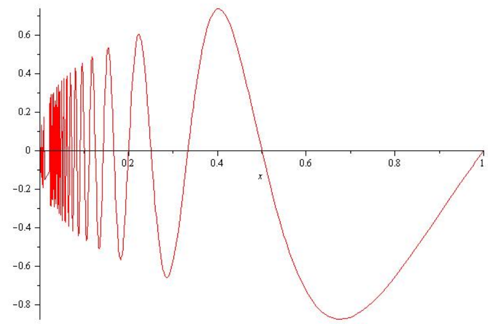

FIGURE 1. The graph of $f(x) = \sqrt[3]{x}\sin(\pi/x)$. This graph was created in Maple.

Theorem 2.6. *If $f : [a, b] \to \mathbb{R}$ is a function of bounded variation then there exist two increasing functions, f_1 and f_2, such that $f = f_1 - f_2$.*

We define an increasing function as a function f such that if $x_1 < x_2$ then $f(x_1) \leq f(x_2)$. Before we begin the proof of Theorem 2.6 we introduce Lemma 2.2 and 2.3.

Lemma 2.2. *For a function f, $V(f, [a, b]) = 0$ if and only if f is constant on $[a, b]$.*

Proof. Suppose that f is constant. Then f is a monotone function and by Theorem 2.2 $V(f, [a, b]) = f(b) - f(a)$. However, $f(b) = f(a)$ and so $V(f, [a, b]) = 0$.

For the other direction we proceed by contraposition. Suppose that f is not constant on $[a, b]$. We wish to show that $V(f, [a, b]) \neq 0$. Since f is not constant on $[a, b]$ there exist an x_1 and an x_2 such that both x_1 and x_2 are between a and b and such that $f(x_1) \neq f(x_2)$. If we take these two points as a partition of $[a, b]$ we have

$$V(f, [a, b]) \geq |f(x_1) - f(a)| + |f(x_2) - f(x_1)| + |f(b) - f(x_2)|.$$

However, we know that $|f(x_2) - f(x_1)| > 0$. Since each other addend is at least zero, we see that the sum must be greater than zero, and thus $V(f, [a, b]) > 0$ and $V(f, [a, b]) \neq 0$. □

Lemma 2.3. *If f is a function of bounded variation on $[a, b]$ and $x \in [a, b]$ then the function $g(x) = V(f, [a, x])$ is an increasing function.*

Proof. We begin by introducing x_1 and x_2 such that $x_1 < x_2$. We wish to show that $g(x_1) \leq g(x_2)$. Because f is of bounded variation on $[a, b]$, by Theorem 2.5

$$\begin{aligned} V(f, [a, x_2]) &= V(f, [a, x_1]) + V(f, [x_1, x_2]) \\ V(f, [a, x_2]) - V(f, [a, x_1]) &= V(f, [x_1, x_2]) \\ g(x_2) - g(x_1) &= V(f, [x_1, x_2]). \end{aligned}$$

Since $V(f, [x_1, x_2]) \geq 0$ we see that $g(x_2) \geq g(x_1)$. Furthermore, by Lemma 2.2 we have equality only if f is constant on $[x_1, x_2]$. □

Now we prove Theorem 2.6.

Proof. We begin by defining $f_1 = V(f,[a,x])$ for $x \in (a,b]$ and $f_1(a) = 0$. We know this function to be increasing by Lemma 2.3. Define f_2 as $f_2(x) = f_1(x) - f(x)$. Then $f = f_1 - f_2$. We need only show that f_2 is increasing.

Suppose that $a \le x < y \le b$. Using Theorem 2.5 we can write

$$\begin{aligned} f_1(y) - f_1(x) &= V(f,[x,y]) \\ &\ge |f(y) - f(x)| \\ &\ge f(y) - f(x). \end{aligned}$$

From this we see that

$$\begin{aligned} f_1(y) - f_1(x) &\ge f(y) - f(x) \\ f_1(y) - f(y) &\ge f_1(x) - f(x) \\ f_2(y) &\ge f_2(x). \end{aligned}$$

This shows that f_2 is increasing on $[a,b]$ and so completes the proof. □

Corollary 2.1 is a refinement of Theorem 2.6.

Corollary 2.1. *If $f : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$ then f is the difference of two strictly increasing functions.*

Proof. We know from Theorem 2.6 that f can be written as the difference of two increasing functions. We call these functions f_1 and f_2 and write $f = f_1 - f_2$ where f_1 and f_2 are increasing.

Create two new functions, $g_1(x) = f_1(x)+x$ and $g_2(x) = f_2(x)+x$. Because both f_i and x are increasing functions, their sum is also increasing. However, since x is a strictly increasing function, the result of this addition is also a strictly increasing function. Thus we write

$$f(x) = f_1(x) - f_2(x) = (f_1(x) + x) - (f_2(x) + x) = g_1(x) - g_2(x)$$

where g_1 and g_2 are strictly increasing functions. □

2.6. **Continuity and Functions of Bounded Variation.** The following theorem gives us some interesting properties of functions of bounded variation involving continuity.

Theorem 2.7. *Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$ and define a function V on $[a,b]$ by $V(a) = 0$ and $V(x) = V(f,[a,x])$ for all $x \in (a,b]$. Then*

(1) *If $a \le x < y \le b$ then $V(y) - V(x) = V(f,[x,y])$.*
(2) *V is increasing on $[a,b]$.*
(3) *If V is continuous at $c \in [a,b]$ then f is continuous at $c \in [a,b]$.*
(4) *If f is continuous at $c \in [a,b]$ then V is continuous at $c \in [a,b]$.*
(5) *If f is continuous on $[a,b]$ then f can be written as the difference of two increasing continuous functions.*

Proof. The proof of (1) is contained in the proof of Lemma 2.3. Part (2) is Lemma 2.3. To begin the proof of (3) let $\epsilon > 0$. By the continuity of V choose δ such that if $|x - c| < \delta$ then $|V(x) - V(c)| < \epsilon$. Notice the following.

If $c < x$ then $V(x) - V(c) = V(f, [c, x])$. If $x < c$ then $V(c) - V(x) = V(f, [x, c])$. Thus

$$|f(x) - f(c)| \leq \begin{cases} V(f, [c, x]) = |V(x) - V(c)| \\ V(f, [x, c]) = |V(x) - V(c)|. \end{cases}$$

This shows that when $|x - c| < \delta$ we have $|f(x) - f(c)| \leq |V(x) - V(c)| < \epsilon$. Thus f is continuous.

To prove (4) we us (2), which tells us that V is an increasing function. Thus, both one sided limits exist at all points in $c \in [a, b]$. We have only to show that $\lim_{x \to c} V(x) = V(c)$. We will do this by showing that the right-hand limit of $V(x)$ as $x \to c$ is equal to $V(c)$. The case for the left-hand limit is similar.

Let $\epsilon > 0$. Choose $\delta > 0$ by the continuity of f at c such that $|f(x) - f(c)| < \epsilon/2$ when $0 < x - c < \delta$.

Find partition $P = \{p_0, p_1, \ldots p_n\}$ of $[c, b]$ as follows. We know, by the definition of $V(f, [c, b])$, that there exists a partition P such that

$$V(f, [c, b]) < \sum_{i=1}^{n} |f(p_i) - f(p_{i-1})| + \frac{\epsilon}{2}. \tag{1}$$

If $p_1 - c < \delta$ then we are finished. If $p_1 - c \geq \delta$ we take a point, x such that $x - c < \delta$ and add it to the partition. By Lemma 2.1 this does not influence the inequality in (1). This is now the partition P, and $x = p_1$. Notice that $V(f, [x, b]) \geq \sum_{i=2}^{n} |f(p_i) - f(p_{i-1})|$.

Consider

$$\begin{aligned} V(x) - V(c) &= V(f, [c, x]) \\ &= V(f, [c, b]) - V(f, [x, b]) \\ &< \sum_{i=1}^{n} |f(p_i) - f(p_{i-1})| + \frac{\epsilon}{2} - \sum_{i=2}^{n} |f(p_i) - f(p_{i-1})| \\ &= |f(x) - f(c)| + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

To prove (5) we begin by assuming that f is continuous on $[a, b]$. Thus, by part (4) we know that V is continuous on $[a, b]$. Since the difference of continuous functions is also continuous, we know that $V - f$ is also continuous. Thus we can write $f = V - (V - f)$ and f is the difference of two continuous, increasing functions. □

Now we turn to possible discontinuities of functions of bounded variation.

Theorem 2.8. *Let f be a function of bounded variation on $[a, b]$.*

1. *The function f has one-sided limits at each point of $[a, b]$.*
2. *The function f has at most countably many discontinuities on $[a, b]$.*

Before we can prove this theorem we need a few theorems which can be found in most real analysis books, or specifically in Gordon's text [1].

Theorem 2.9. *Let I be an interval. If a function $f : I \to \mathbb{R}$ is a monotone function on I then f has one-sided limits at each point of I.*

Theorem 2.10. *If a function $f : [a, b] \to \mathbb{R}$ is monotone, then the set of discontinuities of f in $[a, b]$ is countable.*

Now we are ready to prove Theorem 2.8.

Proof. To prove (1) we begin by assuming that f is of bounded variation on $[a, b]$. By Theorem 2.6f can be written as the difference of two increasing functions, f_1 and f_2, such that $f = f_1 - f_2$. By Theorem 2.9 we know that f_1 and f_2 both have one-sided limits at each point of $[a, b]$. Let $c \in [a, b)$ be an arbitrary point in the domain of f. Because the difference of limits is the limit of the difference we can write

$$\begin{aligned}\lim_{x^+ \to c} f(x) &= \lim_{x^+ \to c} \big(f_1(x) - f_2(x)\big) \\ &= \lim_{x^+ \to c} f_1(x) - \lim_{x^+ \to c} f_2(x)\end{aligned}$$

Because both these limits exist, we see that $\lim_{x^+ \to c} f(x)$ exists as well. A proof for left-handed limits can be found by replacing all the right-handed limits with left-handed limits and considering $c \in (a, b]$.

To prove (2) we once again rely on the fact that f can be written as the difference of two increasing functions such that $f = f_1 - f_2$ where f_1 and f_2 are monotone increasing functions. Thus by Theorem 2.10 we know that f_1 and f_2 each have countably many discontinuities. Let the set $D_1 = \{x | f_1 \text{ is discontinuous at } x\}$ and the set $D_2 = \{x | f_2 \text{ is discontinuous at } x\}$. By Theorem 2, D_1 and D_2 are countable. Then let $D = D_1 \cup D_2$. Now, f can not be discontinuous at a point where neither f_1 nor f_2 was discontinuous. Thus we can conclude that the number of discontinuities of f is at most the number of points in D. Since the union of two countable sets is countable we see that f has a countable number of discontinuities on $[a, b]$. □

3. Absolute Continuity

In this section we discuss absolute continuity and its relationship to bounded variation. We begin by defining uniform continuity and absolute continuity, and show that absolute continuity implies uniform continuity.

3.1. **Introduction to Absolute Continuity.**

Definition 3.1. *Let I be an interval. A function $f : I \to \mathbb{R}$ is* uniformly continuous *on I if for each $\epsilon > 0$ there exists $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ for all $x, y \in I$ that satisfy $|y - x| < \delta$.*

We need the following definition in order to define absolute continuity. Two intervals are *non-overlapping* if their intersection contains at most one point.

Definition 3.2. *Let I be an interval. A function $f : I \to \mathbb{R}$ is* absolutely continuous *on I if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} |f(d_i) - f(c_i)| < \epsilon$ whenever $\{[c_i, d_i] : 1 \le i \le n\}$ is a finite collection of non-overlapping intervals in I such that $\sum_{i=1}^{n} (d_i - c_i) < \delta$.*

Theorem 3.1. *An absolutely continuous function is uniformly continuous.*

Proof. This result is a trivial consequence of the definition of absolute continuity. We choose $n = 1$ in Definition 3.2 and the desired result follows immediately. □

We now consider a specific example of an absolutely continuous function.

Example 3.1. *The function $g(x) = \sqrt{x}$ is absolutely continuous on* $[0, 1]$.

We begin by letting $\epsilon > 0$ and taking $\{[c_i, d_i] : 1 \leq i \leq n\}$ to be a non-overlapping collection of intervals in $[0, 1]$ such that $\sum_{i=1}^{n}(d_i - c_i) < \epsilon^2$. Choose $a = \epsilon^2/4$. Now we break the sum $\sum_{i=1}^{n} |g(d_i) - g(c_i)|$ into two parts, those intervals that are in $[0, a]$ and those that are in $[a, 1]$. If a happens to fall in the middle of an interval we break the interval at a. By Lemma 2.1 this will only make the sum $\sum_{i=1}^{n} |g(d_i) - g(c_i)|$ larger if it has any effect. We will say that $a = d_m$.

Now we consider the sum over the intervals that are in $[0, a]$,

$$\begin{aligned}\sum_{i=1}^{m} |g(d_i) - g(c_i)| &= \sum_{i=1}^{m} |\sqrt{d_i} - \sqrt{c_i}| \\ &\leq \sqrt{a} \\ &= \epsilon/2.\end{aligned}$$

This follows from the fact that $\sqrt{x}$ is an increasing function.

Now we consider the sum over the intervals that are in $[a, 1]$

$$\begin{aligned}\sum_{i=m+1}^{n} |g(d_i) - g(c_i)| &= \sum_{i=m+1}^{n} |\sqrt{d_i} - \sqrt{c_i}| \\ &= \sum_{i=m+1}^{n} |\sqrt{d_i} - \sqrt{c_i}| \cdot \frac{|\sqrt{d_i} + \sqrt{c_i}|}{|\sqrt{d_i} + \sqrt{c_i}|} \\ &= \sum_{i=m+1}^{n} \frac{d_i - c_i}{\sqrt{d_i} + \sqrt{c_i}} \\ &\leq \sum_{i=m+1}^{n} \frac{d_i - c_i}{2\sqrt{a}} \\ &= \frac{1}{2\sqrt{a}} \cdot \sum_{i=1}^{n}(d_i - c_i) \\ &< \frac{1}{\epsilon} \cdot \epsilon^2 \\ &= \epsilon.\end{aligned}$$

Combining these two sums we see that

$$\begin{aligned}\sum_{i=1}^{n} |g(d_i) - g(c_i)| &\leq \sum_{i=1}^{m} |g(d_i) - g(c_i)| + \sum_{i=m}^{n} |g(d_i) - g(c_i)| \\ &< \epsilon/2 + \epsilon \\ &< 2\epsilon.\end{aligned}$$

Thus $g(x) = \sqrt{x}$ is absolutely continuous on $[0, 1]$.

Now we consider some general examples of absolutely continuous functions, such as Lipschitz functions. We also consider some of the algebraic properties of absolute continuity.

Definition 3.3. *Let $f : I \to \mathbb{R}$ be a function with I an interval, and let $k \in \mathbb{R}$ such that $k > 0$. Then f satisfies a* Lipschitz condition *with constant k if $|f(b)-f(a)| \leq k|b-a|$ for all $a, b \in I$. The function f is called a* Lipschitz function.

Theorem 3.2. *If $f : I \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant $k > 0$ then f is absolutely continuous on I.*

Proof. Let $\epsilon > 0$ and choose $\delta = \epsilon/k$. Let $\{[c_i, d_i] : 1 \leq i \leq n\}$ be a finite set of non-overlapping intervals in I such that $\sum_{i=1}^{n}(d_i - c_i) < \delta$. Using the Lipschitz condition we obtain

$$\sum_{i=1}^{n} |f(d_i) - f(c_i)| \leq \sum_{i=1}^{n} k(d_i - c_i) < \frac{\epsilon}{k} < \epsilon$$

Thus f is absolutely continuous on I. □

Notice that a linear function of the form $f(x) = ax + b$ is Lipschitz with $k = |a|$ on all of $\mathbb{R}$ and so linear functions are absolutely continuous.

Theorem 3.3. *If $f : I \to \mathbb{R}$ is absolutely continuous then so is $|f|$.*

Proof. Notice that

$$\sum_{i=1}^{n} \big||f(d_i)| - |f(c_i)|\big| \leq \sum_{i=1}^{n} |f(d_i) - f(c_i)|$$

and because f is absolutely continuous on I we can make $\sum_{i=1}^{n} |f(d_i) - f(c_i)|$ arbitrarily small. Thus $|f|$ is absolutely continuous on I. □

The following two theorems tell us that the sum and product of two absolutely continuous functions are also absolutely continuous.

Theorem 3.4. *If f and g are absolutely continuous on the interval I , then $f + g$ is absolutely continuous on I.*

Proof. Let $\epsilon > 0$. Choose $\delta_f > 0$ and $\delta_g > 0$ according to the definition of absolute continuity such that $\sum_{i=1}^{n} |f(d_i) - f(c_i)| < \epsilon/2$ and $\sum_{i=1}^{n} |g(b_i) - g(a_i)| < \epsilon/2$. Define $\delta = \min\{\delta_f, \delta_g\}$. Let $\{[x_i, y_i] : 1 \leq i \leq n\}$ to be a finite collection of non-overlapping intervals in I such that $\sum_{i=1}^{n}(y_i - x_i) < \delta$. Then, with repeated use of the triangle inequality, we can deduce that

$$\sum_{i=1}^{n} |(f+g)(y_i) - (f+g)(x_i)| \ \leq \ \sum_{i=1}^{n} |f(y_i) - f(x_i)| + \sum_{i=1}^{n} |g(y_i) - g(x_i)| \ < \ \epsilon$$

and so $f + g$ is absolutely continuous on I. □

Theorem 3.5. *If f and g are absolutely continuous on $[a, b]$, then fg is absolutely continuous on $[a, b]$.*

Proof. Notice that because f and g are absolutely continuous on $[a, b]$ both are continuous on $[a, b]$ and so achieve a maximum value on $[a, b]$. Choose M_f and M_g such that $M_f \geq |f(x)|$ and $M_g \geq |g(x)|$ for all $x \in [a, b]$. Let $\epsilon > 0$. Choose $\delta_f > 0$ and $\delta_g > 0$ according to the definition of absolute continuity such that

$\sum_{i=1}^{n} |f(d_i) - f(c_i)| < \epsilon/2M_g$ and $\sum_{i=1}^{n} |g(d_i) - g(c_i)| < \epsilon/2M_f$. Define $\delta = \min\{\delta_f, \delta_g\}$ and let $\{[c_i, d_i] : 1 \leq i \leq n\}$ be a finite collection of non-overlapping intervals in $[a, b]$ such that $\sum_{i=1}^{n}(d_i - c_i) < \delta$. Consider the following:

$$\begin{aligned}
\sum_{i=1}^{n} \left|f(d_i)g(d_i) - f(c_i)g(c_i)\right| &= \sum_{i=1}^{n} \left|g(d_i)[f(d_i) - f(c_i)] + f(c_i)[g(d_i) - g(c_i)]\right| \\
&\leq \sum_{i=1}^{n} \left|g(d_i)\right|\left|f(d_i) - f(c_i)\right| + \sum_{i=1}^{n} \left|f(c_i)\right|\left|g(d_i) - g(c_i)\right| \\
&\leq M_g \sum_{i=1}^{n} \left|f(d_i) - f(c_i)\right| + M_f \sum_{i=1}^{n} \left|g(d_i) - g(c_i)\right| \\
&< M_g \cdot \frac{\epsilon}{2M_g} + M_f \cdot \frac{\epsilon}{2M_f} \\
&= \epsilon.
\end{aligned}$$

Thus fg is absolutely continuous on $[a, b]$. □

3.2. **Connecting Absolute Continuity to Bounded Variation.**

Theorem 3.6. *If a function f is absolutely continuous on the interval $[a, b]$ then f is of bounded variation on $[a, b]$.*

Proof. Suppose that $f : [a, b] \to \mathbb{R}$ is absolutely continuous. Use this fact to find a $\delta > 0$ such that $\sum_{i=1}^{n} |f(d_i) - f(c_i)| < 1$ when $\sum_{i=1}^{n}(d_i - c_i) < \delta$ and $\{[c_i, d_i] : 1 \leq i \leq n\}$ is a finite set of non-overlapping intervals in $[a, b]$. Round up $(b - a)/\delta$ to the nearest integer value and call it k.

Now construct a partition of $[a, b]$ as follows. $\{x_i = a + i(b - a)/k : 0 \leq i \leq k\}$. Now, each subinterval of this partition has length $(b - a)/k \leq \delta$. Thus $V(f, [x_i, x_{i-1}]) \leq 1$ by the absolute continuity condition. There are at most k of these subintervals and so by Theorem 2.5 we know that $V(f, [a, b]) \leq k$ and so f is of bounded variation on $[a, b]$. □

We next use Theorem 3.6 to provide an example of a function that is uniformly continuous but not absolutely continuous. This example is important as it shows that there is indeed a difference between the two kinds of continuity.

Before the next example we need to recall a theorem from analysis.

Theorem 3.7. *Let X be a compact set and f a continuous function on X. Then f is uniformly continuous on X.*

Example 3.2. *The function f defined by $f(x) = \sqrt[3]{x}\sin(\pi/x)$ when $x \neq 0$ and $f(0) = 0$ is uniformly continuous but not absolutely continuous on $[0, 1]$.*

In Example 2.4 we showed that this function is not of bounded variation on $[0, 1]$, and thus by Theorem 3.6 we know that it is not absolutely continuous. However, this function is uniformly continuous. Because f is continuous on $[0, 1]$, a compact set, it follows from Theorem 3.7 that f is uniformly continuous on $[0, 1]$.

Theorem 3.8. *If $f : I \to \mathbb{R}$ is an absolutely continuous function then f can be written as the difference of two increasing, continuous functions.*

Proof. Because f is absolutely continuous on I we know that f is continuous and, by Theorem 3.6, that f is of bounded variation. Thus, by Theorem 2.6, we know that f can be written as the difference of two increasing continuous functions. □

3.3. **Connecting Absolute Continuity and Derivatives.** When the derivative of a continuous function f is bounded, we can conclude that f is absolutely continuous. This provides a tool for showing that a function is absolutely continuous, and thus of bounded variation.

Theorem 3.9. *If f is continuous on $[a, b]$ and f' exists and is bounded on (a, b), then f is absolutely continuous on $[a, b]$.*

Proof. Suppose that $|f'(x)| < M$ for all $x \in (a, b)$. Let $\epsilon > 0$ and consider $\sum_{i=1}^{n} |f(d_i) - f(c_i)|$ where $\{[c_i, d_i] : 1 \leq i \leq n\}$ is a finite collection of non-overlapping intervals in $[a, b]$ such that $\sum_{i=1}^{n} |d_i - c_i| < \epsilon/M$. Then we observe that

$$\sum_{i=1}^{n} |f(d_i) - f(c_i)| = \sum_{i=1}^{n} \frac{|f(d_i) - f(c_i)|}{|d_i - c_i|} |d_i - c_i|.$$

The Mean Value Theorem tells us that for every i there exists a value $x_i \in [c_i, d_i]$ such that

$$\frac{|f(d_i) - f(c_i)|}{|d_i - c_i|} = f'(x_i) < M.$$

Thus we can write

$$\begin{aligned} \sum_{i=1}^{n} \frac{|f(d_i) - f(c_i)|}{|d_i - c_i|} |d_i - c_i| &< \sum_{i=1}^{n} M |d_i - c_i| \\ &= M \sum_{i=1}^{n} |d_i - c_i| \\ &< M \frac{\epsilon}{M} \\ &= \epsilon. \end{aligned}$$

Thus f is absolutely continuous on $[a, b]$. □

Example 3.3. *A continuous function with an unbounded derivative may be absolutely continuous.*

Consider $f(x) = \sqrt{x}$ on $[0, 1]$. In Example 3.1 we saw that this function is absolutely continuous. However, $f'(x) = \frac{1}{2\sqrt{x}}$ which is not bounded on $(0, 1)$.

Example 3.4. *The function $f(x) = x^2|sin(1/x)|$ for $x \neq 0$ and $f(0) = 0$ is absolutely continuous on $[0, 1]$.*

Consider first the function $g(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $g(0) = 0$. Notice that $|g(x)| = f(x)$. By Theorem 3.3 we need only show that g is absolutely continuous on $[0, 1]$.

Now, when $x \neq 0$, $g'(x) = 2x \sin(1/x) - \cos(1/x)$. Notice that $\sin(1/x) \leq 1$ and $-\cos(1/x) \leq 1$ and $x \leq 1$, so

$$|g'(x)| = |2x \sin(1/x) - \cos(1/x)| \leq 2|x||\sin(1/x)| + |cos(1/x)| \leq 3.$$

Thus $g'(x)$ is bounded on $[0, 1]$ by three and by Theorem 3.9 we know that $g(x)$ is absolutely continuous on $[0, 1]$. Thus $|g(x)| = f(x)$ is absolutely continuous on $[0, 1]$.

We can use Example 3.4 to demonstrate that absolute continuity does not hold up under composition.

Example 3.5. *The composition of two absolutely continuous functions need not be absolutely continuous.*

Consider $h = f \circ g$ where $f(x) = \sqrt{x}$ and $g(x) = x^2|\sin(1/x)|$ on $[0,1]$. We know that f and g are absolutely continuous on $[0,1]$ from Examples 3.1 and 3.4, respectively. Now, $h = x\sqrt{|\sin(1/x)|}$ and h is increasing on intervals of the form $[2/(2n+1)\pi, 2/(2n)\pi]$. The intervals over which h is increasing are non-overlapping, so by Definition 2.4

$$V(h,[0,1]) \geq \sum_{i=1}^{n} V(h,[2/(2i+1)\pi, 2/(2i)\pi])$$

for all n. The variation on each of these subintervals is known, however. Because h is increasing from 0 to x we can use Theorem 2.2 to find that $V(h,[0,1]) \geq \sum_{i=1}^{n} 2/2i\pi$. That is

$$\begin{aligned} V(h,[0,1]) &\geq \sum_{i=1}^{n} \frac{2}{2i\pi} \\ &= \sum_{i=1}^{n} \frac{1}{\pi}\frac{1}{i} \end{aligned}$$

which is a harmonic series and so diverges as $n \to \infty$. Thus $V(h,[0,1])$ is unbounded so h is not of bounded variation and thus can not be absolutely continuous on $[0,1]$ by Theorem 3.6.

4. Cantor Ternary Function

In this section we explore the Cantor ternary function. This function provides an interesting example of a function that is uniformly continuous on a closed interval and of bounded variation on the closed interval but is not absolutely continuous. The closed, bounded interval that we work on is $[0,1]$.

First we discuss ternary representations of the numbers in $[0,1]$. For all real numbers $x \in [0,1]$ there is a sequence of integers $t_k \in \{0,1,2\}$ such that

$$x = \frac{t_{x1}}{3} + \frac{t_{x2}}{3^2} + \frac{t_{x3}}{3^3} + \frac{t_{x4}}{3^4} + \cdots.$$

That is to say, x has a ternary expansion. We assume this in our discussion of the Cantor ternary function. The ternary expansion of a number can be written in decimal form as $x = t_{x1}t_{x2}t_{x3}\cdots$. Further, the two ternary expansions

$$\frac{t_1}{3} + \frac{t_2}{3^2} + \frac{t_3}{3^3} + \cdots + \frac{t_n}{3^k} + 0 + 0 + 0 + \cdots$$

and

$$\frac{t_1}{3} + \frac{t_2}{3^2} + \frac{t_3}{3^3} + \cdots + \frac{t_n - 1}{3^k} + \frac{2}{3^{k+1}} + \frac{2}{3^{k+2}} + \frac{2}{3^{k+3}} + \cdots$$

are equal. This is the only way for two distinct ternary expansions to represent the same number. These two representations are similar to representing the number one in base 10 as either 1 or $0.9\overline{9}$.

Definition 4.1. *The* Cantor ternary function *is a function* $f : [0, 1] \to \mathbb{R}$ *such that if the digit* 1 *does not appear in the ternary expansion of* x *then*

$$f(x) = \sum_{k=1}^{\infty} \frac{t_{xk}/2}{2^k}.$$

If the digit 1 *does appear in the ternary expansion of* x*, let* $j_x = \min\{k : t_{xk} = 1\}$ *and then*

$$f(x) = \sum_{k=1}^{j_x-1} \frac{t_{xk}/2}{2^k} + \frac{1}{2^{j_x}}.$$

Observation 4.1. *The Cantor ternary function is well defined.*

Proof. We show that f is well defined by considering two possible representations of a number x and showing that f gives the same value for both. In the ternary expansion of a number, the only way for two numbers to be the same is for them to be of the following forms.

$$x = \frac{t_1}{3} + \frac{t_2}{3^2} + \frac{t_3}{3^3} + \cdots + \frac{t_n}{3^k} + 0 + 0 + 0 + \cdots \tag{2}$$

$$x = \frac{t_1}{3} + \frac{t_2}{3^2} + \frac{t_3}{3^3} + \cdots + \frac{t_n - 1}{3^k} + \frac{2}{3^{k+1}} + \frac{2}{3^{k+2}} + \frac{2}{3^{k+3}} + \cdots \tag{3}$$

Now suppose we have a number x represented in both these ways. If one of the t_k such that $k < n$ is one, then for both representations we have

$$f(x) = \sum_{k=1}^{j_x-1} \frac{t_{xk}/2}{2^k} + \frac{1}{2^{j_x}}.$$

Since in this case j_x is the same for both representations the value of the function at either representation is the same.

Now, suppose that there are no ones before t_n and $t_n = 2$. Then f of the representation in line (2) is

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} \frac{t_{xk}/2}{2^k} \\ &= \sum_{k=1}^{n-1} \frac{t_{xk}/2}{2^k} + \frac{1}{2^n}. \end{aligned}$$

In line (3) $t_n = 1$ and so we have $j_x = n$ giving us

$$f(x) = \sum_{k=1}^{n-1} \frac{t_{xk}/2}{2^k} + \frac{1}{2^n}.$$

Thus the two representations yield the same result.

Finally, suppose that there is no one before t_n and $t_n = 1$. The representation in line (2) gives us

$$f(x) = \sum_{k=1}^{n-1} \frac{t_k/2}{2^k} + \frac{1}{2^n}.$$

The representation in line (3) gives us

$$\begin{aligned} f(x) &= \sum_{k=1}^{n-1} \frac{t_k/2}{2^k} + 0 + \sum_{k=n+1}^{\infty} \frac{1}{2^k} \\ &= \sum_{k=1}^{n-1} \frac{t_k/2}{2^k} + \frac{1}{2^n} \end{aligned}$$

where the second step follows from simplifying the geometric series. Thus we see that the two representations give the same result in any case, and so the Cantor ternary function is well defined. □

Observation 4.2. *The Cantor ternary function, f, has values such that $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$.*

Proof. First notice that all of the addends in either sum are either positive or zero, and thus $f(x) \geq 0$.

If $f(x) = \sum_{k=1}^{\infty} \frac{t_{xk}/2}{2^k}$ then we know

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} \frac{t_{xk}/2}{2^k} \\ &\leq \sum_{k=1}^{\infty} \frac{1}{2^k} \\ &= 1. \end{aligned}$$

Similarly, if $f(x) = \sum_{k=1}^{j_x-1} \frac{t_{xk}/2}{2^k} + \frac{1}{2^{j_x}}$ we know

$$\begin{aligned} f(x) &= \sum_{k=1}^{j_x-1} \frac{t_{xk}/2}{2^k} + \frac{1}{2^{j_x}} \\ &\leq \sum_{k=1}^{j_x-1} \frac{1}{2^k} + \frac{1}{2^{j_x}} \\ &= 1 - \frac{1}{2^{j_x-1}} + \frac{1}{2^{j_x}} \\ &\leq 1. \end{aligned}$$

Thus $f(x)$ is always smaller than 1 and greater than 0 on $[0, 1]$. □

Observation 4.3. *The Cantor ternary function, f, is increasing on $[0, 1]$.*

Proof. If $y > x$ then we know that their ternary representation is the same to some point, and at the digit where they differ the digit in y is larger than the digit in x. Writing x and y in the decimal form of their ternary expansion with $y > x$ we have the following:

$$\begin{aligned} y &= 0.t_1 t_2 \cdots t_n \cdots \\ x &= 0.s_1 s_2 \cdots s_n \cdots \end{aligned}$$

where $t_n > s_n$ and $t_i = s_i$ for $1 \leq i < n$.

If one of the $t_i = 1 = s_i$ for $1 \leq i < n$, then $f(y) = f(x) = \sum_{k=1}^{i-1} \frac{t_{xk}/2}{2^k} + \frac{1}{2^i}$, and so $f(y) \geq f(x)$.

Suppose that there are no ones in the ternary representation of y or x up to n, and suppose further that $t_n = 2$. Then we have

$$f(y) = \sum_{k=1}^{n-1} \frac{t_{xk}/2}{2^k} + \frac{1}{2^n} + K$$

and

$$f(x) = \sum_{k=1}^{n-1} \frac{t_k/2}{2^k} + \frac{1}{2^n}$$

or

$$f(x) = \sum_{k=1}^{n-1} \frac{t_k/2}{2^k} + 0 + C$$

where C and K are the sum of the remaining terms that are not 1. Notice that

$$C \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}$$

and thus $f(y) \geq f(x)$ in this case.

The last remaining case is when $t_n = 1$. We find a similar scenario:

$$\begin{aligned} f(y) &= \sum_{k=1}^{n-1} \frac{t_k/2}{2^k} + \frac{1}{2^n} \\ f(x) &= \sum_{k=1}^{n-1} \frac{t_k/2}{2^k} + 0 + C. \end{aligned}$$

Once again

$$C \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}$$

and $f(y) \geq f(x)$ and so f is increasing on all of $[0, 1]$. □

Example 4.1. *Determine the intervals on which f is constant.*

From Observation 4.3 we know that f is increasing, so we need only find two separate points where f gives the same value to know that it has the same value at every intermediate point. Consider intervals of the form $[0.000 \cdots 0100, 0.000 \cdots 0200]$ where these are the ternary expansions in decimal form. In other words, we are looking at intervals of the form $[1/3^n, 2/3^n]$. If the non-zero digit is in the nth place, f evaluates to $1/2^n$ at both these endpoints. Inspection shows that any value greater than the right endpoint or smaller than the left gives a different value of f. Thus these are the largest intervals where f is constant at the value of $1/2^n$. We can generalize this observation by noticing that inserting a two at any place value gives a similar effect. So intervals of the form $[2/3^k + 1/3^n, 2/3^k + 2/3^n]$ also give constant values of f, and no larger interval will do. Overall, then, f is constant on intervals of this form,

$$\left[\sum_{k \ni t_{xk}=2} \frac{2}{3^k} + \frac{1}{3^n}, \sum_{k \ni t_{xk}=2} \frac{2}{3^k} + \frac{2}{3^n}\right].$$

On a number line this would be the middle third of the interval $[0, 1]$, the middle third of each of the remaining intervals, the middle third of each the remaining thirds after that, and so on.

FIGURE 2. A number line showing some of the intervals for which the Cantor ternary function is constant.

Example 4.2. *Find the sum of the lengths of the intervals where f is constant.*

From Example 4.1 we know the form of these intervals. The length of one interval of this form is

$$\sum_{k \ni t_{kx}=2} \frac{2}{3^k} + \frac{2}{3^n} - \sum_{k \ni t_{kx}=2} \frac{2}{3^k} + \frac{1}{3^n} = \frac{2}{3^n} - \frac{1}{3^n} = \frac{1}{3^n}.$$

Each interval of this length will appear 2^n times in the interval $[0, 1]$, so we multiply by 2^n and sum as n gets large:

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n = 1.$$

Observation 4.4. *The Cantor ternary function is continuous.*

Proof. We know already that the Cantor ternary function is increasing on $[0, 1]$. Thus, since the Cantor ternary function is monotone increasing, the only possible discontinuities are jump discontinuities. If we show that the Cantor ternary function maps from $[0,1]$ onto an interval, then we know that there are no jump discontinuities and that the function must be continuous.

Suppose $y \in [0,1)$. We would like to show that there exists an x such that $f(x) = y$. We know that y must have a binary expansion. We create the ternary expansion of a number by using the digits from the binary expansion of y multiplied by two. Notice that this number, x, has only zeros and twos in the ternary expansion, and also that $x \in [0,1]$. Now consider $f(x)$:

$$f(x) \quad = \quad \sum_{k=1}^{\infty} \frac{t_{xk}/2}{2^k}.$$

This is exactly the number y that we started with. Thus we see that $f([0,1])$ hits every point in $[0,1)$. Fortunately, we also know that $f(1) = 1$, and so we can say that $f([0,1])$ in fact hits everything in $[0,1]$ and thus has no jump discontinuities. Therefor the Cantor ternary function is continuous. □

Theorem 4.1. *The Cantor ternary function is not absolutely continuous.*

Proof. Let $\epsilon = 1/2$. Consider the following sets of intervals.

$$\begin{aligned}
&\Big\{[0,1/3], \quad [2/3,1]\Big\} \\
&\Big\{[0,1/9], \quad [2/9,1/3], \quad [2/3,7/9], \quad [8/9,1]\Big\} \\
&\Big\{[0,1/27], \quad [2/27,1/9], \quad [2/9,7/27], \quad [8/27,1/3], \\
&\qquad [2/3,19/27], \quad [20,27,7/9], \quad [8/9,25/27], \quad [26/27,1]\Big\}
\end{aligned}$$

To obtain one set from the next, we remove the middle third of the previous intervals. Label these intervals $\{[c_i,d_i] : 1 \le i \le n\}$. Then $f(d_i) = f(c_{i+1})$, that is to say the value of f at the right end point of an interval is the same as the value of f at the left end point of the following interval. Because $f(0) = 0$ and $f(1) = 1$ we see that

$$\begin{aligned}
\sum_{i=1}^{n} |f(d_i) - f(c_i)| &= [f(d_1) - f(0)] + [f(d_2) - f(c_2)] + \cdots + [f(1) - f(c_n)] \\
&= -f(0) + [f(d_1) - f(c_2)] + \cdots + [f(d_{n-1}) - f(c_n)] + f(1) \\
&= 0 + 0 + 0 + \cdots + 0 + 1 \\
&= 1
\end{aligned}$$

for any set of intervals chosen this way. Now, if we show that a set of intervals of this type may be made as small as we like, we will have shown that f is not absolutely continuous.

The length of each interval in the set is $1/3^n$ for some n. Also, when the length is $1/3^n$ then there are precisely 2^n such intervals in the set. Thus the length of the sum of these intervals is given by $(2/3)^n$. However, we also know that $\lim_{n\to\infty}(2/3)^n = 0$ and thus, for large enough n we can make this as small as we like. That is to say, for any δ we can find a set of intervals whose lengths sum to less than δ but where the sum of the variance of f over those intervals is always 1. This shows that f is not absolutely continuous. □

5. Arc Length

In this section we explore a connection between bounded variation and arc length and show the equivalence of two definitions of arc length. We begin with a definition of arc length.

Definition 5.1. *Let f be a continuous function defined on $[a, b]$. The* arc length *of the curve $y = f(x)$ on the interval $[a, b]$ is defined by $L = \sup\{S\}$ where*

$$S = \left\{ \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} : \{x_i : 1 \leq i \leq n\} \textit{partitions } [a, b] \right\}$$

If S is unbounded, then f is said to have infinite length on the given interval.

We now state and prove the theorem connecting arc length and the variation of a function.

Theorem 5.1. *The length of a curve is finite if and only if f is of bounded variation on $[a, b]$.*

Proof. Suppose that the arc length is finite, of length L, and $\{x_i : 0 \leq i \leq n\}$ is a partition of $[a, b]$. Then we write

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \;\leq\; \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \;\leq\; L.$$

Thus the variation of f is bounded and f is of bounded variation on$[a, b]$.

Now, suppose that f is of bounded variation. Recall that $\sqrt{x^2 + y^2} \leq |x| + |y|$. Consider that

$$\begin{aligned}
\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} &\leq \sum_{i=1}^{n} |x_i - x_{i-1}| + \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\
&= b - a + \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\
&\leq b - a + V(f, [a, b]). \qquad (4)
\end{aligned}$$

Since the variation of f is finite, line (4) is finite, and so the arc length is finite. □

Now we show that when f has a continuous derivative on $[a, b]$ then Definition 5.1 is equivalent to $L = \int_a^b \sqrt{1 + (f'(x))^2} dx$, another definition of are length. We take this as a series of lemmas.

Lemma 5.1. *The inequality $\sqrt{(d - c)^2 + (f(d) - f(c))^2} \;\leq\; \int_c^d \sqrt{1 + (f'(x))^2} dx$ holds when $[c, d]$ is a subinterval of $[a, b]$ and f is a function with a continuous derivative on $[a, b]$.*

Proof. Let $[c, d]$ be any subinterval of $[a, b]$. Define the following.

$$r = \sqrt{(d-c)^2 + (f(d) - f(c))^2}, \qquad \alpha = \frac{d-c}{r}, \text{ and } \qquad \beta = \frac{f(d) - f(c)}{r}.$$

Consider the following:

$$\begin{aligned}
\int_c^d \alpha + \beta f'(x)dx &= \alpha x + \beta f(x)\Big|_c^d \\
&= \alpha(d-c) + \beta(f(d) - f(c)) \\
&= \frac{d-c}{r} \cdot (d-c) + \frac{f(d) - f(c)}{r} \cdot (f(d) - f(c)) \\
&= \frac{r^2}{r} \\
&= \sqrt{(d-c)^2 + (f(d) - f(c))^2}.
\end{aligned}$$

Also,

$$\begin{aligned}
\int_c^d \alpha + \beta f'(x)dx &\leq \left| \int_c^d \alpha + \beta f'(x)dx \right| \\
&\leq \int_c^d |\alpha + \beta f'(x)|dx.
\end{aligned}$$

Recall that $|x| = \sqrt{x^2}$. Thus we write $\int_c^d |\alpha + \beta f'(x)|dx = \int_c^d \sqrt{(\alpha + \beta f'(x))^2}dx$ In two dimensions, the Cauchy-Schwartz inequality says that

$$(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2).$$

Taking $a = \alpha$, $b = \beta$, $c = 1$, and $d = f'(x)$

$$\int_c^d \sqrt{(\alpha + \beta f'(x))^2}dx \leq \int_c^d \sqrt{\alpha^2 + \beta^2}\sqrt{1 + (f'(x))^2}dx.$$

Finally, notice that $\sqrt{\alpha^2 + \beta^2} = 1$ when we substitute our expressions for α and β. Stringing all these observations together we have the following inequalities:

$$\begin{aligned}
\sqrt{(d-c)^2 + (f(d) - f(c))^2} &= \int_c^d \alpha + \beta f'(x)dx \\
&\leq \int_c^d |\alpha + \beta f'(x)|dx \\
&\leq \int_c^d \sqrt{\alpha^2 + \beta^2}\sqrt{1 + (f'(x))^2}dx \\
&= \int_c^d \sqrt{1 + (f'(x))^2}dx.
\end{aligned}$$

□

Lemma 5.2. *The inequality*

$$\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \leq \int_a^b \sqrt{1 + (f'(x))^2}dx$$

holds where $\{x_i : 0 \leq i \leq n\}$ *is a partition of* $[a, b]$ *and* f *is a function with a continuous derivative on* $[a, b]$.

Proof. Let $P = \{x_i : 1 \leq i \leq n\}$ be a partition of $[a, b]$. Consider the following. By Lemma 5.1 we know that

$$\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \leq \int_{x_i}^{x_{i-1}} \sqrt{1 + (f'(x))^2} dx$$

and summing over i we have

$$\begin{aligned} \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} &\leq \sum_{i=1}^{n} \int_{x_i}^{x_{i-1}} \sqrt{1 + (f'(x))^2} dx \\ &= \int_a^b \sqrt{1 + (f'(x))^2} dx. \end{aligned}$$

□

We use the Mean Value Theorem to define a tagged partition associated with P. We start with a definition.

Definition 5.2. *A* tagged partition, ${}^tP$, *of an interval* $[a, b]$ *is a partition of* $[a, b]$, $P = \{x_i : 0 \leq i \leq n\}$, *and a set of points,* $\{t_i : 1 \leq i \leq n\}$, *such that* $x_{i-1} \leq t_i \leq x_i$ *for* $1 \leq i \leq n$. *We denote this by* ${}^tP = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ *and say that* ${}^tP$ *is a tagged partition associated with* P.

So a tagged partition is a regular partition where we have picked out a particular point within each subinterval. Consider the following.

Lemma 5.3. *For any partition* P *of* $[a, b]$ *there exists a tagged partition* ${}^tP$ *associated with* P *such that*

$$\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} = \sum_{i=1}^{n} (x_i - x_{i-1}) \sqrt{1 + (f'(t_i))^2}.$$

Proof. Let $\{x_i : 0 \leq i \leq n\}$ be a partition P of $[a, b]$. By the Mean Value Theorem we know that there exists a t_i within each subinterval $[x_{i-1}, x_i]$ such that

$$f'(t_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}.$$

Thus we write

$$\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} = \sum_{i=1}^{n} (x_i - x_{i-1}) \sqrt{1 + (f'(t_i))^2}.$$

We take the t_i to form a tagged partition, ${}^tP$ associated with the partition P.

□

Now we can put everything together. By Lemma 5.2 we know that

$$\sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \leq \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

Thus, $L = \sup \left\{ \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \right\} \leq \int_a^b \sqrt{1 + (f'(x))^2} dx$. We need only show that we have equality.

We know that f is Riemann integrable, so there exists $\delta > 0$ such that for all tagged partitions ${}^tP$ of $[a,b]$ with $||{}^tP|| < \delta$

$$\int_a^b \sqrt{1+(f'(x))^2}dx - \epsilon \leq \sum_{i=1}^{n}(x_i - x_{i-1})\sqrt{1+(f'(t_i))^2}.$$

However, we also know by Lemma 5.3 that for one of those tagged partitions

$$\sum_{i=1}^{n}(x_i - x_{i-1})\sqrt{1+(f'(t_i))^2} = \sum_{i=1}^{n}\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}.$$

Thus

$$\int_a^b \sqrt{1+(f'(x))^2}dx - \epsilon \leq \sum_{i=1}^{n}\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

and, as shown above,

$$\int_a^b \sqrt{1+(f'(x))^2}dx \geq \sup\left\{\sum_{i=1}^{n}\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}\right\}.$$

Thus $\int_a^b \sqrt{1+(f'(x))^2}dx$ is indeed the supremum, since nothing smaller will work.

Thus we have shown that

$$L = \sup\left\{\sum_{i=1}^{n}\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}\right\} = \int_a^b \sqrt{1+(f'(x))^2}dx.$$

6. Riemann-Stieltjes Integration

In this section we define Riemann-Stieltjes integration and consider some examples of Riemann-Stieltjes integration. Finally, we present some theorems regarding Riemann-Stieltjes integration and use these theorems to discover a relationship between functions which are of bounded variation and functions which are Riemann-Stieltjes integrable.

Definition 6.1. *Let α be a monotonically increasing function on $[a,b]$. For each partition $P = \{x_i : 0 \leq i \leq n\}$ of $[a,b]$ write $\Delta_i\alpha = \alpha(x_i) - \alpha(x_{i-1})$. For any real function f that is bounded on $[a,b]$ and any partition P we define the* upper and lower sums*, respectively, as*

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i\Delta_i\alpha \quad L(P,f,\alpha) = \textstyle\sum_{i=1}^{n} m_i\Delta_i\alpha$$

where M_i and m_i are the supremum and infimum of f on the interval $[x_{i-1}, x_i]$. Finally we define the upper and lower integrals*, respectively, as*

$$\begin{aligned}\overline{\int_a^b} f d\alpha &= \inf\{U(P,f,\alpha) : P \textit{ is a partition of } [a,b]\}\\ \underline{\int_a^b} f d\alpha &= \sup\{L(P,f,\alpha) : P \textit{ is a partition of } [a,b]\}.\end{aligned}$$

If these two are equal their common value is the Riemann-Stieltjes integral *of f over α, denoted $\int_a^b f d\alpha$. When $\int_a^b f d\alpha$ exists we say that f is Riemann-Stieltjes integrable with respect to α and write $f \in \Re(\alpha)$.*

The Riemann-Stieltjes integral with $\alpha = x$ is the regular Riemann integral. Consider $\alpha = x$. We know that α is monotonically increasing on all intervals $[a, b]$. Also, $\Delta_i \alpha = (x_i - x_{i-1}) = \Delta x$. Then the upper and lower sums reduce to the Riemann sums. If both these sums converge to some common value, we call this value $\int_a^b f dx$ which is the Riemann integral of f.

We now proceed with a series of theorems that lead to a relationship between bounded variation and Riemann-Stieltjes integration. Proofs of Theorems 6.1 and 6.2 can be found in Rudin's text [2]. These proofs are similar to those of similar results involving Riemann integration.

Definition 6.2. *We say the partition P^* is a* refinement *of the partition P if $P \subset P^*$. If P_1 and P_2 are partitions then their* common refinement *is $P = P_1 \cup P_2$.*

Theorem 6.1. *If P^* is a refinement of P then $L(P, f, \alpha) \leq L(P^*, f, \alpha)$ and $U(P, f, \alpha) \geq U(P^*, f, \alpha)$.*

Theorem 6.2. *The upper and lower integrals of f with respect to α over $[a, b]$ are related by $\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha$.*

Theorem 6.3. *A function f is in $\Re(\alpha)$ on $[a, b]$ if and only if for each $\epsilon > 0$ there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.*

Proof. Suppose that for each $\epsilon > 0$ there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$. Let $\epsilon > 0$. By the definition of the upper and lower integrals and Theorem 6.2 there exists a partition P such that

$$L(P, f, \alpha) \leq \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \leq U(P, f, \alpha).$$

Thus

$$\overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

Since $\epsilon > 0$ was arbitrary, we conclude that $\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha$. Thus, by Definition 6.1, $f \in \Re(\alpha)$.

Now suppose that $f \in \Re(\alpha)$ and let $\epsilon > 0$. Because $\int f d\alpha$ is defined as the common value of the supremum and infimum of $U(P, f, \alpha)$ and $L(P, f, \alpha)$, respectively, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$U(P_2, f, \alpha) - \int f d\alpha < \epsilon/2 \quad \text{and} \quad \int f d\alpha - L(P_1, f, \alpha) < \epsilon/2.$$

Let P be the common refinement of P_1 and P_2. By Theorem 6.1 we know that

$$U(P, f, \alpha) \leq U(P_2, f, \alpha) \quad \text{and} \quad L(P_1, f, \alpha) \leq L(P, f, \alpha).$$

Thus

$$U(P,f,\alpha) \le U(P_2,f,\alpha) < \int f d\alpha + \epsilon/2 < L(P_1,f,\alpha) + \epsilon \le L(P,f,\alpha) + \epsilon$$

and

$$U(P,f,\alpha) - L(P,f,\alpha) \le \epsilon.$$

□

Theorem 6.4. *If f is monotonic on $[a,b]$ and α is continuous (and monotonic) on $[a,b]$ then $f \in \Re(\alpha)$.*

Proof. Suppose that f is monotone increasing. The case where f is decreasing is similar. The function α is continuous on $[a,b]$, and because $[a,b]$ is a closed interval α is also uniformly continuous on $[a,b]$. Thus, we choose $\delta > 0$ such that

$$|\alpha(x) - \alpha(y)| < \frac{\epsilon}{f(b) - f(a)} \quad \text{when} \quad |x - y| < \delta.$$

Then we take $n \in \mathbb{N}$ such that $(b-a)/n < \delta$. We now create the partition

$$P = \{x_i : x_i = a + i\frac{b-a}{n},\ 0 \le i \le n\}.$$

Notice that because f is monotone increasing $M_i = f(x_1)$ and $m_i = f(x_{i-1})$. Consider

$$\begin{aligned} U(P,f,\alpha) - L(P,f,\alpha) &= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\Delta\alpha \\ &\le \frac{\epsilon}{f(b) - f(a)} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\ &= \epsilon. \end{aligned}$$

Thus, by Theorem, 6.3 $f \in \Re$.

□

Theorem 6.5. *Suppose that f is bounded on $[a,b]$ and has only finitely many points of discontinuity. Suppose further that α is continuous everywhere that f is discontinuous. Then $f \in \Re(\alpha)$.*

Proof. Let $\epsilon > 0$. Because f is bounded on $[a,b]$, we can set

$$M = \sup\{|f(x)| : x \in [a,b]\}.$$

Let $E = \{e_i : f \text{ is discontinuous at } e_i,\ 0 \le i \le m\}$. Because α is continuous at each point, e_i, in E, we know that there exists an interval $[u_i, v_i]$ such that $e_i \in (u_i, v_i)$ and $\alpha(u_i) - \alpha(v_i) < \epsilon/4mM$. Furthermore, these intervals can be made to be disjoint. The set of intervals $L = \{[u_i, v_i] : 1 \le i \le n\}$ covers E. The set $K = [a,b] \setminus L$ is compact, and f is continuous on K. Because K is compact f is in fact uniformly continuous on K. Thus there exists a δ such that when $s, t \in K$ and $|s - t| < \delta$ then $|f(s) - f(t)| < \epsilon/2(\alpha(b) - \alpha(a))$.

Make a partition $P = \{x_i : 0 \le i \le n\}$ as follows. Each u_i and v_i is in P. No point in the segments (u_i, v_i) is in P. Unless x_i is one of the v_i, $\Delta x_i < \delta$. That is to say, each subinterval formed by the partition points has length less than δ, unless it is an interval of the form $[u_i, v_i]$.

Let M_i and m_i be as in Definition 6.1. Notice that $M_i - m_i < 2M$. Let $\pi_1 = \{i : x_i = v_i \text{ for some } i\}$ and $\pi_2 = \{i : x_i \neq v_i \text{ for some } i\}$. Notice that if $x_i \in \pi_2$ then $M_i - m_i < \frac{\epsilon}{2(\alpha(b)-\alpha(a))}$ by the uniform continuity condition on f.

Consider the following:

$$\begin{aligned} U(P,f,\alpha) - L(P,f,\alpha) &= \sum_{i=1}^{n}(M_i - m_i)\Delta_i\alpha \\ &= \sum_{i\in\pi_1}(M_i - m_i)\Delta_i\alpha + \sum_{i\in\pi_2}(M_i - m_i)\Delta_i\alpha. \end{aligned} \tag{5}$$

Consider the first sum in Equation (5).We know that each of the $\Delta_i\alpha = [u_i, v_i] < \frac{\epsilon}{4mM}$. We also know that $M_i - m_i < 2M$. There are exactly m terms in this sum, because there are exactly m of the intervals $[u_i, v_i]$. Thus

$$\sum_{i\in\pi_1}(M_i - m_i)\Delta_i\alpha < 2Mm\frac{\epsilon}{4mM} = \frac{\epsilon}{2}.$$

Next we attack the second sum in Equation (5). Here we know $M_i - m_i < \frac{\epsilon}{2(\alpha(b)-\alpha(a))}$ by the uniform continuity of f. Thus

$$\sum_{i\in\pi_2}(M_i - m_i)\Delta_i\alpha < \frac{\epsilon}{2(\alpha(b)-\alpha(a))}(\alpha(b)-\alpha(a)) = \frac{\epsilon}{2}.$$

Thus each sum in Equation (5) is smaller than $\epsilon/2$. We conclude that

$$U(P,f,\alpha) - L(P,f,\alpha) \quad < \quad \epsilon$$

and so, by Theorem 6.3, $f \in \Re(\alpha)$. □

Lemma 6.1. *If $f = f_1 + f_2$ and P is any partition of $[a, b]$ then*

$$L(P,f_1,\alpha) + L(P,f_2,\alpha) \leq L(P,f,\alpha) \tag{6}$$
$$\leq U(P,f,\alpha) \leq U(P,f_1,\alpha) + U(P,f_2,\alpha) \tag{7}$$

Proof. We already know that $L(P,f,\alpha) \leq U(P,f,\alpha)$. Thus we have only to prove the inequalities in (6) and (7). We begin with the inequality in line (6).

Recall that if $A = \{a_1, a_2 \cdots a_n : a_i \in \mathbb{R}\}$, $B = \{b_1, b_2, \cdots b_n : b_n \in \mathbb{R}\}$, and $A + B = \{a_1 + b_1, a_2 + b_2, \cdots a_n + b_n\}$ then

$$\inf\{A\} + \inf\{B\} \leq \inf\{A + B\}. \tag{8}$$

Let $P = \{x_0, x_1, \ldots x_n\}$ be a partition of $[a, b]$. Let

$$m_i^1 = \inf\{f_1(x) : x \in [x_i, x_{i-1}]\},$$
$$m_i^2 = \inf\{f_2(x) : x \in [x_i, x_{i-1}]\},$$

and

$$m_i = \inf\{f(x) : x \in [x_i, x_{i-1}]\}.$$

Thus, by the inequality in line (8), $m_i^1 + m_i^2 \le m_i$. Consider

$$\begin{aligned} L(P, f_1, \alpha) + L(P, f_2, \alpha) &= \sum_{i=1}^n m_i^1 \Delta_i \alpha + \sum_{i=1}^n m_i^2 \Delta \alpha \\ &= \sum_{i=1}^n (m_i^1 + m_i^2) \Delta_i \alpha \\ &\le \sum_{i=1}^n (m_i) \Delta_i \alpha \\ &= L(P, f, \alpha). \end{aligned}$$

The proof for the inequality in line (7) is similar. □

Theorem 6.6. *If $f \in \Re(\alpha)$ on $[a, b]$ and c is a constant then $cf \in \Re(\alpha)$.*

Proof. Because $f \in \Re(\alpha)$ there exists a partition $P = \{x_0, x_1, \cdots x_n\}$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon/|c|$. Let

$$M_i = \sup\{f(x) : x \in [x_i - x_{i-1}]\},$$

$$m_i = \inf\{f(x) : x \in [x_i - x_{i-1}]\},$$

$$M_i^c = \sup\{cf(x) : x \in [x_i - x_{i-1}]\},$$

and

$$m_i^c = \inf\{cf(x) : x \in [x_i - x_{i-1}]\}.$$

If $c > 0$ then $M_i^c = cM_i$ and $m_i^c = cm_i$. If $c < 0$ then $M_i^c = cm_i$ and $m_i^c = cM_i$. In either case, the following is true.

$$\begin{aligned} U(P, cf, \alpha) - L(P, cf, \alpha) &= \sum_{i=1}^n M_i^c \Delta_i \alpha - \sum_{i=1}^n m_i^c \Delta_i \alpha \\ &= |c| \left(\sum_{i=1}^n M_i \Delta_i \alpha - \sum_{i=1}^n m_i \Delta_i \alpha \right) \\ &< |c| \frac{\epsilon}{|c|} \\ &= \epsilon. \end{aligned}$$

This proves that $cf \in \Re(\alpha)$.

□

It can further be shown that $\int_a^b cf d\alpha = c \int_a^b f d\alpha$.

Theorem 6.7. *If f_1 and f_2 are in $\Re(\alpha)$ on $[a, b]$ then $f_1 + f_2 \in \Re(\alpha)$.*

Proof. Let $\epsilon > 0$. Then we know that there exist partitions P_1 and P_2 such that $U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \epsilon$ and $U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \epsilon$. Let P be the common refinement of P_1 and P_2. Then by Theorem 6.1

$$U(P, f_1, \alpha) - L(P, f_1, \alpha) < \epsilon \tag{9}$$

$$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \epsilon. \tag{10}$$

By adding the inequalities in (9) and (10) and applying Lemma 6.1 we have $U(P, f, \alpha) - L(P, f, \alpha) < 2\epsilon$. By Theorem 6.3 $f \in \Re(\alpha)$.

□

It can further be shown that $\int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$.

Now we are ready to relate bounded variation and Riemann-Stieltjes integration. Suppose that f is of bounded variation on an interval $[a, b]$. Then f can be written as the difference of two increasing functions, so $f = f_1 - f_2$ where f_1 and f_2 are increasing. By Theorem 6.4 we know that f_1 and f_2 are in $\Re(\alpha)$ whenever α is continuous. Further, by Theorem 6.7 we know that $f \in \Re(\alpha)$ when α is continuous.

Because f_1 and f_2 are increasing on a closed, bounded interval, we know the functions are themselves bounded on $[a, b]$. Thus, if they have finitely many discontinuities on $[a, b]$ and α is continuous at these discontinuities, we can conclude that f_1 and f_2 are in $\Re(\alpha)$. Further, by Theorem 6.7 we know that f is also in $\Re(\alpha)$ when these conditions are met.

7. Conclusion

In this paper we examined functions of bounded variation and provided proofs for some important properties of these functions. Perhaps the most important and interesting property is the fact that a function of bounded variation can be written as the difference of two increasing functions. We then considered three related topics. The first was absolute continuity. We showed that a function that is absolutely continuous is also of bounded variation. To provide an interesting example of a function that is continuous and of bounded variation but not absolutely continuous we explored some properties of the Cantor Ternary Function. The second related topic we considered was arc length. Here we showed that the length of a curve is finite on an interval if and only if the function is of bounded variation on that interval. We then considered Riemann-Stieltjes integration. Here we used the fact that a function of bounded variation can be written as the difference of two increasing functions to find conditions under which functions of bounded variation are Riemann-Stieltjes integrable. For instance, if f is of bounded variation then f is Riemann-Stieltjes integrable over α whenever α is continuous. Similarly, when f is of bounded variation then it is Riemann-Stieltjes integrable over α whenever f has a finite number of discontinuities and α is continuous where f is not.

We have only briefly considered each of these related topics. Any one of them could be explored in more depth by the interested reader. Most especially, there are other connections between Riemann-Stieltjes integration and functions of bounded variation that were not covered in this paper. For instance, if f and α are bounded on $[a, b]$, f is continuous on $[a, b]$, and α is of bounded variation on $[a, b]$, then $f \in \Re(\alpha)$.

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