ON METRIC TOPOLOGIES AND THE REFINABLE CHAIN CONDITION

by

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Certificate of Approval

This is to certify that the accompanying thesis by Dan Crytser has been accepted in partial fulfillment of the requirements for graduation with Honors in Mathematics.

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1. Introduction

A fundamental result in finite Abelian group theory is the so-called structure theorem, which uniquely presents a finite Abelian group as the direct product of cyclic groups of prime-power order. Analogous to this is a result in the theory of vector spaces which reduces the question of vector isomorphism to the question of basis cardinality. These are in fact just two of a collection of theorems which serve to analyze and pick apart algebraic structures by certain size factors, or cardinal invariants. In this paper we consider questions of this type along with a much subtler class of problem which arises from the introduction of topological properties into Abelian groups and vector spaces.

A group is a set equipped with a binary associative operation which also has an identity element and an inverse function. We typically express these conditions symbolically as $(a+b)+c = a+(b+c)$, $a + 0_G = 0_G + a = a$, and $a + (-a) = 0_G$, where a, b, c are arbitrary group elements and 0_G denotes the group identity. An Abelian group is commutative in addition to its associativity, so that the identity $a + b = b + a$ holds identically. A group G is a p-group if for every $g \in G$ there exists a nonnegative integer k such that $p^k g = 0_G$, where $p^k g = g + g + \ldots + g$ the p^k-fold sum . The socle of an Abelian p-group G is the set of all $g \in G$ such that $p^1 g = 0_G$ (this is necessarily a subgroup of G). By using the finite field $\mathbb{Z}_p$ we can define an intuitive vector space structure on the socle K. After all this, we can define the p-height of a socle element g to be the largest integer k such that $g \in p^k G$, where the height is denoted by ∞ if no such integer exists. The p-height function is an excellent example of a vector space valuation. These valuations make up the bulk of the paper. We examine their general definition and their elementary properties before using them to introduce metric topological structure on the socle.

The main problem which we analyze concerns nested sequences of closed subspaces of a valuated vector space. Specifically, the property of *freeness* relays the existence of a vector space basis which is in some sense optimal with regard to the valuation structure. Thus we have to consider both the vector space structure over the field and the valuation structure.

2. Vector Spaces

In this section we outline the basic linear algebra ideas we will use throughout the paper. Let F be a field. An Abelian group $(V, +)$ is called a *vector space over* F if there exists a function $\cdot : F \times V \to V$ which satisfies the following formulas for all α,$\beta \in F$ and all v,$w \in V$:

$$1v = v$$

$$\alpha(v + w) = \alpha v + \alpha w$$

$$(\alpha +_F \beta)(v) = \alpha v + \beta v$$

$$\alpha(\beta v) = (\alpha\beta)v.$$

Be careful in noting the distinction between the addition in the field and the addition in the group V, which should be clear from context. For the rest of this paper, we shall assume that V is a vector space over some arbitrary field F.

3. Subspaces

Given a nonempty set of vectors $H \subseteq V$, we say that H is a *subspace* of V whenever the restrictions of the addition and scalar functions of V to $H \times H$ and $F \times H$ respectively satisfy the vector axioms above.

This is equivalent to the following statement: given any v,$w \in H$ and $\alpha \in F$, we have $\alpha v \in H$ and $v + w \in H$.For a proof, see [1] refer to this form as the 'subspace test.' We sometimes denote the statement 'H is a subspace of V' with the symbols $H \leq V$.

4. Quotient Spaces

Let V be a vector space and let $U \subseteq V$ be a subspace of V. Given any $v \in V$, we define the *left coset of* U *containing* v to be the set of all elements w of V such that $v - w \in U$ (hereafter we simply call this set a *coset*). We denote this by $v + U$. It is straightforward to see that $v + U = \{v + u : u \in U\}$ for any $v \in V$. This creates a partition of V (two cosets of U are either equal or disjoint). For if $v'' \in v + U \cap v' + U$, then $v'' - v$ and $v'' - v'$ are both elements of U. Now suppose that $w \in v + U$. This is equivalent to $w - v$ belonging to U. Then $w - v' = (w - v) + (v - v'') + (v'' - v') \in U$, as U is closed under addition. But then $w \in v' + U$, so that $v + U \subseteq v' + U$. By switching v and v' in this argument, we have the inclusion $v' + U \subseteq v + U$. Thus the two cosets are equal whenever they intersect, and we have that the cosets form a partition of V.

Given a subspace U, we can define vector space operations on the collection of cosets of U. Specifically, we define for any $v, v' \in V$ and any scalar α the following $(v + U) + (v' + U) = (v + v') + U$ and $\alpha(v + U) = (\alpha v) + U$. We can show that the set $V/U = \{v + U : v \in V\}$ is a vector space over F with these operations.

First, we need to show that the operations are well-defined. If $v + U = v' + U$ and $w + U = w' + U$, we need to show that $(v + U) + (w + U) = (v' + U) + (w' + U)$. We know that $v - v' \in U$ and $w - w' \in U$. Subspaces are closed under addition, so $(v - v') + (w - w') \in U$. Then rearranging we have $(v + w) - (v' + w') \in U$. Thus $(v + w) + U = (v' + w') + U$, so that addition is well-defined. Similarly, $\alpha(v - v') \in U$, as U is closed under scaling. Applying right distributivity, we have $\alpha v - \alpha v' \in U$. Then $(\alpha v) + U = (\alpha v') + U$. Thus scaling is well-defined.

Now that we know that coset addition is well-defined, we construct an additive identity element. Specifically, the coset $0 + U$ will serve as additive identity. It is simple to check that for any $v \in V$, we have $(v + U) + (0 + U) = (0 + U) + (v + U) = v + U$.

None of the rest of the vector space axioms require any reference to the subspace U, instead relying exclusively on the vector properties of V. That is, associativity of coset addition only requires that the addition function be well-defined, after which the property follows immediately from associativity of vector addition. Thus, we omit the proofs of the other properties and state that $V/U = \{v + U : v \in V\}$ is a vector space over F under the previously defined operations.

5. Linear Transformations

Let V and W be vector spaces over the same field. A function $\phi : V \to W$ is called a *linear transformation* if for all $v, v' \in V$ and all $\alpha \in F$ we have $\phi(v +_V v') = \phi(v) +_W \phi(v')$ and $\phi(\alpha \cdot_V v) = \alpha \cdot_W \phi(v)$.

Note that we have written the scaling operation slightly differently so that the difference between the two vector spaces is apparent. Essentially, a linear transformation maps V onto a subspace of W in a highly symmetric fashion. This is encompassed by the following theorem.

Theorem 1. *Let $\phi : V \to W$ be a linear transformation between F-vector spaces. Then the kernel of ϕ, $ker(\phi) = \{v \in V : \phi(v) = 0_W\}$ is a subspace of V, and the image of ϕ, $im(\phi) = \{\phi(v) : v \in V\}$ is a subspace of W. For any $w \in \phi(V)$, the set $\phi^{-1}(w)$ is in bijective correspondence with $ker(\phi)$ and the quotient operations $v + ker(\phi) + w + ker(\phi) = v + w + ker(\phi)$ and $\alpha(vker(\phi)) = (\alpha v)ker(\phi)$ define a vector space on the space of cosets of $ker(\phi)$, a vector space which is naturally isomorphic to $\phi(V)$.*

Proof. We prove the first part using the subspace test. Let $v, v' \in ker(\phi)$, so that $\phi(v) = 0_W = \phi(v')$. Then $\phi(v + v') = \phi(v) + \phi(v')$ as ϕ is a linear transformation. But this sum is just 0_W as 0_W is the additive identity for W.

Now suppose that $\alpha \in F$. Then $\phi(\alpha v) = \alpha\phi(v)$. But $\phi(v) = 0_W = 0_W + 0_W = \phi(v) + \phi(v)$, so that we can write $\alpha\phi(v) = \alpha(\phi(v) + \phi(v) = \alpha\phi(v) + \alpha\phi(v)$ by right distribution. Then cancellation gives $\alpha\phi(v) = 0_W$. Thus $\phi(\alpha v) = 0_W$ and $\alpha v \in ker(\phi)$.

Then the subspace test tells us that $ker(\phi)$ is a subspace of V.

We prove that $im(\phi)$ is a subspace using the same method. Note that $0_V \in V$ and $\phi(0_V) = 0_W$ together yield $0_W \in im(\phi)$, so that the set is nonempty. If $w_1 = \phi(v_1)$ and $w_2 = \phi(v_2)$, then $w_1 + w_2 = \phi(v_1 + v_2) \in im(\phi)$. If $\alpha \in F$, then $\alpha w_1 = \alpha\phi(v_1) = \phi(\alpha v_1) \in im(\phi)$. Thus $im(\phi)$ is closed under addition and scaling and $im(\phi)$ is a subspace.

If $w = \phi(v)$ for some $v \in V$, then we define a function Ψ from $\phi^{-1}(w) \subseteq V$ to $ker(\phi)$ by the rule $\Psi(v') = v' - v$. This is a mapping from $\phi^{-1}(w)$ to $ker(\phi)$ for if $\phi(v) = w = \phi(v')$, then $\phi(v' - v) = \phi(v) - \phi(v') = 0_V$. We claim that this a bijection. For if $\Psi(v') = \Psi(v'')$ for two vectors $v', v'' \in \phi^{-1}(w)$, then $v' - v = v'' - v$. Cancellation gives $v' = v''$. If $u \in ker(\phi)$, let $v' = u + v$. Then $\Psi(v') = u + v - v = u$, so that Ψ is surjective.

$\square$

Given two vector spaces V and W, we sometimes want to build a new vector space which contains each of the two as subspaces (or more precisely, which contains *isomorphic copies* of each of V and W, a concept we will define later). One method to do this is by taking pairs (v, w) where $v \in V$ and $w \in W$ and defining the operations componentwise. That is to say, $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ and $\alpha(v, w) = (\alpha v, \alpha w)$. Proving that all the vector space axioms hold in this structure is tedious so we omit. Denote this new structure by $V \oplus W$, the direct sum of V and W. Note that the subspaces $V' = \{(v, 0_W) : v \in V\}$ and $W' = \{(0_V, w) : w \in W\}$ 'look' the same as V and W, respectively (the pairs of spaces are *isomorphic*, which we shall define later).

We can extend this approach from the case with two vector spaces to the case involving an arbitrary family of vector spaces $\{V_i\}_{i\in I}$, where I is an indexing set. We define $\prod_{i\in I} V_i$, the *direct product* of the V_i, as the set of all functions $f : I \to \cup_{i\in I} V_i$ satisfying the inclusion $f(i) \in V_i$ for all $i \in I$.

The direct product has a natural vector space structure, with vector addition and scaling done much the same as in the direct sum. If we have $f, g \in \prod_{i\in I} V_i$, then define $f + g$ pointwise, so that $(f + g)(i) = f(i) +_i g(i)$, where we use the subscript to emphasize that the addition is taking place within the vector space V_i. Similarly define $(\alpha f)(i) = \alpha \cdot_i f(i)$. The vector space axioms are satisfied.

The direct product is very simple to define but it lacks a few useful properties. These properties are satisfied by one of its subspaces. Let $\bigoplus_{i\in I} V_i$ denote the subset of $\prod_{i\in I} V_i$ which contains only those functions which have *finite support*, i.e. $f \in \bigoplus_{i\in I} V_i$ if $f(i) = 0_i$ for all but finitely many $i \in I$. Then this is a subspace which we name the *direct sum* of the V_i.

Note that if I is finite (only finitely many V_i), then the direct sum and the direct product are the same, as there cannot be infinitely many $i \in I$ with $f(i) \neq 0_i$ for some $f \in \prod_i V_i$. Thus if there are only two spaces V_1 and V_2, the two definitions of 'direct sum' agree with one another.

6. Bases

Given a nonempty set S of vectors in V, a *linear combination* in the vectors S is an expression of the form $\alpha_1 v_1 + \ldots + \alpha_k v_k$, where for all i we have $\alpha_i \in F$ and $v_i \in S$.

Now we can precisely formulate the *span* of S. Let $\mathbf{F}$ denote the set of all subspaces of V which contain S as a subset, that is $\mathbf{F} := \{H \leq V : S \subseteq H\}$. Now let $span(S) = \cap_{H\in\mathbf{F}} H$. We claim that this is a subspace of V.

We apply the subspace test listed previously. Suppose that $v, w \in span(S)$ and $\alpha \in F$. Given any $H \leq V$ with $S \subseteq H$ we have that $span(S) \subseteq H$, so that v,$w \in H$. Since H is a subspace by hypothesis, we have $v+w \in H$. Similarly, $\alpha v \in H$. Since H was selected merely as an arbitrary subspace of V containing S, we have $v + w$,$\alpha v \in span(S)$. This completes the proof.

Now we have a characterization of the span in terms of linear combinations: given nonempty $S \subseteq V$, the subspace $span(S)$ is the set of all linear combinations of elements of S. To prove this, let L denote the set of all linear combinations of elements of S. Since $span(S)$ is a subspace of V, it must contain all linear combinations of elements of S. Thus $L \subseteq span(S)$. On the other hand, L contains S and it is straightforward to check that

L is a subspace of V. Thus L contains all linear combinations of elements of S, so that $span(S) \subseteq L$. Thus $L = span(S)$.

It is worth noting that calculating the span of a subspace is trivial, i.e. if $S \leq V$, then $span(S) = S$. Thus $span(span(S)) = span(S)$ for all nonempty subsets $S \subseteq V$. Calculating the span defines a function Φ mapping the family of nonempty subsets of V into itself which is a prime example of a *closure operator*, which is merely a function on the subsets of a space which satisfies the second equation $\Phi(\Phi(S)) = \Phi(S)$.

A *spanning set* for V is a set S of vectors in V satisfying $span(S) = V$. Thus if S is a spanning set for V, *any* vector in V can be written as a linear combination of vectors in S.

Let $S \subseteq V$ be a set of vectors in V. If for any finite subset $\{v_1, \ldots, v_k\}$ the equation $\sum_{i=1}^{k} \alpha_i v_i = 0_V$ implies $\alpha_i = 0_F$ for all i, then we say that S is *linearly independent*.

Now we are able to state the definition of a basis for V. A *basis* for V is a linearly independent spanning set for V.

7. Constructing Bases

Bases are useful when dealing with vector spaces, but we don't as yet know whether or not they always exist. Could there exist a vector space none of whose subsets were bases? No, as the next theorem shows.

Theorem 2. *Let V be a vector space. Suppose that S is a spanning set for V and J is a linearly independent subset of V. Then there exists a subset $S' \subseteq S$ such that $S' \cap J = \emptyset$ and $S' \cup J$ is a basis for V.*

In order to prove this theorem, we must have recourse to something called Zorn's lemma. Recall that a *relation* on a set X is a set R of ordered pairs $(a, b) \in X^2$. A *partial order relation* has the following properties: $(a, a) \in R$ for all $a \in X$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$, and if (a, b) and (b, c) are in R, then so is (a, c). Typically we write $a \leq_R b$ to denote $(a, b) \in R$, writing $a \leq b$ when the context is clear. A *chain* in a partially ordered set X is a subset $C \subseteq X$ such that if $x, y \in C$, then $x \leq y$ or $y \leq x$. An *upper bound* for a set $B \subseteq X$ is an element $y \in X$ such that $b \leq y$ for all $b \in B$. An element $y \in X$ is *maximal* if $y \leq z$ implies $z = y$.

Theorem 3. *(Zorn's lemma) Let X be a partially ordered set such that every chain in X has an upper bound in X. Then given any element $x \in X$, there is a maximal element $y \in X$ such that $x \leq y$.*

We don't prove Zorn's lemma but instead refer to proofs in [2] and [3]. It should be noted, though, that Zorn's lemma is equivalent to the axiom of choice, and by using it we accept some of the liabilities of that axiom. We also equip ourselves with a minor technical lemma about chains.

Lemma 1. *Let X be a partially ordered set and let C be a chain in X. If C is finite, then C contains a greatest element. That is, there exists $c \in C$ such that $c' \leq c$ for all $c' \in C$.*

Proof. We prove this by induction on the size of the chain. If C has just one element, say $C = \{c\}$, then c is trivially a greatest element for C. Now assume that the lemma holds for all chains with size less than or equal to k, where $k \geq 1$. If C is a chain in X with $|C| = k+1$, then we can write $C = \{c\} \cup D$, where c is an arbitrary element of C and $c \notin D$. Then D is also a chain, as it is nonempty, and so it must have a greatest element d, by the inductive hypothesis. If $c < d$, then $c' \leq d$ for all $c' \in C$, and we are finished. Otherwise, $d < c$, and by transitivity of partial orders we can conclude that $c' \leq c$ for all $c' \in C$. Then c' is the greatest element of C.

$\square$

We now prove the basis theorem.

Proof. Remember that in the hypothesis of the theorem, J is a linearly independent subset of V and S is a spanning set for V. If the set J is a spanning set for V, we are done. If J is not a spanning set, there must be some $s \in S$ such that $\{s\} \cup J$ is linearly independent. Otherwise, all the elements of S are included in the span of J and J is a spanning set for

V. Thus we must have a (nonempty) collection T of subsets of S such that for all $E \in T$ have the properties that $E \cup J$ is linearly independent and $E \cap J = \emptyset$. This set T is a partially ordered set under normal subset inclusion. Does T fit the hypotheses of Zorn's lemma? Suppose that C is a chain in T. Can we find an upper bound for C in T? Let $P = \cup_{E \in C} E$ be the union over all elements in the chain. We claim that P is an element of T. For if we have a linear combination $0_V = \sigma_{k=1}^{n} \alpha_k v_k$ for some $v_1, \ldots, v_n \in P$, then there must be elements $E_1, \ldots, E_n$ such that $v_k \in E_k$. But one of these E_k must include all the others, by the technical lemma. Call this set E_i. This set E_i contains all the sets $E_1, \ldots, E_n$, so that all the elements v_k are contained in E_i. But E_i is linearly independent (because the superset $E_i \cup T$ is linearly independent), so that we must have that $\alpha_k = 0$ for all k. To prove that $P \cap J = \emptyset$, we only point out that if $w \in P$, there must be an element of C which contains w. But all of the sets in C are disjoint from J. Thus, every chain in T has an upper bound contained in T, and we can apply Zorn's lemma. Specifically, there is an element of T containing s which is maximal under the inclusion relation. Label this element A. Remember that this element of T is a subset of S with $A \cup J$ linearly independent. We claim that $A \cap J = \emptyset$ and $A \cup J$ is a basis. The first claim follows from the fact that A is an element of T, and all elements of T are disjoint from J. From $A \in T$ we can also conclude that $A \cup J$ is linearly independent. All that remains to be shown is that $A \cup J$ is a basis for V.

Suppose that some $v \in V$ is not included in the span of $A \cup J$. We claim that $A \cup v$ is an element of T which properly includes A, contrary to the maximality of A under subset inclusion. We know that $(A \cup v) \cap J = (A \cap J) \cup (A \cap v) = A \cap v$ by the distributive laws of set theory. Thus if $(A \cup v) \cap J \neq \emptyset$, we have $v \in A$, contradicting our hypothesis that $v \notin span(A \cup J)$. If $(A \cup v) \cup J$ is linearly dependent, we must have some linear dependence relation in the elements of $A \cup J$ and the element v (because $A \cup J$ is linearly indepedent. But we can solve this dependence relation for v to show that $v \in span(A \cup J)$. Thus $(A \cup v) \cup J$ is linearly independent. But this is a contradiction, so $A \cup J$ is a spanning set for V. So $A \cup J$ is a basis for V.

□

Theorem 4. *Every vector space has a basis. Any linearly independent subset J of vector space V can be extended to a basis. That is, there exists a basis B for V containing J.*

Proof. The set $\emptyset$ is vacuously linearly independent, and V is a spanning set for V. So let $J = \emptyset$ and $V = S$ in the above theorem.

□

8. Direct Sums and Direct Products

Recall the definition of a *direct product*: we are given a set I such that V_i is a vector space over F for all $i \in I$. The direct product $\prod_{i \in I} V_i$ is the set of all functions $f : I \to \cup_{i \in I} V_i$ such that $f(i) \in V_i$ for all $i \in I$. We compute the vector operation (addition and scaling) pointwise: $(f + g)(i) = f(i) +_i g(i)$ and $(\alpha \cdot f)(i) = \alpha \cdot_i f(i)$. We do not include the tedious verification that this is a vector space (that the addition is commutative and associative, the scaling distributes over addition, etc.). Recall also the *direct sum* of the V_i, which is the subspace of $\prod_{i \in I} V_i$ consisting of all functions of finite support.

For each $j \in I$, let $\tau_j : V_j \to \oplus_{i \in I} V_i$ be defined by $\tau(x) = (v_i)_{i \in I}$, where $v_i = x$ if $i = j$ and $v_i = 0_i$ otherwise. The restriction of the codomain to $\oplus_{i \in I} V_i$ is not vital–we can think of each τ_j as a map $\tau_j : V_j \to \prod_{i \in I} V_i$. We sometimes refer to these as the set of *inclusion mappings*. We state without proof that these are linear transformations. In a dual fashion, for each $j \in I$ we have a function $\pi_j : \prod_{i \in I} V_i \to V_j$ by $\pi_j((v_i)_{i \in I}) = v_j$. These are the *projection mappings* (they are also linear transformations). Note that for any $j \in I$, the composition $\pi_j \circ \tau_j = 1_j$ the identity transformation on V_j.

The utility of the inclusion and projection mappings becomes apparent in the light of the following theorem.

Theorem 5. *Suppose that V_j is a vector space for every $i \in I$. Let W be a vector space and for each $j \in I$ let $\phi_j : V_j \to W$ be a linear transformation. Also, let $\psi_j : W \to V_j$*

be a linear transformation for each $j \in I$. Then there exists unique $\phi : \oplus_{i \in I} V_i \to W$ and $\psi : W \to \prod_{i \in I} V_i$ such that $\phi \circ \tau_j = \phi_j$ and $\pi_j \circ \psi = \psi_j$ for all $j \in I$.

Proof. We define $\phi(x)$ by 'adding up the images' of all the nonzero coordinates of x. More precisely let $x = \sum_{i=1}^{k} x_i$ where each x_i is nonzero and all the other coordinates of x are zero. Then let $\phi(x) = \sum_{i=1}^{k} \phi_i(x_i)$. Then $\phi \circ \tau_j = \phi_j$ because the function τ_j zeros out the coordinates besides the j-th coordinate. If two such functions exist, say ϕ and ϕ', then we have $\phi \circ \tau_j = \phi' \circ \tau_j$. But then

$$\phi(\sum_{i=1}^{k} v_i) = \sum_{i=1}^{k} (\phi \circ \tau_j)(v_i) = \sum_{i=1}^{k} (\phi' \circ \tau_j)(v_i) = \phi'(\sum_{i=1}^{k} v_i).$$

So $\phi = \phi'$.

It is easier to prove the analogous facts for ψ. We define $\psi(w) \in \prod_{i \in I} V_i$ by $(\psi(w))_j = \psi_j(w)$. Then $(\pi_j \circ \psi)(w) = (\psi(w))_j = \psi_j(w)$, so that ψ satisfies the equation $\pi_j \circ \psi = \psi_j$ for all $j \in I$. If there existed another solution, say $\pi' : W \to \prod_{i \in I} V_i$, then the two would be forced to agree coordinate by coordinate and so would coincide.

□

9. Internal Direct Sums

Suppose that we are given a vector space V and a collection of subspaces of V, say $\{U_i\}_{i \in I}$. Then for each $i \in I$, we have the inclusion mapping $\iota_i : U_i \to V$ which maps vectors to themselves. These mappings are quickly seen to be linear transformations. This fits the hypotheses of the direct sum theorem of the previous section. Thus, we have the adding transformation $\phi : \oplus_{i \in I} U_i \to V$ as previously defined. We use the adding transformation to define two important ideas. We say that the collection spaces $\{U_i\}$ is *linearly independent* whenever the function ϕ is injective. We say that the spaces $\{U_i\}$ *span* V if the function ϕ is surjective (this is equivalent to saying that their union is a spanning set for V). If the spaces are linearly independent and they span V, we say that V is the internal direct sum of the spaces $\{U_i\}$ and write $V = \oplus_{i \in I} U_i$. We now prove a simple criterion for determining when a vector space is the direct sum of two subspaces.

Theorem 6. *Suppose that V is a vector space and U and W are two subspaces of V. Then V is the internal direct sum of U and W if and only if $V = U + W = \{u + w : u \in U, w \in W\}$ and $U \cap W = \{0_V\}$.*

Proof. Suppose that $V = U \oplus W$ and suppose $x \in U \cap W$. Then $\phi(0, x) = \phi(x, 0) = x$ by definition of ϕ. But ϕ must be injective, so that $x = 0_V$. If $v \in V$, then there exists $(u, w) \in U \oplus W$ (the vector direct sum) such that $\phi(u, w) = v$, as ϕ is surjective. But the definition of ϕ then gives $u + w = v$.

Now suppose that $U + W = V$ and $U \cap W = \{0_V\}$. If $\phi(u, w) = \phi(u', w')$, then $u + w = u' + w'$. On rearrangement, we have $u - u' = w' - w \in U \cap W = \{0_V\}$. Thus $u - u' = w' - w = 0_V$, and $(u, w) = (u', w')$. So ϕ is injective. If $v \in V$, there must exist $u, w \in V$ such that $u + w = v$, so that $\phi(u, w) = v$. Thus ϕ is surjective. So $V = U \oplus W$.

□

We now prove a useful structural theorem about general vector spaces. Specifically, this theorem allows us to divide up a vector space along a given subspace.

Theorem 7. *Let V be a vector space and let $U \subseteq V$ be a subspace of V. Then there exists a subspace W of V such that V is the internal direct sum of W and U, i.e. $V = U \oplus W$.*

Proof. By the basis theorem, we can pick a basis B for U. This is a linearly independent subset of V by definition, so we can extend this to a basis B' for V. Let $W = span(B - B')$. Then $B' - B$ is a basis for W. Every element of V can be written uniquely as a linear combination of elements of B. If we divide this linear combination into a sum of two vectors, one a linear combination in B and the other a linear combination in $B' - B$. Thus every element of V has a unique representation as a sum of an element of U and an element of W.

□

10. Some set theory

Now that we have laid out the basic definitions for the linear algebra in the paper, we turn to the set theoretic aspects. Let X and Y be two sets. A function $\phi : X \to Y$ is said to be *injective* (or an *injection*) if $\phi(x) = \phi(x')$ implies $x = x'$. To say that $\phi : X \to Y$ is injective is equivalent to asserting the existence of a *left inverse* for ϕ, which is a function $\kappa : Y \to X$ which satisfies $\kappa \circ \phi = 1_X$ (the identity function on X).

A function $\psi : X \to Y$ is said to be *surjective* (or an *surjection*) if given any $y \in Y$, there exists an $x \in X$ satisfying the equation $\psi(x) = y$. This is equivalent to the existence of a *right inverse* for ψ, which is a function $\epsilon : Y \to X$ satisfying $\psi \circ \epsilon = 1_Y$.

A function $\tau : X \to Y$ which is both injective and surjective is called *bijective* (or a *bijection*). These functions necessarily possess both left and right inverses and in fact these inverses coincide.

There are a few elementary properties of injections, surjections, and bijections which we will use. Most importantly, if $f : X \to Y$ and $g : Y \to Z$ are injections (alternately, surjections or bijections), then the composite function $gf : X \to Z$ is necessarily injective (surjective, bijective).

What does it mean to say that the set $\{1, 2, 3\}$ and the set $\{2, 3, 4\}$ have the same size? Or, in more elementary terms, the set $\{G, C, S\}$ and the set $\{grape, cherry, strawberry\}$.

In the first case, we have an easy bijection f from the first to the second set given by $f(x) = x + 1$. In the second case we match the fruits to the first letters of their names.

Two sets X and Y have the same *cardinality* if there exists a bijection $\phi : X \to Y$.

This is quickly seen to be an equivalence relation–it partitions the universe of sets into classes by cardinality. Often it is difficult to construct directly a bijection $\tau : X \to Y$. In this case we exploit a related but weaker notion. Let X and Y be sets. If there exists an injection $\phi : X \to Y$ then we say that Y is at least as large as X. This is made into a useful idea by the following theorem.

Theorem 8. *(Cantor-Schroeder-Bernstein theorem). Given sets X,Y and injections $\phi_1 : X \to Y$ and $\phi_2 : Y \to X$, there exists a bijection $\tau : X \to Y$.*

For a proof, see [2]. This is truly powerful, because it allows us to reduce the construction of a bijection to the construction of two injections, which are often easier to describe.

If X and Y are sets with $\phi : X \to Y$ a bijection, then we write $|X| = |Y|$ and say that the two sets share the same cardinality. We define operations on cardinalities, $|X||Y| = |X \times Y|$ and $|X| + |Y| = |(X \times \{0\}) \cup (Y \times \{1\})|$, where $A \times B$ is the Cartesian product of A and B.

A fundamental theorem of set theory says that if $X \cup Y$ is infinite, then $|X||Y| = max\{|X|, |Y|\} = |X| + |Y|$ (the long proof is contained in [2]). An inductive argument tells us that if $X_1, \ldots, X_k$ are sets with infinite union, then $\Pi_{i=1}^k |X_i| = max_{i=1}^k |X_i| = \sigma_{i=1}^k |X_i|$. This allows us to quickly calculate the cardinality of finite products and unions of sets, provided we know which set has the greatest cardinality. For example, the Cartesian product of $\mathbb{R}$ with itself, $\mathbb{R} \times \mathbb{R}$ has cardinality $|\mathbb{R}|$, as $\mathbb{R}$ is infinite. Note that this rule fails dramatically if the union of the sets is finite. For example, if $A = \{1, 2, 3\}$, then $|A \times A| = 9 > 3 = |A|$.

The 'smallest' infinite cardinality is $|\mathbb{N}|$. To prove this, we must show that given any infinite set X there is an injection mapping $\mathbb{N}$ into X. We construct this injection inductively. Let $\phi(1)$ be any arbitrary element of X (there must exist such an element or X is empty). Now suppose that $\phi(1), \ldots, \phi(n)$ have all been selected such that no two coincide. There must exist an element $\phi(n + 1)$ in $X - \{\phi(1), \ldots, \phi(n)\}$, or else X is a finite set. Thus we have defined ϕ over the entire set $\mathbb{N}$ as an injection. So any infinite set is at least as large as the positive integers.

11. Set Theory and Linear Algebra

Let $\phi : V \to W$ be an injective linear transformation. Then for any linearly independent set $S \subset V$, we have that the image set $\phi(S)$ is linearly independent in W.

For if $\alpha_1\phi(v_1) + \ldots + \alpha_k\phi(v_k) = 0_W$ for $v_i \in V$ and $\alpha_i \in F$, then we can use the linear properties of ϕ to write $\phi(\alpha_1 v_1 + \ldots + \alpha_k v_k) = 0_W$. Since $\phi(0_V) = 0_W = \phi(\alpha_1 v_1 + \ldots + \alpha_k v_k)$, and ϕ is injective, we must have $\sum_{i=1}^{k} \alpha_i v_i = 0_V$. But S is linearly independent, so that $\alpha_i = 0_F$ for all i. Thus $\phi(S)$ is linearly independent.

We describe a similar interaction between surjections and spanning sets. Let $\psi : V \to W$ be a surjective linear transformation, and suppose that $S \subseteq V$ is a spanning set for V. Then the image set $\phi(V)$ is a spanning set for W. For if we take an arbitrary element $w \in W$, we can always find some $v \in V$ which satisfies $\psi(v) = w$, as ψ is a surjection. Since S spans V, we can find a linear combination in S equaling V, that is $\sum_{i=1}^{k} \beta_i v_i = v$, where $\beta_i \in F$ and $v_i \in S$. Then $w = \psi(v) = \psi(\sum_{i=1}^{k} \beta_i v_i) = \sum_{i=1}^{k} \beta_i \phi(v_i)$. Thus we can take any element of W and write it as a linear combination in $\phi(V)$. This proves that $\phi(V)$ is a spanning set for W.

Combining these two results, we can see that if $\tau : V \to W$ is a bijective linear transformation, and $B \subset V$ is a basis for V, then $\tau(B)$ is a basis for W.

12. All Finite Subsets

Theorem 9. *Suppose that* $\mathbf{S}$ *is an infinite collection of sets, each of which is finite. Then* $|\mathbf{S}| = |\cup_{A\in\mathbf{S}} A|$. *(An immediate consequence of this is that for any infinite set* X*, if* X_f *is the collection of all finite subsets of* X*, then* $|X_f| = |X|$*).*

Proof. Define $X = \cup_{A\in\mathbf{S}} A$. Then X is infinite, for if X is finite, then the power set $P(J)$ has finite order $2^{|X|}$. Thus there are only finitely many subsets of X, and thus only finitely many elements of $\mathbf{S}$, contradicting the hypothesis that $\mathbf{S}$ is infinite. Thus we can apply our facts from set theory to write $|X \times X| = |X^2| = |X|$. Then induction and the equation $|(X^k \times X)| = |X^{k+1}|$ (trivial to verify) give us that $|X^n| = |X|$ for any positive integer n.

Let $Y = \cup_{n\in\mathbb{N}} X^n$. For every positive integer k, let $\phi_k : X^k \to X$ be a bijection. This allows us to define a bijection from Y to $\mathbb{N} \times X$ by sending the element $(x_1, \ldots, x_m)$ to $(m, \phi_m(x_1, \ldots, x_m))$. Thus $|Y| = |\mathbb{N} \times X| = |X|$ as X is infinite and $\mathbb{N}$ is the smallest infinite set.

Now we can show that $|\mathbf{S}| \leq |X|$. Using the axiom of choice (for the second time), for each $A \in \mathbf{S}$ we take the finitely many elements of A and list them in any order we like, forming a vector v of size $|A|$. Now map A to v, an element of Y. Our mapping must be injective, for if the lists for two sets coincide, they must share all elements and therefore be the same set. Thus we have an injection mapping $|\mathbf{S}|$ into Y, so that $|\mathbf{S}| \leq |X|$.

Since $\mathbf{S}$ is infinite, we have the equation $|\mathbf{S}| = |\mathbf{S} \times \mathbb{N}|$. Apply the axiom of choice once more to select for each set $A \in \mathbf{S}$ an enumeration $\{x_1, \ldots, x_m\}$. Let $\psi : \mathbf{S} \times \mathbb{N} \to X$ be given by $\phi(S, i) = x_i$, if S has at least i elements, and x_1, if S doesn't have i elements. Then every element of X is in the image of ψ. This follows from the definition of X: we just pick A containing $x \in X$ and then some integer j must have $\phi(A, j) = x$.

This surjection gives us the relation $|X| \leq |\mathbf{S} \times \mathbb{N}| = |\mathbf{S}|$. Then the previous relation $|\mathbf{S}| \leq |X|$ and the Cantor theorem give us $|\mathbf{S}| = |X|$.

□

13. Invariance of Basis Size

Theorem 10. *Suppose that V is a vector space over F and S and T are two bases for V. Then $|S| = |T|$; that is, there exists a bijection $\phi : S \to T$.*

Proof. For a given $x \in S$, let $f(x)$ be any finite subset of T such that x is contained in the span of $f(x)$ (this requires the axiom of choice). We can always manage to find such a finite subset because T is a spanning set for V. Define $f(S) = \{f(x) : x \in S\}$. Let $\mathbf{C} = \{f^{-1}(Y) : Y \in f(S)\}$. We claim that this is a partition of S into finite sets.

First, we show that the sets $f^{-1}(Y)$ are finite. If some $x' \in S$ has $f(x') = Y$ with $f^{-1}(Y)$ infinite, then the subspace generated by Y contains infinitely many linearly independent vectors (the elements of $f^{-1}(Y)$). But Y consists of only finitely many vectors in T, and so it cannot have infinitely many linearly independent vectors in its span, by elementary linear algebra. Thus all the sets $f^{-1}(Y)$ are finite.

Now we show that the sets $f^{-1}(Y)$ form a partition of S. This is trivial: if $x \in f^{-1}(Y) \cap f^{-1}(Y')$, then $f(x) = Y$ and $f(x) = Y'$. Also, $x \in f^{-1}(f(x))$.

So we have shown that the sets $f^{-1}(Y)$ form a partition of S into finite sets. This gives us $S = \cup_{Y=f(x):x\in S} f^{-1}(Y)$. The function f induces a bijection from the collection $\mathbf{C}$ to the set $f(S)$ given by $f^{-1}(f(x)) \to f(x)$. Since $f(S)$ is just a collection of finite subsets of T, we have $|f(S)| \leq |T|$. The previous proposition tells us that $|S| = |\mathbf{C}| = |f(S)| \leq |T|$.

Then we can merely switch S for T in the preceding discussion and get $|T| \leq |S|$. The Cantor theorem then applies: because $|S| \leq |T|$ and $|T| \leq |S|$, we have $|S| = |T|$.

$\square$

Now a particularly nice formula regarding the direct sum of two vector spaces is possible. If we denote the dimension of a vector space V by $dim(V)$, then we have the following:

$$dim(V_1 \oplus V_2) = (dim(V_1)) + (dim(V_2)).$$

We prove this by first letting B be a basis for V_1 and C be a basis for V_2. Then the set $\{(b, 0) : b \in B\} \cup \{(0, c) : c \in C\}$ has cardinality $|B| + |C|$ by definition. It is also a spanning set for $V \oplus W$, because if $v_1 = \sum_{i=1}^{k} \alpha_i b_i$ and $v_2 = \sum_{i=1}^{k} \beta_i c_i$ (adding zeros to make the number of indices equal if necessary), then $(v_1, v_2) = \sum_{i=1}^{k} \alpha_i (b_i, 0) + \sum_{i=1}^{k} \beta_i (0, c_i)$. Linear independence is trivial to see: if $\sum_{i=1}^{k} \gamma_i (b_i, 0) + \sum_{i=1}^{k} \delta_i (0, c_i) = (0, 0)$, then the linear independence conditions on B and C give $\gamma_i = \delta_i = 0$. Thus S is a basis for $V_1 \oplus V_2$ and $dim(V_1 \oplus V_2) = (dim(V_1)) + (dim(V_2))$.

14. The Well-Ordered Set T

Let $T := \{0, 1, 2, \ldots\} \cup \{\infty\}$ be ordered with the integer order, with the additional rule that if k is an integer, then $k < \infty$. Then every nonempty subset of T has a least element. For if a nonempty subset A of T contains no integers, it must have only ∞ as an element. Then ∞ is the least element. Otherwise $A \cap \{0, 1, \ldots\} \neq \emptyset$. The claim then follows from the *well-ordering property of the integers*: every nonempty set of integers which is bounded from below has a least element. This is proven in [2].

15. Valuations

Let V be a vector space over a field F. A function $\nu : V \to T$ is called a *valuation* if it satisfies the following conditions for all $\alpha \in F - \{0_F\}$, $x, y \in V$:

$$\nu(x+y) \geq min\{\nu(x), \nu(y)\}$$

$$\nu(\alpha x) = \nu(x)$$

$$\nu(0_V) = \infty.$$

An ordered pair (V, ν) where V is a vector space and $\nu : V \to T$ is a valuation is referred to as a *valuated vector space*. Sometimes we will refer to V as a valuated vector space, unless the context is unclear.

Now we prove the subadditivity of valuations: if $\nu(x) < \nu(y)$, then $\nu(x+y) = \nu(x)$. Again we note that $\nu(-y) = \nu(y)$ and $\nu(x+y) \geq \nu(x)$. Then we have $\nu(x) = \nu(-y+x+y) \geq min\{\nu(y), \nu(x+y)\} \geq \nu(x)$. Then we must have equality in all cases: $min\{\nu(y), \nu(x+y)\} = \nu(x)$ and so $\nu(x) = \nu(x+y)$. We will frequently call this the additive rule for valuations.

16. Homogeneous Valuations

Given a valuated vector space (V, ν), we define the *value spectrum* of V to be the set $\nu(V^*) = \{\nu(v) : v \neq 0_V\} \subseteq T$. If the value spectrum of V consists of a single element, then we say that V is *homogeneous*. If the value spectrum is finite, we say that the space is *finite spectral*.

Constructing these spaces is almost the same as defining them: for a given $k \in T$ and a vector space V, we can endow V with a homogeneous valuation by defining $\nu_k : V \to T$ by $\nu(x) = k$ for x nonzero and $\nu(0_V) = \infty$. We omit the straightforward proof that this defines a valuation.

17. Subspaces

Let $k \in T$. Then we define $V(k) = \{v \in V : \nu(v) \geq k\} \subseteq V$. Since we are assuming $\nu(0_V) = \infty$, we have $\infty \in V(k)$ for all $k \in T$. We prove that $V(k)$ is a subspace of V for all $k \in T$. Suppose that $x, y \in V(k)$, so that $min\{\nu(x), \nu(y)\} \geq k$. Then $\nu(x+y) \geq min\{\nu(x), \nu(y)\} \geq k$. So $x + y \in V(k)$. If $\alpha \in F$ and $x \in V(k)$, we distinguish two cases. If $\alpha = 0_F$, then $\alpha x = 0_V$, and $\nu(\alpha x) = \infty \geq k$. So $\alpha x \in V(k)$. If $\alpha \neq 0_F$, then $\nu(\alpha x) = \nu(x)$ by definition of a vector space valuation. So $\alpha x \in V(k)$.

Thus $V(k)$ is a subspace of V for any $k \in T$.

18. Direct Sums and Direct Products

Let $\{(V_i, \nu_i)\}_{i \in I}$ be a collection of valuated vector spaces. We can construct their vector space direct sum and product, $\bigoplus_{i \in I} V_i$ and $\prod_{i \in I} V_i$ as described above. We can also define a valuation structure on each of these structures by the rule $\nu_\pi(x) = min(\{\nu_i(x_i)\}_{i \in I})$. This quantity always exists because T is well-ordered, so we know that $\nu_{\prod}$ defines a function from $\prod_{i \in I} V_i$ to T.

We prove that $\nu_{\prod}$ is a valuation on $\prod_{i \in I} V_i$. We can then restrict $\nu_{\prod}$ to get a valuation on $\bigoplus_{i \in I} V_i$.

If α is a nonzero scalar and $x = (x_i)_{i \in I}$ is an element of $\prod_{i \in I} V_i$, then $\nu_i(\alpha x_i) = \nu_i(x_i)$ for any i, as the functions ν_i are all valuations. Thus $\nu_{\prod}(\alpha x) = \nu_{\prod}(x)$ for any nonzero scalar α and vector $x = (x_i)_{i \in I}$.

Now suppose that $y = (y_i)_{i \in I}$ is another, not necessarily distinct, element of $\prod_{i \in I} V_i$. Again we have by the definition of valuation that $\nu_i(x_i + y_i) \geq min(\{\nu_i(x_i), \nu_i(y_i)\})$ for all $i \in I$. But this second quantity, $K = min(\{\nu_i(x_i), \nu_i(y_i)\})$, is at least equal to $J = min(\{\nu_{\prod}(x), \nu_{\prod}(y)\}$. Thus if we take the minimum over all the indices i, $\nu_{\prod}(x + y)$, it must also be greater than or equal to J. But this is just the conditin $\nu_{\prod}(x + y) \geq min(\{\nu_{\prod}(x), \nu_{\prod}(y)\})$. Thus $\nu_{\prod}$ is a valuation.

19. T-HOMOMORPHISMS

Let (V_1, ν_1) and (V_2, ν_2) be two valuated vector spaces. A linear transformation $\phi : V_1 \to V_2$ is called a *T-homomorphism* if it satisfies $\nu_2(\phi(x)) \geq \nu_1(x)$ for all $x \in V_1$. We see that T-homomorphisms are closed under composition, i.e. if $\phi_1 : V_1 \to V_2$ and $\phi_2 : V_2 \to V_3$ are two T-homomorphisms (where V_i is valuated under ν_i), then

$$\nu_3((\phi_2 \circ \phi_1)(x)) = \nu_3(\phi_2(\phi_1(x))) \geq \nu_2(\phi_1(x)) \geq \nu_1(x).$$

Of course, linear transformations are closed under composition (see [1]). Linear transformations are also closed under linear combinations, i.e. if $\{f_i : V \to W\}_{i=1}^n$ is a finite set of linear transformations and $\{\alpha_i\}_{i=1}^n$ is a set of scalars, then $\sum_{i=1}^n \alpha f_i$ is also a linear transformation. Do T-homomorphisms enjoy this same property? Yes.

Proving that T-homomorphisms are closed under linear combinations is equivalent to proving that the collection A of T-homomorphisms between V_1 and V_2 forms a subspace of the space of linear transformations between V_1 and V_2. So we must prove that A is nonempty and closed under both scaling and addition.

First, note that the zero transformation $v_1 \to 0_{V_2}$ is a T-homomorphism. The linearity of this function is trivial and the valuation property follows from the condition that $\nu_2(0_{V_2}) = \infty$. So A is nonempty. Now we prove that A is closed under addition of functions. If $\phi : V_1 \to V_2$ and $\psi : V_1 \to V_2$ are both T-homomorphisms, then so is their sum $\psi + \phi$. For we have the inequality

$$\nu_{V_2}(\phi(x) + \psi(x)) \geq min\{\nu_{V_2}(\phi(x)), \nu_{V_2}(\psi(x))\} \geq \nu_{V_1}(x).$$

Scaling by zero scalar is equivalent to showing that the zero transformation is a T-homomorphism, which we have already shown to be true. If $\alpha \in F - \{0_F\}$, then we have $\nu_{V_2}((\alpha\phi)(x)) = \nu_{V_2}(\phi(x)) \geq \nu_{V_1}(x)$. Thus we have proven that T-homomorphisms are closed under linear transformations.

20. Embeddings and Isometry

We distinguish a special class of T-homomorphisms. In group theory, an *embedding* is an injective group homomorphism. The name refers to the fact that an embedding $\phi : G \to H$ 'sticks' G into H without losing any structure. An embedding between valuated vector spaces (V, ν_1) and (W, ν_2) is an injective linear transformation $\psi : V \to W$ which preserves valuations, i.e. $\nu_2(\psi(v)) = \nu_1(v)$ for all vectors $v \in V$. An *isometry* (the analogue of an isomorphism) is a surjective embedding. If there exists an isometry $\tau : V \to W$ then we say that V and W are *isometric* or have the same *isometry class* (because isometries determine an equivalence relation).

We can exhibit a case of isometry and embedding. Suppose that V is a valuated vector space under $\nu : V \to T$ and U is a proper subspace of V. Then $\nu|_U$ is a valuation on U, and the inclusion map $i : U \to V$ is trivially an embedding.

More interestingly, suppose that V_1 and V_2 are both homogeneous vector spaces. Then we have the following theorem:

Theorem 11. *Homogeneous spaces V_1 and V_2 are isometric if and only if they have the same dimension (are isomorphic) and value spectrum.*

Proof. We prove the forward direction first. Since an isometry ϕ is necessarily a vector space isomorphism, we only need to show that isometric homogeneous spaces have the same value spectra. But if we take a nonzero vector $x \in V_1$, we have that the value spectrum of V_1 is equal to $\{\nu_1(x)\}$. But an isometry preserves valuation and carries nonzero vectors to nonzero vectors, so the value spectra agree. Thus the dimensions and value spectra agree.

Now suppose that the dimensions agree and both value spectra equal $\{\beta\}$. Then we have a vector space isomorphism $\phi : V_1 \to V_2$. We claim that this must also be an isometry. Again, note that the function ϕ maps nonzero vectors to nonzero vectors, and 0_{V_1} to 0_{V_2}. If $x \neq 0_{V_1}$, then $\nu_2(\phi(x)) = \beta = \nu_1(x)$. Similarly, $\nu_2(\phi(0_{V_1})) = \nu_2(0_{V_2}) = \infty = \nu_1(0_{V_1})$. Thus ϕ is an isometry.

□

21. Valuated Quotient Spaces

Let (V, ν) be a valuated vector space and let U be a subspace. Then we can form the quotient space V/U as described earlier. Then we can define a valuation $\mu' : V/U \to T$ on V/U as follows: for any $v \in V$, let $\mu(v + U) = \sup(\{\nu(v + u) : u \in U\})$. Here sup denotes the least upper bound of a set, the minimal element of T greater than or equal to all elements of that set. We might first ask if this is defined for arbitary $v \in V$. But we can remark that for any $v \in V$, the set $\{\nu(v + u) : u \in U\} \subseteq T$ must satisfy one and only one of three possible cases. If $\infty \in \{\nu(v + u) : u \in U\}$, then $\infty = \sup(\{\nu(v + u) : u \in U\})$. If $\infty \notin \{\nu(v + u) : u \in U\}$, but the set $\{\nu(v + u) : u \in U\}$ contains arbitrarily large integers (equivalently, contains infinitely many elements), then $\infty = \sup(\{\nu(v+u) : u \in U\})$, because no integer will suffice as an upper bound, yet ∞ must as any integer is less than ∞. In the last case, $\{\nu(v + u) : u \in U\}$ consists of a finite set of integers. But an elementary inductive argument shows that every finite set of integers has a maximum, which is necessarily the sup of the set. So the function $\mu : V/U \to T$ is well-defined. Is it necessarily a valuation?

First we let α be a nonzero scalar and let $v \in V$. Does $L := \mu(v + U)$ equal $L' := \mu(\alpha(v + U))$? Yes, which we prove by first showing that L' is an upper bound for the set $\{\nu(v + u) : u \in U\}$, so that $L \leq L'$. We then reverse the argument, so that $L' leq L$, from which $L = L'$ follows. Let $w \in v + U$. Then $\nu(w) = \nu(\alpha w)$, as α is nonzero. So $\nu(w) \leq L'$, as L' is an upper bound for the set $\{\nu((\alpha v) + u) : u \in U\}$. Thus L' is an upper bound for $\{\nu(v + u) : u \in U\}$, and so $L \leq L'$. We can reverse this argument, noting that $v + U = \alpha^{-1}((\alpha v) + U)$ (where α^{-1} is of course nonzero), to see that $L' \leq L$. Thus $\mu(\alpha(v + U)) = \mu(v + U)$.

Now suppose that v and v' are vectors in V. Do we have the inequality $\mu(v + v' + U) \geq min\{\mu(v + U), \mu(v' + U)\}$? Without loss of generality, let $K = min\{\mu(v + U), \mu(v' + U)\} = \mu(v + U)$.

Again we distinguish three cases. First, suppose both cosets $v + U$ and $v' + U$ contain elements of maximum valuation (that is, elements $v+u$ and $v'+u'$ such that $\nu(v+u'') \leq \nu(v+$

u) and $\nu(v'+u''') \leq \nu(v'+u')$ for all $u'', u''' \in U$). Then $\mu(v+v'+U) \geq \nu(v+v'+(u+u')) \geq K$, as $u+u' \in U$.

Second, suppose that only one of the two cosets attains its supremum. Then we must have that the value range of this coset is an infinite set of integers, the only case where a coset can fail to meet its supremum. Note that in this case, the μ-value of the coset is ∞. If this coset is the smaller of the two, then they both must have infinite valuation. Then given any integer K, we can find elements $v+u_1$ and $v'+u_2$ of the two cosets such that $\nu(v+u_1)$ and $\nu(v'+u_2)$ are both greater than or equal to K. Then

$$\mu((v+v')+U) \geq \nu(v+u_1+v'+u_2) \geq min\{\nu(v+u_1), \nu(v'+u_2)\} \geq K.$$

So $\mu((v+v')+U) \geq K$ for any integer K, and we conclude $\mu((v+v')+U) = \infty$. If the coset which fail to meet its valuation is the larger of the two, then we can repeat the argument, substituting in a maximizer for the value range of the other coset in place of $v+u_1$ or $v'+u_2$.

Lastly, suppose that both cosets fail to meet their supremum. Then as above we can find elements of each coset with valuation greater than any fixed integer K. Their sum also has valuation greater than K, and is an element of the coset sum. Thus the inequality follows, and we have the μ defines a valuation on V/U.

22. The Valuation Pseudometric

Let (V, ν) be a valuated metric space. Define d mapping $V \times V$ into the nonnegative reals by the rule $d(x, y) = 2^{-\nu(x-y)}$ (where we define $2^{-\infty} = 0$). Typically we write $d(x, y)$ instead of $d((x, y))$ for brevity's sake. Then this defines a *pseudometric*, which has the following properties for all $x, y, z \in V$:

$$d(x, x) = 0$$

$$d(x, y) = d(y, x)$$

$$d(x, y) + d(y, z) \geq d(x, z).$$

The names of the properties are identity, symmetry, and triangle inequality. By hypothesis $\nu(0_V) = \infty$, so that we have $d(x,x) = 2^{-\nu(x-x)} = 2^{-\nu(0_V)} = 2^{-\infty} = 0$ for any $x \in V$. Since $-1 \neq 0_F$, we have $\nu(x-y) = \nu(-1(x-y)) = \nu(y-x)$ for all $x, y \in V$. Then we have $d(x,y) = 2^{-\nu(x-y)} = 2^{-\nu(y-x)} = d(y,x)$. Let $k = min\{\nu(x-y), \nu(y-z)\}$ and let $K = max\{\nu(x-y), \nu(y-z)\}$. Then $\nu(x-z) = \nu((x-y)+(y-z)) \geq k$, so that $d(x,z) \leq 2^{-k} \leq 2^{-k} + 2^{-K} = d(x,y) + d(y,z)$.

The pseudometric determined by a valuation distinguishes a collection of subsets of V called a *topology on* V. A subset $U \subseteq V$ is in this collection (or is called *open*) if and only if for every $x \in U$ there exists some $r > 0$ so that $d(x,y) < r$ implies $y \in U$. The *open ball of radius* $r > 0$ *centered at* $x \in V$, denoted $B_r(x)$, is the set of all $y \in V$ satisfying $d(x,y) < r$. Thus we can say that a subset of V is open if and only if it is a union of open balls. Note that $\emptyset$ and V are both trivially open, the latter satisfying $V = \cup_{v \in V} B_1(v)$. Importantly, open balls themselves are open sets. Also, arbitrary unions and finite intersections of open sets are themselves open sets. We do not prove these facts but instead refer to [3].

Extremely important to consider are the *closed sets* determined by a topology. These are the complements of the open sets, $\{V - U : U\, open\}$. By the DeMorgan Laws for set complements and the previously stated results on the unions and finite intersections of open sets, we can say that arbitrary intersections and finite unions of closed sets are themselves closed. In order to show that a subset N of V is closed, it suffices to show that the complement $V - N$ is an open subset of V.

23. Hausdorff Valuations

The valuation pseudometric is defined for *any* valuated vector space (V, ν). However, the pseudometric lacks an important property which we define now. A function $d : V \times V \to \mathbb{R}$ with nonnegative range has the property of *positive definition* if $d(x,y) = 0$ implies $x = y$ for all $x, y \in V$.

A pseudometric which has positive definition is called a *metric* or a *distance function.* This is important because the topology determined by a metric is *Hausdorff*: given any distinct

$x, y \in V$ there exist open sets U_x and U_y such that $x \in U_x$ and $y \in U_y$, and $U_x \cap U_y = \emptyset$ (sometimes this is referred to as a T_2 space). This is fairly easy to prove: suppose that $x \neq y$, so that $d(x, y) = r > 0$. Then the open balls of radius $\frac{r}{4}$ centered at x and y respectively satisfy the requirements for a Hausdorff space. For if $z \in B_{r/4}(x) \cap B_{r/4}(y)$, then $d(x, y) \leq d(x, z) + d(z, y) < r/4 + r/4 < r = d(x, y)$, a contradiction.

What are the Hausdorff valuation pseudometrics, or valuation metrics? This question obviously involves the set $V(\infty) = \{x \in V : \nu(x) = \infty\}$. By hypothesis, $0_V \in V(\infty)$. What happens when nonzero $y \in V(\infty)$? Then for any $x \in V$, we have $d(x + y, x) = 2^{-\nu(y)} = 0$. Thus we cannot obtain a valuation metric if $V(\infty)$ is nontrivial. Is the converse true: if $V(\infty)$ is trivial, can we obtain a valuation metric?

The answer is yes, because of the way we defined d. The pseudometric d vanishes (satisfies $d(x, y) = 0$) if and only $\nu(x-y) = \infty$, because $2^{-k} > 0$ for all integers k. Since $V(\infty) = \{0_V\}$, we have $x - y = 0_V$, so that $x = y$. Thus the valuation metrics are those exactly those whose vector spaces satisfy $V(\infty) = \{0_V\}$.

We designate those valuated metric spaces (V, ν) satisfying $V(\infty) = \{0_V\}$ as *Hausdorff valuated vector spaces.*

24. Topological Properties

Here is an example of the connection between the valuation structure and the topological structure which arises from the valuation metric. Specifically, we show that if the valuation is more or less 'trivial,' then the topology is similarly trivial.

Suppose that (V, ν) is a homogeneous Hausdorff valuated vector space with $\nu(V^*) = k < \infty$. Then the topology determined by the valuation metric is uninteresting–it gives us no interesting structure at all. For if we take any $x \in V$, we can form the open ball B of radius 2^{-k} about x. This is an open set in the metric topology by definition. But this open ball contains only x, for any other $y \in V$ has $d(x, y) = 2^{-\nu(x-y)} = 2^{-k}$ so that $y \notin B$. Thus the singleton x is open for every $x \in V$. Then given any subset $W \subseteq U$, the union $\cup_{w \in W}\{w\} = W$ is open, as unions of open sets are open. Thus every subset of V is open in this topology, and the valuation topology on V is in this case just the set of all subsets of V (the power set of V). This topology is called the *discrete topology*, and it gives us no interesting structure or information.

25. A splitting lemma

Recall that if V is a vector space and U is a subspace of V, then there exists a subspace W such that $V = W \oplus U$ (although this W is by no means unique). The following lemma shows that the valuation structure is preserved over internal direct sums.

Lemma 2. *Suppose that V is a T-valuated vector space and $\beta \in T$. If V is the internal direct sum $W \oplus V(\beta)$, then this decomposition preserves T-valuations. That is, the mapping $(w, v) \to w + v$ from the valuated direct sum to the space V preserves valuations.*

Proof. For all nonzero $w \in W \subseteq V$, we have that $\nu(w) < \beta$, or else the sum is not internal direct. Thus if $v = w + v'$, where $w \in W$ and $v' \in V(\beta)$, we have $\nu(v) = \nu(w) = \nu_{\oplus}(w, v')$. Thus the decomposition preserves T-valuations.

□

26. Freeness

We prove the following simple lemma:

Lemma 3. *Let V be a homogeneous vector space with basis $\{b_i\}_{i \in I}$ and value spectrum $\{\beta\}$. Then V is isometric to $\oplus_{i \in I} \langle b_i \rangle$ under the mapping ϕ which carries $\alpha_{i_1} b_{i_1} + \ldots + \alpha_{i_k} b_{i_k}$ to the function which equals $\alpha_{i_j} b_{i_j}$ for i_j in the linear combination and 0_V everywhere else.*

Proof. We omit the proof that ϕ is a vector space isomorphism, as it is only tedious. It essentially repeats the proof that $\alpha_1 b_1 + \alpha_2 b_2 \to (\alpha_1 b_1, \alpha_2 b_2)$ is a linear transformation. This follows from $\{b_1, b_2\}$ being a basis. Since both spaces are homogeneous with value spectrum $\{\beta\}$, we have that they are isometric.

□

A valuated vector space V is called *free* if it is isometric to the valuated direct sum of homogeneous spaces. That is, $V \approx \bigoplus_{i \in I} V_i$, where each V_i is homogeneous. This allows us to generalize the proof that two homogeneous vector spaces are isometric if and only if they are isomorphic with identical value spectra.

First we prove a closure-type lemma.

Lemma 4. *Let $\{V_i\}_{i \in I}$ be a collection of free valuated vector spaces. Then the valuated direct sum $\bigoplus_{i \in I} V_i$ is also free.*

Proof. Each V_i can be written as $\bigoplus_{j \in J_i} W_j$ for a set $\{W_j\}_{j \in J_i}$ of homogeneous vector spaces. We label the corresponding isometries as $\phi_i : V_i \to \bigoplus_{j \in J_i} W_j$ for every $i \in I$. Let $x \in \bigoplus_{i \in I} V_i$. We want to write x as a function x' from $K = \cup_{i \in I} J_i$ to $\cup_{i \in I} \cup_{j \in J_i} W_j$ such that $x(a) \in W_a$ for all $a \in K$. Thus if $a \in J_i$, we define $x'(a) = \phi_i(x(a))$. This fulfills the requirements for the direct sum definition. The mapping $x \to x'$ is linear and isometric, but we omit the proof. $\square$

Theorem 12. *A valuated vector space is free if and only if it is isometric to a T-valuated direct sum $\oplus_{\beta \in T} V_\beta$, where each V_β is homogeneous with value spectrum $\{\beta\}$.*

Proof. This is simple in the 'if' direction, for each homogeneous space is of course free and so is their direct sum by the previous proposition. The opposite direction, we assume that $V = \oplus_{i \in I} V_i$ (where each V_i is one-dimensional) and we create $C_\beta = \{i \in I : \nu_i(V_i^*) = \beta\}$ and then form the direct sum $V_\beta = \oplus_{i \in C_\beta} V_i$. Then V_β is homogeneous and V is isometric to $\oplus_{\beta \in T} V_\beta$.

$\square$

For each homogeneous vector space in the above theorem V_β, we have some basis B_β. Then let $B = \cup_{\beta \in T} B_\beta$. This is a basis for the direct sum, so its preimage under the isomorphism ϕ is a basis for V. We call such a basis a *free basis* and note that if $v = a_1 b_1 + \ldots + a_k b_k$ for $b_i \in B$, then $\nu(v) = min(\{\nu(b_i)\}_{i=1}^k$. If we can construct such a basis, then we have in effect created an isometry using the universal property of direct sums from $\oplus_{b \in B} \langle b \rangle$ to V. Thus proving that a space is free is the same as proving that it has a free basis (i.e. a basis which has the above additive property on valuations).

We prove the following lemma which we apply later.

Lemma 5. *Suppose that V is a vector space with finite value spectrum. Then V is free.*

Proof. We induct on the size of the value spectrum $|\nu(V^*)|$. If the size is 1, then the space is homogeneous, and isometric to $\oplus_{i\in I}\langle b_i\rangle$, where $\{b_i\}_{i\in I}$ is a basis for V. Assume that for $k \leq N$, a valuated vector space with k elements in its value spectrum is free. Now assume that V has $N+1$ elements in its value spectrum, and let β be the maximum of these, $\beta = max(\nu(V^*))$. Then we can make the internal direct sum decomposition $V = W_\beta \oplus V(\beta)$, which preserves valuations. The space $V(\beta)$ is homogeneous with value spectrum $\{\beta\}$, and the space W_β has a strictly smaller value spectrum, as all of the elements of W_β have valuation less than β by the internal direct sum criterion. Thus W_β is free by induction, and V is a direct sum of free spaces, as the homogeneous space $V(\beta)$ is free.

$\square$

27. Vector Space Aside

Suppose that V_i for $i \in \{1,2,3\}$ are vector spaces, and let $V = V_1 \oplus V_2 \oplus V_3$. Let $U_1 = \{(x,y,z) \in V : x = 0_{V_1}\}$ and $U_2 = \{(x,y,z) \in V : x = 0_{V_1}, y = 0_{V_2}\}$. Then $U_2 \subseteq U_1$ are subspaces of V and U_1/U_2 is isomorphic to V_2. U_1 is the kernel of the projection transformation $\pi_1 : V \to V_1$ and U_2 is the image of the inclusion transformation $\tau_3 : V_3 \to V$. Of course, images and kernels are always subspaces and the inclusion is trivial.

Why is U_1/U_2 isomorphic to V_2? Let B be a basis for V_2 and let $B' = \{(0,b,0)+U_2 : b \in B\}$. These are distinct, because $(0,b,0) + U_2 = (0,b',0) + U_2$ implies $b - b' = 0_{V_2}$ Then B' is a basis for U_1/U_2, because it is trivially spanning and linearly independent. So the classification theorem applies.

28. Induced Isometries

We have already defined isometry. The following lemma shows that if ϕ is an isometry from V to W, then ϕ maps a subspace $V(\alpha)$ onto the corresponding subspace $W(\alpha)$.

Lemma 6. *If $\beta \in T$ and $\beta \neq \infty$, then the set $V(\beta^+) = \{v \in V : \nu(v) > \beta\}$ is a subspace of V. Also, if $\phi : V \to W$ is an isometry of T-valuated vector spaces, then the restrictions $\phi|_{V(\beta)} : V(\beta) \to W(\beta)$ and $\phi|_{V(\beta^+)} : V(\beta^+) \to W(\beta^+)$ are both isometries.*

Suppose that $\phi : V \to W$ is an isometry of T-valuated vector spaces. Suppose that $v \in V(\beta)$. Then $\nu_2(\phi(v)) = \nu_1(v) \geq \beta$. Thus ϕ maps isometrically $V(\beta)$ onto $W(\beta)$. Since the image of $V(\beta^+)$ is contained in $W(\beta^+)$ and the preimage of $W(\beta^+)$ is necessarily contained in $V(\beta^+)$, we can see that ϕ maps $V(\beta^+)$ onto $W(\beta^+)$. Thus the restrictions are isometries.

29. Ulm Spaces

Let V be a valuated vector space, and let $\beta \in T$. As above, we have the subspace $V(\beta)$. For $\beta < \infty$, we can construct the valuated quotient space $U_\beta = V(\beta)/V(\beta^+)$. Recall that the valuation on this space is given by $\nu_{V(\beta)/V(\beta^+)}(x+V(\beta^+)) = \sup(\{\nu(x+y) : y \in V(\beta^+)\})$.

We denote the β-th Ulm space $V(\beta)/V(\beta^+)$ as U_β. We show that U_β is homogeneous. To prove homogeneity, we take $x \in V(\beta) - V(\beta^+)$ and calculate $\nu_{U_\beta}(x + V(\beta^+))$. If $y \in V(\beta^+)$, then $\nu(x) < \nu(y)$ and so $\nu_V(x+y) = min(\{\nu(x), \nu(y)\}) = \beta$. Then the entire coset $x+V(\beta^+)$ has valuation β, and so $\nu_{U_\beta}(x + V(\beta^+)) = \beta$. Then U_β is homogeneous with value spectrum $\{\beta\}$. If $\beta < \infty$, then we define the βth Ulm invariant $U_\beta(V) := dim(V(\beta)/V(\beta^+))$, and if $\beta = \infty$, we define $U_\beta(V) := dim(V(\infty))$. The Ulm invariants are used to describe free valuated vector spaces very precisely.

Suppose that V is a free T-valuated vector space and as above we have $V = \oplus_{\beta\in T} V_\beta$, where each V_β is homogeneous with value spectrum $\{\beta\}$. For any $\beta \in T$, define $W_{\beta^-} = \oplus_{\delta<\beta} V_\delta$ and $W_{\beta^+} = \oplus_{\delta>\beta} V_\delta$. Then $V = W_{\beta^-} \oplus V_\beta \oplus W_{\beta^+}$. If U_1^β and U_2^β are defined as above, then $U_\beta^1 = V(\beta)$ and $U_\beta^2 = V(\beta^+)$, so that U_β is isometric to $U_1/U_2 \approx V_\beta$. We prove that the βth Ulm space of V is isometric to U^1_β/U^2_β. But if $V \approx W$, then $V(\alpha) \approx W(\alpha)$ and $V(\alpha^+) \approx W(\alpha^+)$. Then the quotients are isometric, and we are done.

We can prove the following theorem, which provides a nice generalization of the classification of vector spaces by dimension:

Theorem 13. *Suppose that V and W are free T-valuated vector spaces. Then V and W are isometric if and only if $U_\beta(V) = U_\beta(W)$ for all $\beta \in T$.*

Proof. Suppose that V and W are free T-valuated vector spaces. Following the earlier proposition we have $V = \oplus_{\beta \in T} V_\beta$ and $W = \oplus_{\beta \in T} W_\beta$, with appropritate homogeneity conditions (ie $\nu_V(V^*_\beta) = \{\beta\}$ and so on). If the spaces are isometric under $\phi : V \to W$, we have the isometries induced on their subspaces which forces identical Ulm invariants. If the Ulm invariants are equal, then we have isometry for the summands, $V_\beta \approx W_\beta$, because the dimension and value spectra agree for two homogeneous spaces. This compels isometry by the universal property of T-homomorphisms. □

30. Basic Subspaces

Now we construct a sequence of homogeneous subspaces of V. By the linear algebra work, for each $\beta \in T - \{\infty\}$ we have a subspace B_β such that $V(\beta)$ is the internal direct sum $B_\beta \oplus V(\beta^+)$. By the previous lemma, this sum preserves T-valuations. Thus B_β is isometric to $V(\beta)/V(\beta^+)$ for all $\beta < \infty$, because the spaces are vector space-isomorphic and share the same value spectra. Note that while there may be many possible subspaces which 'work' for B_β, they are all unique up to isometry.

We can use the inclusion homomorphisms $B_\beta \to V$ to construct the unique adding homomorphism $\phi : \bigoplus_{\beta \in T} B_\beta \to V$. By the homogeneity of the spaces B_β and the rule that if $\nu(x_1) < \nu(x_2) < \ldots < \nu(x_k)$ then $\nu(x_1 + \ldots + x_k) = \nu(x_1)$, we can conclude that the transformation ϕ is injective.

Thus we have an injective linear transformation $\phi : \bigoplus_{\beta \in T} B_\beta \to V$, which we can consider quite naturally as inclusion $((a, b) \to a + b)$.

If $\beta \in T$ and we look at the decomposition $V = W_\beta \oplus V(\beta)$, we can assume that the direct sum $\bigoplus_{\delta<\beta} B_\delta$ is contained in W_β. We are allowed this because we can inductively form the subspaces B_β so that they all belong to W_β. If $\beta \in T^*$, let $\gamma_\beta : V \to B_\beta$ be the projection for the decomposition $V = W_\beta \oplus B_\beta \oplus V(\beta^+)$, given by $w + b + v \to b$, mapping onto the space B_β. Let $B_\infty = V(\infty)$, and using $V = W_\infty \oplus B_\infty$, let $\gamma_\infty : V \to B_\infty$ be the projection onto B_∞, mapping $w + b \to b$.

Then we can can apply the universal property of direct products to create a mapping $\psi : V \to \prod_{\beta\in T} B_\beta$. This takes v to a vector whose coordinates are the images of v under the projection mappings γ_β for $\beta \in T$.

Lemma 7. *The mapping ψ is an embedding.*

Proof. First we prove that ψ is injective. Since ψ is a linear transformation, all we have to do is prove that the kernel is trivial, i.e. $ker(\psi) = \{0_V\}$.

Suppose that $v \in ker(\psi)$. Then we can write $v = w + b$, where $w \in W_\infty$ and $b \in B_\infty = V(\infty)$. Since $\psi(v)$ has ∞-th coordinate equal to zero, we have $v = w$. Thus $v = 0_V$ or v has finite valuation. Suppose that v has finite valuation k. Then $\gamma_k(v) = 0$, because $v \in ker(\psi)$. But we have a decomposition $v = w + v' + b$, where $w \in W_k$, $v' \in B_k$ and $b \in V(\beta^+)$. Thus $v' = 0_V$. But then $w = 0_V$, for otherwise we would have a smaller valuation for v, by the additive rule. But of coure then we have $b = 0_V$, or else the valuation of v is larger than k.

So ψ is injective. Now we prove that ψ preserves valuations.

The case where $v = 0_V$ is trivial, as ψ is linear. Suppose that v is nonzero, and let j be minimal such that $\psi(v)_j$ is nonzero. In the first case, assume that j is finite. Then suppose that $\psi(v)_j = y$. We have $v = w+y+b$ in the decomposition $W_j \oplus B_j \oplus V(j^+)$. We must have $w = 0_V$ or the additive rule applies and we get a contradiction. But then the valuation of $\psi(v)$ is merely $\nu(y) = j$. If j is infinite, then we refer to the decomposition $v = w + b$, where $w \in W_\infty$ and $b \in B_\infty$. The additive rule forces $w = 0_V$, so that $v = b$. Then $\nu(\psi(v)) = \nu(b)$. Thus ψ is an isometry.

$\square$

We call $B = \bigoplus_{\beta \in T} B_\beta$ a *basic subspace* for V. Clearly, B is a free valuated vector space. The crucial property of B is that when we consider it as a subspace of a free, Hausdorff valuated vector space V, it must be *dense* in the topology determined by the metric. That is, given any $v \in V$ and $\epsilon > 0$, there exists $b \in B$ such that $d(b, v) < \epsilon$. We record this as a lemma.

Lemma 8. *Suppose that V is a Hausdorff valuated vector space and $B \subseteq V$ is a basic subspace for V. Then B is dense in V*

Proof. By the above, we have $V \subseteq \prod_{\beta \in T} B_\beta$. Since B is a subspace of V, we know that $0_V \in B$. For any nonzero $x \in V$ and $\epsilon > 0$, we take an integer $K > 0$ such that $2^{-K} < \epsilon$. Then we 'zero out' the first K terms of the product notation $x = (x_1, x_2, \ldots, x_K, \ldots)$ by subtracting the element $x' = (x_1, x_2, \ldots, x_K, 0, 0, 0, \ldots)$. Then the difference $x - x'$ has valuation at least $K + 1$, and the distance $d(x, x') \leq 2^{-(K+1)} < 2^{-K} < \epsilon$. Thus B is dense in V.

□

31. Nice Homomorphisms

We distinguish a special class of T-homomorphisms. A *nice* T-homomorphism f mapping from V to W satisfies the equation $f(V(\beta)) = W(\beta)$ for all $\beta \in T$. Note that by the definition of T-homomorphism, we already have $f(V(\beta)) \subseteq W(\beta)$. This allows us to define nice homomorphism as one satisfying the inclusion $W(\beta) \subseteq f(V(\beta))$. Every element w of W must have a preimage v such that $\nu_V(v) = \nu_W(w)$, by definition of T-homomorphism. Also, all nice T-homomorphisms are surjective. We make a quick observation in the form of the following lemma.

Lemma 9. *Let N be a subspace of V. Then the natural homomorphism $\phi : V \to V/N$ given by $\phi(v) = v + N$ is nice if and only if every coset of N has an element of maximum value.*

Proof. Suppose that ϕ is nice. Let $x + N$ be a given coset of N in V. Then there exists $n \in N$ such that $\nu_V(x+n) = \nu_{V/N}(x+N)$, as ϕ is nice. But $\nu_{V/N}(x+N) \geq \nu_V(x+n')$ for all $n' \in N$. Thus $x+n$ is an element of maximum valuation.

Now suppose that every coset of N has an element of maximum value. Also, suppose that $\nu_{V/N}(x+N) \geq \beta$. Pick an element $x+n$ of $x+N$ of maximum value. Then $\beta \leq \nu_{V/N}(\phi(x)) = \nu_{V/N}(x+N) = \nu_V(x+n)$. Thus $x+N \in \phi(V(\beta))$. So ϕ is a nice T-homomorphism.

□

If a subspace N induces a nice natural homomorphism $V \to V/N$, then we say that N itself is *nice*. We follow this observation with a quick topological characterization of nice subspaces, applicable only to Hausdorff spaces.

Lemma 10. *Suppose that V is a Hausdorff valuated vector space. Then a subspace $N \subseteq V$ is nice if and only if it is closed in the topology determined by the metric.*

Proof. Suppose that N is nice. We prove that N is a closed subspace of V. For suppose that $x \in V - N$. Then the coset $x+N$ has an element $x+n$ of maximum value, say β (β is finite as $V(\infty) = \{0_V\}$ and $x \notin N$). Let $\epsilon = 2^{-(\beta+1)}$. Then the open ball about x of radius ϵ must not intersect N. For if $d(x,n) < \epsilon$ for some $n \in N$, we must have $2^{-\nu(x-n)} < \epsilon < 2^{-\beta}$. But then $\nu(x-n) > -\beta$, contradicting maximality of β.

Now suppose that N is not nice. We prove that N is not closed. For if N is not nice, there must exist some coset $x+N$ of N with no element of highest value. Thus we would have a sequence $x+n_1, x+n_2, \ldots$ such that $\nu(x+n_k) \geq k$. Now let $\epsilon > 0$. By calculus, we know that there exists some integer $K > 0$ such that $2^{-K} < \epsilon$. But then $d(x, -n_K) < \epsilon$. Thus $-n_k \in B_\epsilon(x)$, and every open ball about x contains an element of N. But then $V - N$ is not open, for there is no open ball about x strictly contained within $V - N$. So N is not closed.

Thus the equivalence follows.

□

Now we can prove a useful splitting result that allows us to break up a valuated vector space according to the behavior of a nice T-homomorphism.

Theorem 14. *Suppose that $f : V \to W$ is a nice surjective T-homomorphism with kernel N and W free. Then $N \oplus W'$ is isometric to V for some $W' \leq V$.*

Proof. Since W is free, we have some set of summands $\{w_i\}_{i\in I}$. For each w_i, let $\alpha_i = \nu(w_i)$. Then the nice f must have some $x_i \in V(\alpha_i)$ satisfying $f(x_i) = w_i$. But $\nu_V(x_i) \leq \nu_V(w_i) = \alpha_i$, so that $\nu_V(x_i) = \nu_W(w_i)$.

Let $W' = \text{span}\{x_i\}_{i\in I}$. If we form the direct sum $N \oplus W'$, we want an isometry to or from V. Let $v \in V$. Then $f(v) = \oplus_{k=1}^{n} s_k w_k$, the sum taken over the finite support of $f(v)$, by the freeness of W.

Then let

$$\phi(v) = (\sum_{k=1}^{n} s_k x_k, v - \sum_{k=1}^{n} s_k x_k) \in W' \oplus N.$$

If $\phi(w) = (\sum_{k=1}^{n} s'_k x'_k, v - \sum_{k=1}^{n} s'_k x'_k)$ satisfies $\phi(w) = \phi(v)$, then the sums of the two coordinates are in each case equal and so $w = v$. Thus, ϕ is injective.

If $a \in W'$, $b \in N$, then $\phi(a + b) = (a, b)$. So ϕ is surjective.

Previous results show that ϕ is linear, in particular the fact that f is linear itself.

Let $f(v) = \sum_{i=1}^{k} a_i y_i$. We have that $\nu_{\oplus}(\phi(v)) = min(\{\nu(v - \sum_{i=1}^{k} a_i x_i, \nu(\sum_{i=1}^{k} x_i))$. But $\nu(v) \leq \nu_W(f(v))$ as f is a T-homomorphism. The right hand side of this inequality is just $\nu(\sum_{i=1}^{k} a_i y_i) = min(\{\nu_W(y_i)\}_{i=1}^{k})$. Then $\nu(x) \leq \nu(\sum_{i=1}^{k} a_i x_i)$ and we have $\nu(v) = \nu_{\oplus}(\phi(v))$ by the additive rule.

Thus ϕ is an isometry.

□

We apply this theorem and results from the previous section towards a version of Kulikov's theorem, which we prove now.

Theorem 15. *(Kulikov's Theorem). Let V be a T-valuated vector space. Then V is free if and only if there exists a chain of subspaces $U_1 \subseteq U_2 \subseteq \ldots\ldots$ whose union is V and whose value spectra are all finite.*

Proof. Suppose that $V = \oplus_{i \in I} \langle x_i \rangle$ is free. For each $k \in T$, let $c(k) = \{i \in I : \nu(x_i) \leq k\}$, the set of all summands of V with valuation less than or equal to k. We

Then let $U_k = \oplus_{i \in c(k)} \langle x_i \rangle \cup V(\infty)$ for each $k \in T$. Each U_k must be a subspace of V. We prove this by observing that scaling elements of either of the two sets in the union does not require us to leave the set, and each of the two sets in the union is closed under addition. The only possible problem arises when we add an element from the subspace $\oplus_{i \in c(k)} \langle x_i \rangle$ to an element from the subspace $V(\infty)$. But the nonzero elements of the first subspace all have finite value, and the elements of the second subspace all have infinite value, so that if x is from the first subspace and y is from the second subspace, then $\nu(x + y) = \nu(x)$. The finite support condition on direct sums insures that any element v of V is contained in some U_s. That is, for any $v = (v_i)_{i \in T}$, let $J = \{i \in T : v_i \neq 0\}$. Since there are only finitely many elements in J by definition of the direct sum, J must have a greatest element s or no elements. In the first case, we have $v \in U_s$. In the second case, $v = 0_V \in U_k$ for all k as U_k is a subspace for all $k \in T$. All of these spaces are of finite value spectrum, specifically $\nu(U_k^*) \subseteq \{1, \ldots, k\}$.

Now suppose that we have a chain $U_1 \subseteq U_2 \subset \ldots$ with each U_k having finite value spectrum.

We can construct quotients U_2/U_1,$U_3/U_2, \ldots$. Since each U_k has finite value spectrum, and the quotient valuation of a coset is defined by the maximum value of any element within the coset, each coset has an element of maximum value and U_k is a nice subspace of U_{k+1}. Then the natural T-homomorphism $\phi_k : U_{k+1} \to U_{k+1}/U_k$ given by $\phi(x) = x + U_k$ is nice by a previous proposition.

Since the quotient group has value spectrum equal to a subset of the value spectrum of the space from which it was constructed, all of these such quotients have finite value spectrum. But then they are free by lemma 3.

The nice natural homomorphisms ϕ_{k+1} map the U_{k+1} onto U_{k+1}/U_k. But the factors are free, and so we have some W_k satisfying $U_{k+1} = ker(\phi_{k+1}) \oplus W_k$. But the kernel of the natural homomorphism $\phi_{k+1} : U_{k+1} \to U_{k+1}/U_k$ is U_k. So we have some W_k satisfying $U_{k+1} = U_k \oplus W_k$. Define $W_0 = V_1$. Each of the W_k has value spectrum a subset of $\nu(U_{k+1})$, which is finite. Thus, all the W_k are finite.

Then inductively we have a sequence $W_0, W_1, W_2, ...$ of subspaces of V satisfying $U_k = \oplus_{j<k} W_j$. Let $V' = \oplus_{k=0}^{\infty} W_k$, regarding this as an inclusion within V. Now no vector shows up more than once in this sum, because the 'partial sums' contain any U_m and are internal direct. For the same reason, each vector $v \in V$ is contained once.

Since V is the direct sum of free subspaces, it is itself free.

□

Corollary 1. *Let V be a free valuated vector space and suppose that W is a subspace of V. Then W is free.*

Proof. Kulikov's theorem tells us that we have a chain of finite-spectral subspaces $U_1 \subseteq U_2 \dots$ covering V, that is $\cup_{n=1}^{\infty} U_n = V$. Then $W = W \cap V = W \cap (\cup_{n=1}^{\infty} U_n) = \cup_{n=1}^{\infty} (W \cap U_n)$, by the distributive laws of set theory (see [3]). But $W \cap U_n$ is a subspace of a finite-spectral space and therefore finite spectral for all $n \in \mathbb{N}$. So Kulikov's theorem applies and W is free.

□

Notice how powerful Kulikov's theorem is. To prove that a valuated vector space V is free *from the definition*, we must construct a direct sum of homogeneous spaces D and an isometry $\phi : V \to D$. If we apply Kulikov's theorem, all that is required is a chain of finite-spectral subspaces covering V.

32. Primary Abelian Groups

Let G be an Abelian group with operation written $+$ and identity element 0_G (the subscript distinguishes from $0 \in \mathbb{Z}$ and $0 \in T$). Let $g \in G$. If there exists $k \in \mathbb{N}$ such that the k-fold sum $g+g+\ldots+g = k \cdot g = 0_G$, then there exists a *least* positive integer j satsifying $j \cdot g = 0_G$ by the well-ordering property of $\mathbb{N}$. In this case, we say that the *order* of g is j. Otherwise, no such k exists and we say that g has infinite order. We write $|g|$ for the order of $g \in G$.

Now suppose that p is a prime, and every $g \in G$ satisfies $|g| = p^k$ for some $k \in \{0\} \cup \mathbb{N}$. Then we say that G is *p-primary.*

Now we consider an idea which is in some sense dual to the idea of the order of an element in G. Let $x \in G$ and let $n \in \mathbb{Z}$. If there exists $y \in G$ such that $n \cdot y = x$, then we write $n|x$ and say that n divides x.

We have some results on this definition which mirror properties of integers. For example, if $m, n \in \mathbb{Z}$ satisfy the divisibility relation (in $\mathbb{Z}$) $m|n$, and if $n|g$ for a given $g \in G$, then $m|g$ as well. For if $n = mt$ for some $t \in \mathbb{Z}$ and $g = nh$ for some $h \in G$, then $g = n \cdot h = (mt) \cdot h = m \cdot (t \cdot h)$, so that $m|g$.

If G is p-primary, then for any $g \in G$ we are interested in this question: what is the largest power of p dividing g, if such exists? Whenever such an integer exists, we call it the *p-height* of g, written $ht(g)$. If infinitely many powers of p divide g, the result in the last paragraph tells us that all powers of p divide g. In this case, we say that g has *infinite height* and write $ht(g) = \infty$.

Note that this defines a function $ht : G \to T$, where T is the same well-ordered set that we have been using in the previous journals.

The function $ht : G \to T$ has many of the trappings of a vector space T-valuation, even if G itself is potentially not a vector space over any immediately recognizable field.

33. The Socle

Continuing the last section, G is an abelian p-primary group. Then we can define the *socle* of G as the set $G[p] = \{g \in G : pg = 0_G\}$. Since the function $\phi : G \to G$ given by $\phi(g) = pg$ is a homomorphism for abelian G, this set, its kernel, is a subgroup of G.

If $F_p = \mathbb{Z}_p$ is the finite field of order p, then we can consider $G[p]$ as a vector space over F. Denote the residue class in F_p containing n by $\overline{n}$. The addition is already defined within the group, and we define the scaling by the function $\sigma : F_p \times G[p] \to G[p]$ given by $\sigma(\overline{n}, g) = ng$.

We don't check the vector space axioms here, as they follow easily from elementary properties of Abelian groups, but we do check whether or not the scaling function is well-defined. Let $\overline{x} = \overline{y}$ and let $g = h$. Then $x = y + sp$ for some $s \in \mathbb{Z}$, because they share a residue class modulo p. Then $\overline{x}g = xg = (y + sp)g = yg + s(pg) = yg = yh = \overline{y}h$, where the third equality comes from $g \in S$ and the fifth from addition being well-defined on G. Note that the socle is the largest subgroup of G where $\overline{n}x = nx$ defines a scaling operation on the residue classes modulo p. This follows from the equation $\overline{p + 1} = \overline{1}$ in the residue system modulo p, and so $(p + 1)g = (1)g$ for all elements g in any vector space with $\mathbb{Z}_p$ as scalars. But then subtraction gives $pg = 0_G$. So there is no larger subgroup of G over which n-fold addition defines a scaling operation from the residue classes modulo p.

Since we have the function $ht(g)$ defined on all of G, we can restrict the domain to obtain $ht|_{G[p]} : G[p] \to T$. Our claim is that $ht|_{G[p]}$ is a valuation on the F_p-vector space $G[p]$.

It is true that $ht(0_G) = \infty$, because if $k \in \mathbb{N}$, then $p^k 0_G = 0_G$. Thus there can be no greatest positive integer j satisfying some equation of the form $p^j y = 0_G$. So $(ht|_S)(0_G) = \infty$.

Let $x, y \in G[p]$ and let $k = min\{ht(x), ht(y)\}$. Assume that k is finite. Then we write $x = p^k x'$ and $y = p^k y'$ by the definition of infinite height and the result on divisibility in the section on height. Then by commutativity, $x+y = p^k x' + p^k y' = p^k(x' + y')$, so that $p^k|(x+y)$. If k is infinite, then for any $l \in \mathbb{N}$, we can write $x = p^l x''$ and $y = p^l y''$, so that $p^l|(x + y)$. Then any p^l divides $x + y$. Thus in each of the two cases, $ht(x + y) \geq min\{ht(x), ht(y)\}$.

Now we suppose that we have nonzero $\alpha \in F_p$, with $\alpha^{-1} \in F_p$ its multiplicative inverse modulo p. Let $k \in \mathbb{N}$. Suppose there exists $h \in G$ satisfying $p^k h = g$. Scaling both sides by α, we obtain $\alpha(g) = \alpha(p^k h) = p^k(\alpha h)$. Then $p^k | \alpha g$, so that $ht(\alpha g) \geq ht(g)$. Applying the same argument with the scalar α^{-1} and the vector $\alpha(g)$ gives the inequality $ht(g) \geq ht(\alpha g)$, so that the two valuations are in fact equal.

34. Analyzing $G[p]$

We prove a theorem which links the twin ideas of freeness in valuated vector spaces and freeness in group theory. (A group is free if and only if it is group-theoretically isomorphic to a direct sum of cyclic groups).

Theorem 16. *Let G be a p-primary, abelian group. Then G is isomorphic to a direct sum of cyclic groups if and only if $G[p]$ is a free, Hausdorff, T-valuated vector space.*

Proof. Suppose that $G \approx \oplus_{i \in I} C_i$ where each C_i is a cyclic group and the isomorphism is $\phi : G \to \oplus_{i \in I} C_i$. Since G is p-primary, no element of G may have infinite order and all the C_i are finite cyclic. The orders of elements are preserved under group isomorphism by elementary abstract algebra, so the orders of the elements of the C_i must all be powers of p. Thus the orders of the groups C_i must all be prime-power, by Cauchy's theorem. Thus we can write $G \approx \oplus_{i \in I} Z_{p^{k_i}}$.

For any of these direct summands $H_i = Z_{p^{k_i}}$ we know $H_i[p]$ is a cyclic subgroup of H_i and hence an image of a cyclic subgroup G_i of G. Then $H_i[p]$ is homogeneous and therefore free.

Our next claim is this: $G[p] \approx \oplus_{i \in I} H_i[p]$. Restrict ϕ to $G[p]$. Then $\phi|_{G[p]}$ is a vector space isomorphism with $\phi(G[p])$.

We want to show that $\phi|_{G[p]}$ is surjective. If we write the unique sum with nonzero terms $h_1 + h_2 + \ldots + h_k$ for $\overline{h} \in \oplus_{i \in I} H_i[p]$, then we have some $g \in G$ satisfying $\phi(g) = \sum_{i=1}^{k} h_i$. Then $\phi(pg) = p\phi(g) = p(\sum_{i=1}^{k} h_i) = 0_{\oplus}$. But this implies $pg = 0_{\oplus}$ and $pg = 0_G$ because ϕ is injective.

Is $\phi|_{G[p]}$ an isometry? Trivially, because $\phi(p^r g) = p^r\phi(g)$, holds for all nonnegative integers r and group elements g. So $ht(\phi(g)) = ht(g)$ for any $g \in G[p]$. This is the definition of an isometry, given vector space isomorphism ϕ.

So we have a vector isomorphism of $G[p]$ with $\oplus_{i\in I} H_i[p]$. Yet each of these summands $H_i[p]$ is free, hence $G[p]$ is free.

We record some remarks on the converse. Suppose G is abelian and p-primary and that $G[p]$ is a free, Hausdorff, T-valuated vector space. We aim to show that G is isomorphic to a direct sum of cyclic groups.

Because $G[p]$ is free, we have an internal direct sum representation $G[p] = \oplus_{i\in I}\langle g_i\rangle$, where each $g_i \in G[p]$. This set $\{g_i\}$ is a free basis for $G[p]$, so that $\nu(\sum_{i=1}^{n} \alpha_i g_i) = min(\{\nu(g_i)\}_{i=1}^{n})$.

We can write $g_i = p^{ht(g_i)} h_i$ for some $h_i \in G$. Informally, we want to represent every element of G as a unique linear combination of elements in the sets $\langle h_i\rangle$. Let $H = \{h_i\}$.

Let $g \in G$.

Suppose that $|g| = p^b$. Then $p(p^{b-1}g) = 0_G$, implying that $p^{b-1}g \in G[p]$ and so we have a representation $p^{b-1}g = \sum_{i=1}^{j} a_i g_i = \sum_{i=1}^{j} a_i p^{ht(g_i)} h_i$.

We can divide all the terms in the sum by p^{b-1} because the g_i form a free basis for $G[p]$. This means that the lower bound on the height on the left of the equation is also a lower bound for the height of each g_i, so that $p^{b-1}|g_i$ for each $i \in [j]$. If we divide all of the terms in the sum by p^{b-1}, we get the new sum $s_1 = \sum_{i=1}^{j} a_i p^{ht(g_i)-(b-1)} h_i$, a linear combination in the set H.

If $g = s_1$, then we are done (as far as exhibiting g in the span of H).

If not, we still know that $p^{b-1}(g - s_1) = 0_G$, so that we have $p^{b-2}(g - s_1) \in G[p]$.

If $p^{b-2}(g-s_1) \in G[p]$, then it equals some linear combination of the free basis for $G[p]$. But again, we can divide by the leading power of p (in this case, p^{b-2}) to find a linear combination in H, call it s_2, satisfying $p^{b-2}(g - s_1 - s_2) = 0_G$.

Then $p^{b-3}(g - s_1 - s_2) \in G[p]$. If $g - s_1 - s_2 = 0_G$, then we are finished finished as g is a sum in H. Otherwise, we continue the process, adding more and more s_n and reducing the exponent of p each time. Eventually we either get $g - s_1 - s_2 - \ldots - s_k = 0_G$ for some k or finally exhaust the exponent on p.

If $p^{b-b}(g - s_1 - s_2 - \ldots - s_j) = 0_G$, though, we immediately have $g = s_1 + s_2 + \ldots + s_j$. So we eventually create some sum in H equal to g.

So we have a representation of $g \in G$ in the form $g = \sum_{i=1}^{k} \alpha_i h_i$. Is this unique?

If g also equals $\sum_{i=1}^{k} \beta_i h_i$, where we expand the indexing to accomodate both supports. Then $\sum_{i=1}^{k}(\alpha_i - \beta_i)h_i = 0_G \in G[p]$. Say that we modify the sum to range exclusively over the i satisfying $\alpha_i - \beta_i \neq 0$. If the maximum order of the h_i for the i in question is p^t, then multiplying by p^{t-1} we see the all terms with smaller order vanish. But then we have a linear combination in the $\{g_i\}_{i \in I}$ in the form of the terms which do not vanish (at least one does not vanish–the term with maximal order).

But then the coefficient on all these is 0, by uniqueness of representation by the free basis for $G[p]$. Thus we can winnow down the coefficients $\alpha_i - \beta_i$ 'from the top down' and show them to be identically zero.

So the representation is unique. Thus G is isomorphic to the direct sum $\oplus_{i \in I} \langle h_i \rangle$.

□

35. Refinable Chain Condition

Suppose that V is a Hausdorff T-valuated vector space. We say that V satisfies the *refinable chain condition* if for every ascending sequence of subspaces $U_1 \subseteq U_2 \subseteq \ldots$ satisfying $\cup_{n=1}^{\infty} U_n$ there exists a sequence of *closed* subspaces $C_1 \subseteq C_2 \subseteq \ldots$ with $C_n \subseteq U_n$ and $\cup_{n=1}^{\infty}$. We sometimes abbreviate this by RCC.

Theorem 17. *Suppose that V is a free Hausdorff T-valuated vector space. Then V satisfies RCC.*

Proof. If V is free then it contains a free basis W. For any $\beta \in T$, let W_β consist of those elements of T whose orders are not greater than β. Let $C_i = span(W_i \cap U_i)$. Is C_i closed?

Let $v = a_1w_1 + \ldots + a_nw_n$, as the w_i form a (free) basis for V. We want to show that the coset $v + C_i$ has an element of maximum value, but this is equivalent to showing that if $v \notin C_i$, then $\nu_{V/C_i}(v + C_i)$ is bounded by an integer. Let $k = max(\{\nu(w_i)\})_{i=1}^n$, an integer as V is Hausdorff and therefore none of the nonzero vectors w_i have infinite valuation. We claim that $\nu_{V/C_i}(v + C_i) \leq k + i \in \mathbb{N}$. For any element of C_i can only have valuation up to i, so that the definition of a free basis and the additive rule for valuations give us the inequality.

To see that the C_i cover V, we only need to show that the C_i cover W, and then the subspace condition on C_i gives $V = span(W) \subseteq \cup_{i=1}^{\infty} C_i$. If $w \in W$, then $w \in U_j$ for some $j \in \mathbb{N}$. If we let c equal $max\{\nu(w), j\}$, a finite number, then $w \in C_c$.

So the C_i cover V. $\square$

36. Strong RCC

We give a condition which is equivalent to the RCC. We state and prove this as a theorem.

Theorem 18. *(Strong RCC) Suppose that whenever V is covered by a sequence of* dense *subspaces $U_1 \subseteq U_2 \subseteq \ldots$, there exists closed subspace $C_i \subseteq U_i$ with $\cup_{i=1}^{\infty} C_i = V$. Then V satisfies RCC.*

Proof. Suppose that the sets $U_1 \subseteq U_2 \subseteq \ldots$ cover V and the weakened RCC holds for V.

Recall that a basic subspace of a space V is a free space which is dense in V, which also has the same Ulm invariants as V.

Let B_1 be a basic subspace of U_1 and let B' be a basic subspace of U_2.

We aim to show that B_1 is isometric to a subspace of B', so that we can eventually split B' isometrically into $B_1 \oplus B_2$.

Let $\alpha \in T$. Define $\phi : U_1(\alpha) \to (U_2(\alpha)/U_2(\alpha^+))$ by $\phi(u) = u + U_2(\alpha^+)$. This induces an embedding of the αth Ulm space of U_1 into the αth Ulm space of U_2.

This means that if we take a basis E_α for the αth Ulm space, Z_α, of U_2, we can select a subset $E'_\alpha \subseteq E_\alpha$ with $|E'|$ equal to the αth Ulm invariant of U_1. Then $B_1(\alpha)/B_1(\alpha^+) \approx span(E')$.

This gives us a decomposition $B'(\alpha)/B'(\alpha^+) \approx span(E'_\alpha) \oplus span(E_\alpha - E'_\alpha)$. These can be combined over all the $\alpha \in T$ to give a T-isometric decomposition $B' \approx B_1 \oplus B_2$. Since B' is basic, and the two share all Ulm invariants, the direct sum is a basic subspace for U_2.

Repeating this procedure we get a nested sequence of basic subspaces: $B_1 \subseteq B_1 \oplus B_2 \subseteq B_1 \oplus B_2 \oplus B_3 \subseteq \ldots$. Let $B = \oplus_{n=1}^{\infty} B_n$. Claim: B is dense in V. But this follows immediately from $V = \cup_{n=1}^{\infty} U_n$. If $v \in V$, then $v \in U_j$ for some positive integer j, and so we have a sequence in $B_1 \oplus \ldots \oplus B_j$ converging to v.

Thus the dense sets $W_n := U_n + B$ form a cover for V. The weakened RCC implies that there exist closed and nested $C_n \subseteq W_n$ covering V. Let $C'_n = U_n \cap C_n$. The fact that the sets U_n and C_n are nested sequences gives $\cup_{n=1}^{\infty}(U_n \cap C_n) = (\cup_{n=1}^{\infty} U_n) \cap (\cup_{n=1}^{\infty} C_n) = V$.

We need to show that the C'_n are closed. Let v_k be a sequence in C'_n converging to $v \in V$. Since this sequence is in C_n, a closed subspace of V, the limit v necessarily belongs to C_n. But $C_n \subseteq U_n \oplus B_{n+1} \oplus \ldots$.

Let $v = u + \sum_{i=1}^{j} b_{\alpha_i}$, where $b_{\alpha_i} \in B_{\alpha_i}$ (for $\alpha_i > n$) are all nonzero. The elements of the sequence $\{v_k\}_{k=1}^{\infty}$ are all contained in U_n. So the distance from any element of the sequence to v is always at least $2^{-min(\nu(b_{\alpha_i}))}$ and we obtain a contradiction to to $v_k \to v$.

So all the sets C'_n are closed and we have the desired covering. □

37. Freeness and Dimension

We now attempt to prove a special case of a statement whose truth is unknown, which is a converse to theorem 17.

Statement 1. *If V satisfies RCC then V is free.*

First we must outline a principle of set theory which can neither be proven nor disproven. Recall that given a set A we denote its cardinal by $|A|$ and we denote the existence of an injection $f : A \to B$ by $|A| \leq |B|$. The existence of a bijection from A to B is denoted by $|A| = |B|$.

Statement 2. *(The Continuum Hypothesis) Let A be set with* $\aleph_0 = |\mathbb{N}| \leq |A| \leq |2^{\mathbb{N}}|$. *Then* $|A| = |\mathbb{N}|$ *or* $|A| = |2^{\mathbb{N}}|$. *(Equivalently,* $\aleph_1 = 2^{\aleph_0}$*).*

For an examination of the continuum hypothesis and some of the other set theory terminology in this section, particularly the aleph ($\aleph$) notation, see [2]. Now we can state the partial converse to theorem 17 that we wish to prove.

Theorem 19. *Suppose that V is a valuated vector space over $\mathbb{Z}_p$ (where p is some prime), and suppose that V has a countable basic subspace. Then if V satisfies RCC and we assume the continuum hypothesis, V is free.*

Proof. If V has a countable basic subspace, then every other subspace has a countable basic subspace by an argument similar to the previous section (embedding Ulm spaces). However such a V only has $\aleph_1$ countable subsets. If N is closed in V, then any of one of its basic subspaces B generate N as the closure $\overline{B}$. However, such a countable set has unique closure, and so there can be only as many closed subspaces of V as there are countable subsets of V (this is actually a very rough bound in the finite case!). There are $\aleph_1$ countable subsets of V. Since any finite subset of a basis generates a closed subspace of V, and there are $\aleph_1$ finite subsets of V, we have that there are exactly $\aleph_1$ closed subspaces of V.

Also, there are $\aleph_1$ uncountable closed subspaces. This is because we can remove any of the $\aleph_1$ basis elements for V and get a basis with the same size, which spans a subspace isomorphic to V.

We are going to select a sequence of elements from each uncountable closed subspace in such a way that the sequences are 'independent' in two senses: they are linearly independent subsets of their respective spaces, and any sequence is distinct from the span of the others.

Apply the well-ordering theorem to the uncountable closed subspaces. We label elements with the minimal uncountable ordinal ω_1 as $A_1, A_2, \ldots$. Let A_0 be the smallest element under the ordering. Pick a linearly independent countable sequence from C_0. Let $\mathbf{Y}$ denote the subset of ω_1 containing all those i such that for all $k \leq i$, there exists a set of sequences $\overline{x_k}$ whose union $\cup_{k\leq i}\overline{x_k}$ is linearly independent.

We show that $\mathbf{Y}$ is inductive. If $x \in \omega_1$, then the section $S_x = \{j \in \omega_1 : A_j < A_x\}$ of ω_1 by x is countable because ω_1 is countable. If we take the union $\cup_{n\in\omega_x}\overline{x_n}$ the result is countable. Thus we can take a linearly independent sequence of elements from A_x which is independent from the previous union by calculating $|(\cup_n\overline{x}_n)| = \aleph_0 < \aleph_1$. Since the previous elements are linearly independent, the intersection $I = (\cup_n\overline{x}_n) \cap A_x$ is linearly independent. We expand this to a basis for A_x and take a sequence from the complement of the union. Thus $x \in \mathbf{Y}$. Thus we have a collection $C = \{\overline{x_i} : i \in \omega_1\}$ of countably infinite sequences whose union is linearly independent.

Let their union be B'. This is linearly independent so we expand it to a basis $B = B' \cup B''$ for B'' disjoint from B'.

Now we define for any positive integer j the set $B_j = \{x_{k,i} : k \leq j, i \in \omega_1\}$. The union of the B_j is B', so the sets $U_j = span(B_j \cup B'')$ form a nested subspace cover for V. However, any uncountable subspace is necessarily excluded from these sets U_j, because the uncountable set A_i includes the sequence $x_{i,1}, x_{i,2}, \ldots$ and B_j only contains finitely many of these. The sequences were selected to be linearly independent, so the span of these cannot contain these missing elements of the ith sequence. Also the span of B'' cannot contain these elements by construction of the basis. So no uncountable subspace can be contained in any of these sets B_j. But every nested chain of closed subspaces, if it is to cover uncountable V, must contain an uncountable closed subspace. Why? Countable unions of countable sets are countable.

Thus the RCC is violated, contradicting to our hypothesis that V satisfies RCC. □

38. Conclusion

In this paper, we defined most of the basic notions of linear algebra and set theory, along with some elementary ideas from topology. After obtaining a handful of crucial early results in these areas, we moved on to investigate the class of valuated vector spaces. After developing their elementary theory, we defined the critical notion of free valuated vector spaces. We classified free valuated vector spaces up to isomorphism using a Kulikov-type theorem. After that, we investigated more of the topological properties of valuated vector spaces. Specifically, we defined the refinable chain condition (RCC), and we were able to prove a weak form of it dependent on the continuum hypothesis. After that, we applied the theory of valuated vector spaces to the study of the socles of Abelian p-groups, and we were able to find analogues between free Abelian groups and free valuated vector spaces. Finally, we proved a realization theorem which allowed us to present a free valuated vector space as the socle of an Abelian p-group.

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