THE FIBONACCI NUMBERS

TYLER CLANCY

1. Introduction

The term "Fibonacci numbers" is used to describe the series of numbers generated by the pattern

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144...,$$

where each number in the sequence is given by the sum of the previous two terms.

This pattern is given by $u_1 = 1$, $u_2 = 1$ and the recursive formula

$$u_n = u_{n-1} + u_{n-2},\ n > 2.$$

First derived from the famous "rabbit problem" of 1228, the Fibonacci numbers were originally used to represent the number of pairs of rabbits born of one pair in a certain population. Let us assume that a pair of rabbits is introduced into a certain place in the first month of the year. This pair of rabbits will produce one pair of offspring every month, and every pair of rabbits will begin to reproduce exactly two months after being born. No rabbit ever dies, and every pair of rabbits will reproduce perfecctly on schedule.

So, in the first month, we have only the first pair of rabbits. Likewise, in the second month, we again have only our initial pair of rabbits. However, by the third month, the pair will give birth to another pair of rabbits, and there will now be two pairs. Continuing on, we find that in month four we will have 3 pairs, then 5 pairs in month five, then 8,13,21,34,...,etc, continuing in this manner. It is quite apparent that this sequence directly corresponds with the Fibonacci sequence introduced above, and indeed, this is the first problem ever associated with the now-famous numbers.

Now that we have seen one application of the Fibonacci numbers and established a basic definition, we will go on to examine some of the simple properties regarding the Fibonacci numbers and their sums.

2. Simple Properties of the Fibonacci Numbers

To begin our research on the Fibonacci sequence, we will first examine some simple, yet important properties regarding the Fibonacci numbers. These properties should help to act as a foundation upon which we can base future research and proofs.

The following properties of Fibonacci numbers were proved in the book <u>Fibonacci Numbers</u> by N.N. Vorob'ev.

Lemma 1. *Sum of the Fibonacci Numbers*

The sum of the first n Fibonacci numbers can be expressed as

$$u_1 + u_2 + ... + u_{n-1} + u_n = u_{n+2} - 1.$$

Proof. From the definition of the Fibonacci sequence, we know

$$\begin{aligned} u_1 &= u_3 - u_2, \\ u_2 &= u_4 - u_3, \\ u_3 &= u_5 - u_4, \\ &\vdots \\ u_{n-1} &= u_{n+1} - u_{n+2}, \\ u_n &= u_{n+2} - u_{n+1}. \end{aligned}$$

We now add these equations to find

$$u_1 + u_2 + ... + u_{n-1} + u_n = u_{n+2} - u_2.$$

Recalling that $u_2 = 1$, we see this equation is equivalent to our initial conjecture of

$$u_1 + u_2 + ... + u_{n-1} + u_n = u_{n+2} - 1.$$

□

Lemma 2. *Sum of Odd Terms*
The sum of the odd terms of the Fibonacci sequence

$$u_1 + u_3 + u_5 + ...u_{2n-1} = u_{2n}.$$

Proof. Again looking at individual terms, we see from the definition of the sequence that

$$\begin{aligned} u_1 &= u_2, \\ u_3 &= u_4 - u_2, \\ u_5 &= u_6 - u_4, \\ &\vdots \\ u_{2n-1} &= u_{2n} - u_{2n-2}. \end{aligned}$$

If we now add these equations term by term, we are left with the required result from above. □

Lemma 3. *Sum of Even Terms*
The sum of the even terms of the Fibonacci sequence

$$u_2 + u_4 + u_6 + ...u_{2n} = u_{2n+1} - 1.$$

Proof. From lemma 1, we have

$$u_1 + u_2 + ... + u_{n-1} + u_{2n} = u_{2n+2} - 1.$$

Subtracting our equation for the sum of odd terms, we obtain

$$u_2 + u_4 + ... + u_{2n} = u_{2n+2} - 1 - u_{2n} = u_{2n+1} - 1,$$

as we desired. □

Lemma 4. *Sum of Fibonacci Numbers with Alternating Signs*
The sum of the Fibonacci numbers with alternating signs

$$u_1 - u_2 + u_3 - u_4 + ... + (-1)^{n+1}u_n = (-1)^{n+1}u_{n-1} + 1.$$

Proof. Building further from our progress with sums, we can subtract our even sum equation from our odd sum equation to find

$$u_1 - u_2 + u_3 - u_4 + ... + u_{2n-1} - u_{2n} = -u_{2n-1} + 1. \tag{1}$$

Now, adding u_{2n+1} to both sides of this equation, we obtain

$$u_1 - u_2 + u_3 - u_4 + ... - u_{2n} + u_{2n+1} = u_{2n+1} - u_{2n-1} + 1,$$

or

$$u_1 - u_2 + u_3 - u_4 + ... - u_{2n} + u_{2n+1} = u_{2n} + 1. \tag{2}$$

Combining equations (1) and (2), we arrive at the sum of Fibonacci numbers with alternating signs:

$$u_1 - u_2 + u_3 - u_4 + ... + (-1)^{n+1}u_n = (-1)^{n+1}u_{n-1} + 1.$$

□

Thus far, we have added the individual terms of simple equations to derive lemmas regarding the sums of Fibonacci numbers. We will now use a similar technique to find the formula for the sum of the squares of the first n Fibonacci numbers.

Lemma 5. *Sum of Squares*
The sum of the squares of the first n Fibonacci numbers

$$u_1^2 + u_2^2 + ... + u_{n-1}^2 + u_n^2 = u_n u_{n+1}.$$

Proof. Note that

$$u_k u_{k+1} - u_{k-1}u_k = u_k(u_{k+1} - u_{k-1}) = u_k^2.$$

If we add the equations

$$\begin{aligned} u_1^2 &= u_1 u_2, \\ u_2^2 &= u_2 u_3 - u_1 u_2, \\ u_3^2 &= u_3 u_4 - u_2 u_3, \\ &\vdots \\ u_n^2 &= u_n u_{n+1} - u_{n-1} u_n \end{aligned}$$

term by term, we arrive at the formula we desired. □

Until now, we have primarily been using term-by-term addition to find formulas for the sums of Fibonacci numbers. We will now use the method of induction to prove the following important formula.

Lemma 6. *Another Important Formula*

$$u_{n+m} = u_{n-1}u_m + u_n u_{m+1}.$$

Proof. We will now begin this proof by induction on m. For $m = 1$,

$$\begin{aligned} u_{n+1} &= u_{n-1} + u_n \\ &= u_{n-1}u_1 + u_n u_2, \end{aligned}$$

which we can see holds true to the formula. The equation for $m = 2$ also proves true for our formula, as

$$\begin{aligned} u_{n+2} &= u_{n+1} + u_n \\ &= u_{n-1} + u_n + u_n \\ &= u_{n-1} + 2u_n \\ &= u_{n-1}u_2 + u_n u_3. \end{aligned}$$

Thus, we have now proved the basis of our induction. Now suppose our formula to be true for $m = k$ and for $m = k + 1$. We shall prove that it also holds for $m = k + 2$.

So, by induction, assume

$$u_{n+k} = u_{n-1}u_k + u_n u_{k+1}$$

and

$$u_{n+k+1} = u_{n-1}u_{k+1} + u_n u_{k+2}.$$

If we add these two equations term by term, we obtain

$$\begin{aligned} u_{n+k} + u_{n+k+1} &= u_{n-1}(u_k + u_{k+1}) + u_n(u_{k+1} + u_{k+2}) \\ u_{n+k+2} &= u_{n-1}u_{k+2} + u_n u_{k+3}, \end{aligned}$$

which was the required result. So, by induction we have proven our initial formula holds true for $m = k + 2$, and thus for all values of m. □

Lemma 7. *Difference of Squares of Fibonacci Numbers*

$$u_{2n} = u_{n+1}^2 - u_{n-1}^2.$$

Proof. Continuing from the previous formula in Lemma 7, let $m = n$. We obtain

$$u_{2n} = u_{n-1}u_n + u_n u_{n+1},$$

or

$$u_{2n} = u_n(u_{n-1} + u_{n+1}).$$

Since

$$u_n = u_{n+1} - u_{n-1},$$

we can now rewrite the formula as follows:

$$u_{2n} = (u_{n+1} - u_{n-1})(u_{n+1} + u_{n-1}),$$

or

$$u_{2n} = u_{n+1}^2 - u_{n-1}^2.$$

Thus, we can conclude that for two Fibonacci numbers whose positions in the sequence differ by two, the difference of squares will again be a Fibonacci number. □

Now that we have established a series of lemmas regarding the sums of the Fibonacci numbers, we will take a brief look at some other interesting properties of the Fibonacci numbers.

2.1. **Fibonacci Numbers and Pascal's Triangle.** The Fibanacci numbers share an interesting connection with the triangle of binomial coefficients known as Pascal's triangle.

Pascal's triangle typically takes the form:

$$\begin{array}{lllll} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \\ & & \cdots & & \end{array} \tag{3}$$

In this depiction we have oriented the triangle to the left for ease of use in our future application. Pascal's triangle, as may already be apparent, is a triangle in which the topmost entry is 1 and each following entry is equivalent to the term directly above plus the term above and to the left.

Another representation of Pascal's triangle takes the form:

$$\begin{array}{lllll} C_0^0 & & & & \\ C_1^0 & C_1^1 & & & \\ C_2^0 & C_2^1 & C_2^2 & & \\ C_3^0 & C_3^1 & C_3^2 & C_3^3 & \\ C_4^0 & C_4^1 & C_4^2 & C_4^3 & C_4^4. \end{array} \tag{4}$$

In this version of Pascal's triangle, we have $C_j^i = \frac{k!}{i!(k-i)!}$, where i represents the column and k represents the row the given term is in. Obviously, we have designated the first row as row 0 and the first column as column 0.

Finally, we will now depict Pascal's triangle with its rising diagonals.

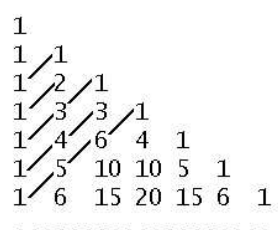


FIGURE 1. Pascal's Triangle with Rising Diagonals

The diagonal lines drawn through the numbers of this triangle are called the "rising diagonals" of Pascal's triangle. So, for example, the lines passing through $1, 3, 1$ or $1, 4, 3$ would both indicate different rising diagonals of the triangle. We now go on to relate the rising diagonals to the Fibonacci numbers.

Theorem 1. *The sum of the numbers along a rising diagonal in Pascal's triangle is a Fibonacci number.*

Proof. Notice that the topmost rising diagonal only consists of 1, as does the second rising diagonal. These two rows obviously correspond to the first two numbers of the Fibonacci sequence.

To prove the proposition, we need simply to show that the sum of all numbers in the $(n-2)^{nd}$ diagonal and the $(n-1)^{st}$ diagonal will be equal to the sum of all numbers in the n^{th} diagonal in Pascal's triangle.

The $(n-2)^{nd}$ diagonal consists of the numbers

$$C^0_{n-3}, C^1_{n-4}, C^2_{n-5}, \ldots$$

and the $(n-1)^{st}$ diagonal has the numbers

$$C^0_{n-2}, C^1_{n-3}, C^2_{n-4}, \ldots$$

We can add these numbers to find the sum

$$C^0_{n-2} + (C^0_{n-3} + C^1_{n-3}) + (C^1_{n-4} + C^2_{n-4}) + \ldots$$

However, for the binomial coefficients of Pascal's triangle,

$$C^0_{n-2} = C^0_{n-1} = 1$$

and

$$\begin{aligned} C^i_k + C^{i+1}_k = & \quad \tfrac{k(k-1)\cdots(k-i+1)}{1\cdot 2\cdots i} \\ + & \quad \tfrac{k(k-1)\cdots(k-i+1)(k-i)}{1\cdot 2\cdots i\cdot(i+1)} \\ = & \quad \tfrac{k(k-1)\ldots(k-i+1)}{1\cdot 2\cdots i}(1 + \tfrac{k-i}{i+1}) \\ = & \quad \tfrac{k(k-1)\cdots(k-i+1)}{1\cdot 2\cdots i} \cdot \tfrac{i+1+k-1}{i+1} \\ = & \quad \tfrac{(k+1)k(k-1)\cdots(k-i+1)}{1\cdot 2\cdots i\cdot(i+1)} \\ = & \quad C^{i+1}_{k+1}. \end{aligned}$$

We therefore arrive at the expression

$$\begin{aligned} & C^0_{n-2} + C^1_{n-2} + C^2_{n-3} + \ldots \\ = \quad & C^0_{n-1} + C^1_{n-2} + C^2_{n-3} + \ldots \end{aligned}$$

to represent the sum of terms of the n^{th} rising diagonal of Pascal's triangle. Indeed, if we look at diagram (4) of Pascal's triangle, this corresponds to the correct expression. Thus, as we know the first two diagonals are both 1, and we now see that the sum of all numbers in the $(n-1)^{st}$ diagonal plus the sum of all numbers in the $(n-2)^{nd}$ diagonal is equal to the sum of the n^{th} diagonal, we have proved that the sum of terms on the n^{th} diagonal is always equivalent to the n^{th} Fibonacci number. □

Example 1. *Let us look at the* 7*th rising diagonal of Pascal's triangle. If we add the numbers* $1, 5, 6,$ *and* 1*, we find that the sum of terms on the diagonal is* 13*. As we know that* $u_7 = 13$*, we can see that the sum of terms on the* 7*th rising diagonal of Pascal's Triangle is indeed equal to the* 7*th term of the Fibonacci sequence.*

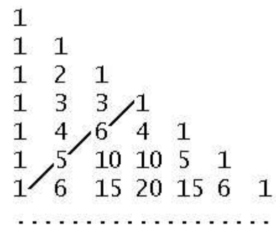


Figure 2. 7th Rising Diagonal of Pascal's Triangle

3. Geometric Properties of the Fibonacci Numbers and the Golden Ratio

3.1. **The Golden Ratio.** In calculating the ratio of two successive Fibonacci numbers, $\frac{u_{n+1}}{u_n}$, we find that as n increases without bound, the ratio approaches $\frac{1+\sqrt{5}}{2}$.

Theorem 2.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1+\sqrt{5}}{2}$$

Proof. Since

$$u_{n+1} = u_n + u_{n-1},$$

by definition, it follows that

$$\frac{u_{n+1}}{u_n} = 1 + \frac{u_{n-1}}{u_n}.$$

Now, let

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L.$$

We then see that

$$\lim_{n\to\infty} \frac{u_{n-1}}{u_n} = \frac{1}{L}.$$

We now have the statement

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1 + \lim_{n\to\infty} \frac{u_{n-1}}{u_n},$$

which is equivalent to the the equation

$$L = 1 + \frac{1}{L}.$$

This equation can then be rewritten as

$$L^2 - L - 1 = 0,$$

which is easily solved using the quadratic formula. By using the quadratic formula, we have

$$L = \frac{1 \pm \sqrt{5}}{2}.$$

Thus, we arrive at our desired result of

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1+\sqrt{5}}{2}.$$

□

Even for relatively low values of n, this ratio produces a very small error. For example

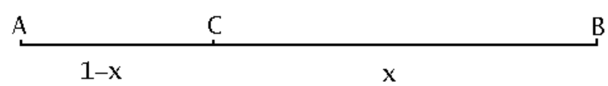
$$\frac{u_{11}}{u_{10}} = \frac{89}{55} \approx 1.6182,$$

and

$$\frac{1+\sqrt{5}}{2} \approx 1.6180.$$

The value $\frac{1+\sqrt{5}}{2}$ is the positive root of the equation $x^2 - x - 1 = 0$ and is often referred to as α. It arises often enough in mathematics and has such interesting properties that we also frequently refer to it as the *golden ratio*. We will now apply this ratio to a few interesting geometric scenarios.

3.2. **The Golden Section.** Let us begin by drawing a line segment, $\overline{AB}$, of length 1 and dividing it into two parts, $\overline{AC}$ and $\overline{CB}$. We will divide this segment such that the ratio of the whole segment to the larger part is equal to the ratio of the larger part to the smaller.



We will denote the length of the larger portion x, while the smaller segment will then obviously be $1 - x$. We have thus produced the proportion:

$$\frac{1}{x} = \frac{x}{1-x},$$

which can be rewritten as

$$x^2 = 1 - x.$$

By using the quadratic formula, we find that the postive root of this equation is $\frac{-1+\sqrt{5}}{2}$, and thus the proportion of the ratios is equal to

$$\frac{1}{x} = \frac{2}{-1+\sqrt{5}} = \frac{2(1+\sqrt{5})}{(-1+\sqrt{5})(1+\sqrt{5})} = \frac{1+\sqrt{5}}{2} = \alpha.$$

As we can see, the resulting ratio is the *golden ratio* we found in the previous section. Furthermore, the division of this line at point C is called the *median section* or *golden section*.

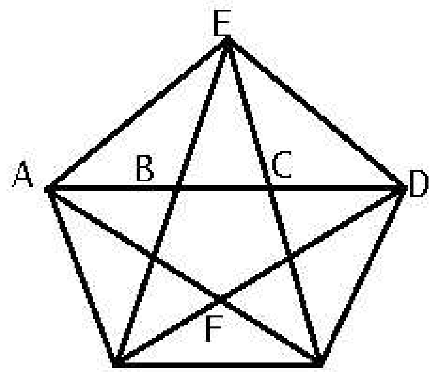


FIGURE 3. A Regular Pentagon with its Diagonals

3.3. **The Golden Ratio and a Regular Pentagon.** Let us now look at a regular pentagon with its diagonals forming a pentagonal star.

From the figure, we see $m\angle AFD$ is $108\,^\circ$, and $m\angle ADF$ is $36\,^\circ$. So, by the sine rule

$$\frac{AD}{AF} = \frac{\sin 108\,^\circ}{\sin 36\,^\circ} = \frac{\sin 72\,^\circ}{\sin 36\,^\circ} = 2\cos 36\,^\circ = 2\frac{1+\sqrt{5}}{4} = \alpha.$$

Obviously, $AF = AC$, so

$$\frac{AD}{AF} = \frac{AD}{AC} = \alpha,$$

and we see that the line segment AD is thus divided at C as a golden section.

From the definition of golden section, we know that $\frac{AC}{CD} = \alpha$, and noting that $AB = CD$, we find

$$\frac{AC}{AB} = \frac{AB}{BC} = \alpha.$$

Thus, we see that of the segments BC,AB,AC, and AD, each is α times greater than the preceding one.

3.4. **A Rectangle and the Golden Ratio.** Let us draw a rectangle in which the sides are to each other as neighboring Fibonacci numbers. If we divide this rectangle into squares, we will see that the side of each square is also equivalent to a Fibonacci number, and the two smallest squares are of the same size. This rectangle is remarkably similar to what is known as a "golden section rectangle," in which the ratio of the sides of the rectangle are equal to α. Using Figure 5, we will now prove that if we inscribe the largest possible square within the golden section rectangle, the resulting space will again be a golden section rectangle.

Since it was our first stipulation, obviously

$$\frac{AB}{AD} = \alpha,$$

and

$$AD = AE = EF,$$

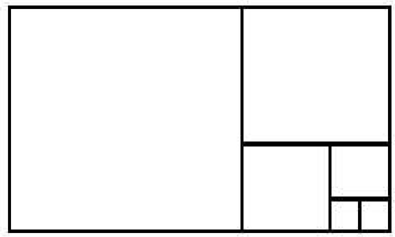


Figure 4. Fibonacci-Based Rectangle

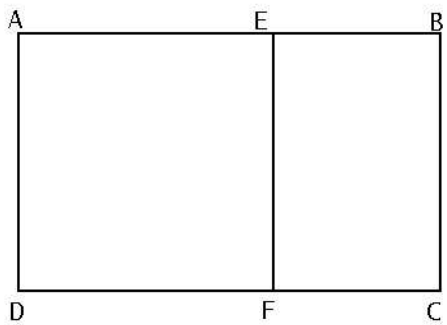


Figure 5. Golden Rectangle with Inscribed Square

since $AEFD$ is a square. So, it follows that

$$\frac{EF}{EB} = \frac{AB - EB}{EB} = \alpha^2 - 1.$$

However, $\alpha^2 - 1 = \alpha$, so we find

$$\frac{EF}{EB} = \alpha.$$

Thus, we see that we do indeed have another golden section rectangle.

It should be obvious that this process of breaking the golden section rectangle down into a series of smaller squares can continue indefinitely. Unlike the golden section rectangle, however, we saw that the rectangle based on Fibonacci numbers did not continue in this inexhaustible manner. Although the ratio of two successive Fibonacci numbers converges towards α, it is not a highly accurate estimate for very low-valued Fibonacci numbers. For this reason, we cannot assume the Fibonacci-based rectangle will continue inexhaustibly, as with the Golden Rectangle. So, as we come to the smallest Fibonacci value, 1, we will find that we have two squares of side length 1, and no more squares can be produced further.

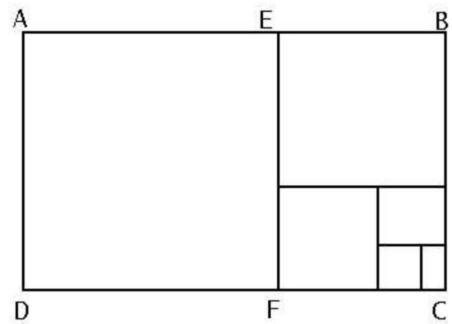
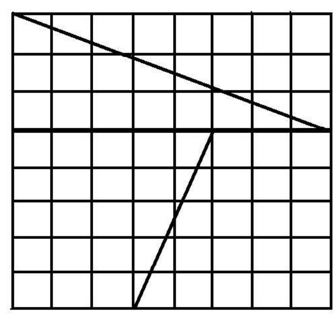
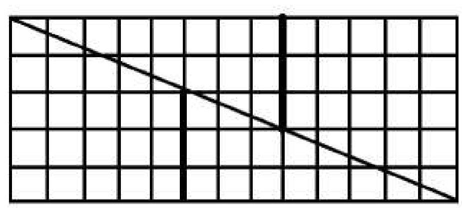


FIGURE 6. Golden Rectangle with Many Inscribed Squares

3.5. **An Interesting Trick.** We shall now go on to "prove" that $64 = 65$.

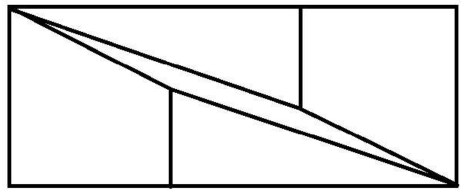


First, let us take an 8x8 square and cut it into four parts, as shown above.



Now, if we rearrange these four parts into a 13x5 rectangle, we see that we now have a total of 65 squares. This does not correspond to our initial value of 64 squares.

The explanation of this dilemna is actually quite simple. While it appears that we correctly realligned the four pieces, the fact is that their vertices do not actually all lie on the same line. If we were to use a larger Fibonacci number to represent the side of our square, we could see that indeed, there is a small gap in between these shapes.



The width of the slit is so miniscule for small Fibonacci numbers that it goes virtually unnoticed.

This trick, while a nice diversion, has little application beyond simple fun.

4. Binet's Formula

Using the method of combinatorics and generating functions, we shall now show that the n^{th} term of the Fibonacci sequence

$$f_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}},$$

which is known as Binet's formula in honor of the mathematician who first proved it.

Since we are proving this formula by means of generating functions, it is important to first give a brief explanation as to what a generating function is.

Definition 1. *A generating function is a function in which the coefficients of a power series give the answers to a counting problem.*

Our method of proving Binet's formula will thus be to find the coefficients of a Taylor series that directly correspond to the Fibonacci numbers.

Proof. By definition, we have $f_n = f_{n-1} + f_{n-2}$, and in this proof we will start with the terms $f_0 = f_1 = 1$. To begin, we shall start with a basic function $f(x)$ giving the general coefficients of the Taylor series.

$$\begin{aligned} f(x) &= f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \dots \\ -x \cdot f(x) &= -f_0 x - f_1 x^2 - f_2 x^3 \dots \\ -x^2 \cdot f(x) &= -f_0 x^2 - f_1 x^3 \dots . \end{aligned}$$

Combining these equations, we find

$$\begin{aligned} f(x) - x \cdot f(x) - x^2 \cdot f(x) &= f_0 + (f_1 - f_0)x \\ &= f_0 \\ &= 1. \end{aligned}$$

Using basic algebra we see

$$f(x) = \frac{1}{1 - x - x^2} = \frac{-1}{x^2 + x - 1}.$$

We will now use the quadratic equation to find the roots of x^2+x-1, which are $x=\frac{-1+\sqrt{5}}{2}$ and $x=\frac{-1-\sqrt{5}}{2}$. Next, we can use the method of partial fractions to break the equation down further.

$$\begin{aligned} f(x) &= \frac{-1}{x^2+x-1} \\ &= \frac{-1}{(x-\frac{-1+\sqrt{5}}{2})(x-\frac{-1-\sqrt{5}}{2})} \\ &= \frac{A}{x-\frac{-1+\sqrt{5}}{2}}+\frac{B}{x-\frac{-1-\sqrt{5}}{2}}. \end{aligned}$$

Solving this equation, we find $A=\frac{-1}{\sqrt{5}}$ and $B=\frac{1}{\sqrt{5}}$. So, we now have the equation

$$\begin{aligned} f(x) &= \frac{-1}{x^2+x-1} \\ &= \frac{-1/\sqrt{5}}{x+\frac{1-\sqrt{5}}{2}}+\frac{1/\sqrt{5}}{x+\frac{1+\sqrt{5}}{2}}. \end{aligned}$$

We shall now use more combinatoric methods to complete this proof. Furthermore, it will first be important to note that

$$\binom{-1}{k}=\frac{(-1)(-2)(-3)\dots(-1-k+1)}{k!}=(-1)^k.$$

Let us now proceed to finish this proof using combinatorics.

$$\begin{aligned} f(x) &= \frac{\frac{-1}{\sqrt{5}}}{x+\frac{1-\sqrt{5}}{2}}+\frac{\frac{1}{\sqrt{5}}}{x+\frac{1+\sqrt{5}}{2}} \\ &= \frac{-1}{\sqrt{5}}\sum_{k=0}^{\infty}\binom{-1}{k}x^k\left(\frac{1-\sqrt{5}}{2}\right)^{-1-k}+\frac{1}{\sqrt{5}}\sum_{k=0}^{\infty}\binom{-1}{k}x^k\left(\frac{1+\sqrt{5}}{2}\right)^{-1-k} \\ &= \frac{1}{\sqrt{5}}\sum_{k=0}^{\infty}\left[(-1)^{k+1}\left(\frac{1-\sqrt{5}}{2}\right)^{-1-k}+(-1)^k\left(\frac{1+\sqrt{5}}{2}\right)^{-1-k}\right]x^k \\ &= \frac{1}{\sqrt{5}}\sum_{k=0}^{\infty}\left[\left(\frac{-2}{1-\sqrt{5}}\right)^{k+1}-\left(\frac{-2}{1+\sqrt{5}}\right)^{k+1}\right]x^k. \end{aligned}$$

From this work, we see that the k^{th} coefficient of x^k is equal to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}.$$

As this generating function was calculated for the recursive formula $f_n=f_{n-1}+f_{n-2}$, this value also corresponds to the k^{th} term of the Fibonacci sequence. Furthermore, if we let $f_1=f_2=1$ rather than $f_0=f_1=1$, we can further simplify this equation to our initial formula of

$$f_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}.$$

Thus, we have proven Binet's formula using the method of combinatorics.

□

5. Using Logarithmic Tables to Calculate Fibonacci Numbers

Theorem 3. *The Fibanacci number u_n is the nearest whole number to the nth term α_n of the geometric progression whose first term is $\frac{\alpha}{\sqrt{5}}$ and whose common ratio is α.*

That is, u_n is the nearest whole number to $\alpha_n = \frac{\alpha^n}{\sqrt{5}}$.

Proof. In proving this theorem, it is sufficient to show that the absolute value of the difference between u_n and a_n will always be less than $\frac{1}{2}$. Let α and β be equal to $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$, respectively, representing the roots of the equation $x^2 - x - 1 = 0$, which we introduced earlier. It is also important to note that $u_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$, from Binet's formula. Then

$$|u_n - \alpha_n| = \frac{\alpha^n - \beta^n}{\sqrt{5}} - \frac{\alpha^n}{\sqrt{5}} = |\frac{\alpha^n - \alpha^n - \beta^n}{\sqrt{5}}| = \frac{|\beta|^n}{\sqrt{5}}.$$

As $\beta = -.618\ldots$, obviously $|\beta| < 1$. So, for any n, $|\beta|^n < 1$, and since $\sqrt{5} > 2$, $\frac{|\beta|}{\sqrt{5}} < \frac{1}{2}$. Thus, we have proven our theorem. □

Now, using this theorem, we can go on to calculate the Fibonacci numbers by using a logarithmic table.

For example, let us calculate u_{13}.

Example 2.

$$\sqrt{5} \approx 2.2361, \quad log\ \sqrt{5} \approx .34949;$$

$$\alpha = \frac{1+\sqrt{5}}{2} \approx 1.6180, \quad log\ \alpha \approx .20898;$$

$$log\ \frac{\alpha^{13}}{\sqrt{5}} = 13 \cdot .20898 - .34949 = 2.36725, \quad \frac{\alpha^{13}}{\sqrt{5}} \approx 232.94318.$$

The closest whole number to 232.94318 is 233, which is indeed u_{13}, the 13th term of the Fibonacci sequence.

While it is not necessary to use *log*s to make this calculation, it allows us to approximate u_n for very large values of n. In most cases a large n value would prohibit us from calculating $u_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ without using a program such as Maple, but using logarithms will at least allow us to evaluate how many digits are in u_n. It is important to note that when we calculate Fibonacci numbers with very large suffixes, we can no longer rely upon available tables of logarithms to calculate all the figures of the number; we can only indicate the first few figures of the number, and the calculation is only approximate.

6. Fibonacci Numbers Under Modular Representation

6.1. **Introduction to Modular Representation.** We will now examine the Fibonacci numbers under modular addition.

First, we will familiarize ourselves with modulo notation. Given the integers a, b and m, the expression $a \equiv b(mod\ m)$ (pronounced "a is congruent to b modulo n") means that $a - b$ is a multiple of m. For $0 \leq a < n$, the value a is equivalent to the remainder, or residue, of b upon division by n.

So, for example,

$$3 \equiv 13(mod\ 10),$$

or

$$2 \equiv 17(mod\ 5).$$

It is also convenient to note that

$$a(mod\ m) + b(mod\ m) = (a + b)(mod\ m).$$

Subtraction and multiplication work similarly.

Now that we are comfortable with basic modular operations, we shall examine an example of the first 12 Fibonacci numbers $(mod\ 2)$.

Example 3. *Fibonacci numbers:*

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144...$$

Fibonacci numbers $(mod\ 2)$*:*

$$1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0...$$

It should be apparent that only the pattern of $1, 1, 0$ repeats throughout the Fibonacci series $(mod\ 2)$. So, we can say that the series is periodic, with the period being 3 in this case, since there is a repetition of only three terms. We will later go on to prove that all modular representaions of the Fibonacci numbers are periodic. Furthermore, we will show that this period is solely determined by the two numbers directly following the first 0 within the series.

6.2. **The Fibonacci Numbers Modulo m.** Before attempting to prove any major conclusions about the Fibonacci numbers modulo m, it may help us to first examine the Fibonacci series for many values of m. Let us look at the first 30 terms of the Fibonacci series $(mod\ m)$, where m ranges from 2 to 10.

First of all, the first 30 Fibonacci numbers are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040.$$

So, by using Maple, we find

$$\begin{aligned}
F(mod\ 2) &= 1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0\\
F(mod\ 3) &= 1,1,2,0,2,2,1,0,1,1,2,0,2,2,1,0,1,1,2,0,2,2,1,0,1,1,2,0,2,2\\
F(mod\ 4) &= 1,1,2,3,1,0,1,1,2,3,1,0,1,1,2,3,1,0,1,1,2,3,1,0,1,1,2,3,1,0\\
F(mod\ 5) &= 1,1,2,3,0,3,3,1,4,0,4,4,3,2,0,2,2,4,1,0,1,1,2,3,0,3,3,1,4,0\\
F(mod\ 6) &= 1,1,2,3,5,2,1,3,4,1,5,0,5,5,4,3,1,4,5,3,2,5,1,0,1,1,2,3,5,2\\
F(mod\ 7) &= 1,1,2,3,5,1,6,0,6,6,5,4,2,6,1,0,1,1,2,3,5,1,6,0,6,6,5,4,2,6\\
F(mod\ 8) &= 1,1,2,3,5,0,5,5,2,7,1,0,1,1,2,3,5,0,5,5,2,7,1,0,1,1,2,3,5,0\\
F(mod\ 9) &= 1,1,2,3,5,8,4,3,7,1,8,0,8,8,7,6,4,1,5,6,2,8,1,0,1,1,2,3,5,8\\
F(mod\ 10) &= 1,1,2,3,5,8,3,1,4,5,9,4,3,7,0,7,7,4,1,5,6,1,7,8,5,3,8,1,9,0.
\end{aligned}$$

6.3. **The Period of F(mod m).**

Lemma 8. *The Fibonacci series under modular representation is always periodic.*

Proof. Since the Fibonacci series is recursive, we know that any pair of consecutive terms will completely determine the rest of the series. Furthermore, under modular representation, we know that each Fibonacci number will be represented as some residue $0 \leq F(mod\ m) < m$. Thus, there are only m possible values for any given $F(mod\ m)$ and hence $m \cdot m = m^2$ possible pairs of consecutive terms within the sequence. Since m^2 is finite, we know that some pair of terms must eventually repeat itself. Also, as any pair of terms in the Fibonacci sequence determines the rest of the sequence, we see that the Fibonacci series modulo m must repeat itself at some point, and thus must be periodic. □

Now, we will let $k(m)$ denote the period of $F(mod\ m)$. That is, $k(m)$ represents the number of terms of $F(mod\ m)$ before the cycle starts to repeat again.

So, analyzing the above data, we can determine the period of all but the last series.

$$\begin{aligned}
k(2) &= 3\\
k(3) &= 8\\
k(4) &= 6\\
k(5) &= 20\\
k(6) &= 24\\
k(7) &= 16\\
k(8) &= 12\\
k(9) &= 24.
\end{aligned}$$

While it appears there may be some connection between these values, it may be more convenient to analyze a larger sample size. To attain this larger sample size, we will make use of a list of periods of $F(mod\ m)$ given by Marc Renault, associate professor of Mathematics at Shippensburg University. From this list, we find:

$$k(10) = 60$$
$$k(11) = 10$$
$$k(12) = 24$$
$$k(13) = 28$$
$$k(14) = 48$$
$$k(15) = 40$$
$$k(16) = 24$$
$$k(17) = 36$$
$$k(18) = 24$$
$$k(19) = 18$$
$$k(20) = 60.$$

With these values, we can begin to analyze patterns and formulate some hypotheses relating to the period of $F(mod\ m)$.

6.4. **Important Properties of k(m).** Before we approach some of the more complicated properties regarding the period of the Fibonacci numbers modulo m, it will be helpful to introduce and prove some general properties. These results are not only interesting in themselves, but will help us in our future proofs.

The following proofs were detailed by Marc Renault in his master's thesis.

Let us first note that it is sometimes convenient to extend the Fibonacci sequence backward by using negative subscripts. So, the Fibonacci recurrence relation can be written as $u_n = u_{n+2} - u_{n+1}$, which will allow us to use this notation.

The following chart will illustrate this new notation:

n value	u_n
-5	5
-4	-3
-3	2
-2	-1
-1	1
0	0
1	1
2	1
3	2
4	3
5	5.

Inspecting this list, as well as u_n for other values of n, we are presented with the following identity.

Identity 1. $u_{-n} = (-1)^{n+1}u_n$.

Using this identity, we can prove our first theorem regarding $k(m)$, the period of $F(mod\ m)$.

For ease of notation, let $k = k(m)$ and let all congruences be taken *mod m*. That is, let $u_n \equiv u_n(mod\ m)$.

Theorem 4. *For $m > 2$, $k(m)$ is even.*

Proof. From Identity 1, we know $u_n = u_{-n}$ when t is odd and $u_n = -u_{-n}$ when n is even. We will now assume k is odd and prove that m must equal 2.

We know $u_1 = u_{-1} \equiv u_{k-1}$. Now, since $k-1$ is even, $u_{k-1} = -u_{1-k} \equiv -u_1$. Thus, $u_1 \equiv -u_1$ and so we see $n = 2$. Since m must equal 2 for odd values of k, we see that all other k values must be even. □

Theorem 5. *If $j|m$, then $k(j)|k(m)$.*

Proof. Let $k = k(m)$. We will now show that $F(mod\ j)$ must repeat in blocks of length k. We shall do this by showing that $u_i \equiv u_{i+k}(mod\ j)$ for any integer i. We already know that $u_i \equiv u_{i+k}(mod\ m)$, so for some $0 \leq a < m$, there exist u_i and u_{i+k} such that $u_i = a + mx$ and $u_{i+k} = a + my$, for some x, y.

Now assume $m = jr$ and substitute accordingly in the above equations. We now have $u_i = a + jrx$ and $u_{i+k} = a + jry$. We can also say that $a = a' + jw$ (for $0 \leq a' < m$) and again substitute this value into our equation. Now, $u_i = a' + j(w + rx)$ and $u_{i+k} = a' + j(w + ry)$. This implies that $u_i \equiv u_{i+k}(mod\ j)$, and hence we have proven our theorem. □

Now that we have intruduced some basic identities and theorems regarding the period of $F(mod\ m)$, we will proceed to analyze the results from our list of periods. After forming some hypotheses from our information, we will go on to prove rules regarding $k(m)$ for certain m values.

We first consider values of m where m is the product of distinct primes ($m = r \cdot s \cdot t \cdots$, for r, s, t distinct primes). Analyzing our data, there is an apparent pattern developing. It seems as if for any product of primes m, $k(m)$ is equivalent to the least common multiple of $k(r)$, $k(s)$, $k(t)$,. . . .

So, $k(r \cdot s \cdot t \cdots) = lcm(k(r), k(s), k(t), \dots)$.

For example,

$$k(6) = k(2 \cdot 3) = 24 = lcm(3, 8) = lcm(k(2), k(3)),$$

or

$$k(15) = k(3 \cdot 5) = 40 = lcm(8, 20) = lcm(k(3), k(5)).$$

If we proceed further down Professor Renault's list, we can see that this hypothesis continues to hold true even for larger values of n. For example,

$$\begin{aligned} k(210) = k(2 \cdot 3 \cdot 5 \cdot 7) = 240 &= lcm(3, 8, 20, 16) \\ &= lcm(k(2), k(3), k(5), k(7)). \end{aligned}$$

Furthermore, the product of powers of primes seems to work in much the same way. In fact, it appears that, for $n = p_1^{e_1} \cdots p_i^{e_i}$, $k(m) = lcm(p_1^{e_1}, \cdots, p_i^{e_i})$. As an example, we see this formula holds for

$$k(400) = k(16 \cdot 25) = k(2^4 \cdot 5^2) = 600 = lcm(24, 100) = lcm(k(16), k(25)).$$

These results lead us to our first important theorem regarding the period of m, $k(m)$.

Theorem 6. *Let m have the prime factorization $n = \Pi p_i^{e_i}$. Then $k(m) = lcm[k(p_i^{e_i})]$, the least common multiple of the $k(p_i^{e_i})$.*

Proof. From our previous theorem, we know $k(p_i^{e_i})|k(n)$ for all i. It follows that $lcm[k(p_i^{e_i})]|k(m)$. Now, since $k(p_i^{e_i})|lcm[k(p_i^{e_i})]$, we know that $F(mod\ p_i^{e_i})$ repeats in blocks of length $lcm[k(p_i^{e_i})]$. So, $F_{lcm[k(p_i^{e_i})]} \equiv F_0$ and $F_{lcm[k(p_i^{e_i})]+1} \equiv$

$F_1(mod\ p_i^{e_i})$ for all i. Since all the $p_i^{e_i}$ are relatively prime, a theorem known as the Chinese Remainder Theorem shows us that $F_{lcm[k(p_i^{e_i})]} \equiv F_0$ and $F_{lcm[k(p_i^{e_i})]+1} \equiv F_1(mod\ m)$. Thus $F(mod\ m)$ repeats in blocks of length $lcm[k(p_i^{e_i})]$ and we see that $k(m)|lcm[k(p_i^{e_i})]$. This concludes our proof. □

This result leads us to another similar, important theorem.

Theorem 7. $k[lcm(m,j)] = lcm[k(m), k(j)]$.

Proof. As $n|lcm(m,j)$ and $m|lcm(m,j)$, we know that $k(m)|k[lcm(m,j)]$ and $k(j)|k[lcm(m,j)]$. It follows that $lcm[k(m), k(j)]|k[lcm(m,j)]$.

Now say we have the prime factorization $lcm(m,j) = p_1^{e_1} \cdots p_t^{e_t}$. Then we know $k[lcm(m,j)] = k(p_1^{e_1} \cdots p_t^{e_t}) = lcm[k(p_1^{e_1}), \ldots, k(p_t^{e_t})]$. Since $p_i^{e_i}$ divides m or j for all i, $k(p_i^{e_i})$ must divide $k(m)$ or $k(j)$ for all i. Thus $lcm[k(p_1^{e_1}), \ldots, k(p_t^{e_t})]|lcm[k(m), k(j)]$, or $k[lcm(m,j)]|lcm[k(m), k(j)]$. Thus, we see that $k[lcm(m,j)] = lcm[k(m), k(j)]$. □

An example of this theorem follows:

$$k[lcm(6,8)] = k(24) = lcm(12,14) = lcm[k(6), k(8)].$$

Now, let us proceed to m values that are squares of primes. So, let $m = p^2$ for some prime p. Then, it appears as if $k(m) = p \cdot k(p)$. For instance,

$$k(49) = 112 = 7 \cdot 16 = 7 \cdot k(7).$$

Furthermore, if n is equivalent to any power of a prime number, that is $n = p^i$, we can see that $k(n) = p^{i-1} \cdot k(p)$. So

$$k(16) = 24 = 2^3 \cdot 3 = 2^3 \cdot k(2)$$

and

$$k(125) = 500 = 5^2 \cdot 20 = 5^2 \cdot k(5).$$

This trend appears to hold for all values of p and i. In fact, a closely-related theorem regarding these values does exist.

Theorem 8. *If t is the largest integer such that $k(p^t) = k(p)$, then $k(p^e) = p^{e-t}k(p)$ for all $e > t$.*

Proof. Insert proof □

The conjecture that $t = 1$ for all primes has existed since 1960, yet there are still no proofs nor counterexamples to completely prove or disprove this hypothesis.

Now that we have established some rules relating to the period of $F(mod\ m)$, we will go on to introduce some new definitions regarding the zeros of $F(mod\ m)$. These new concepts will lead to more interesting features of the Fibonacci numbers under modular representation.

7. The Zeros of $F(mod\ m)$

The following section comes from an article by Marc Renault, associate professor of mathematics at Shippensburg University.

Definition 2. *Let $a(m)$ denote the index of the first Fibonacci number divisible by m. Equivalently, this will also be the position of the first zero in the sequence of $F(mod\ m)$. We call this the restricted period of $F(mod\ m)$.*

Definition 3. *Let $s(m)$ denote the residue that appears after the first zero in $F(mod\ m)$. We will also refer to this as the multiplier of $F(mod\ m)$.*

Definition 4. *Let $b(m)$ denote the order of $s(m)$ modulo m.*

As an example, we will now examine these values for the sequence $F(mod\ 7)$.

Example 4.

$F(mod\ 7) = 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6...$

So the period, $k(7) = 16$.
The restricted period, $a(7) = 8$.
The multiplier, $s(7) = 6$.
The order of 6, mod 7, i.e. $b(7) = 2$, as $6^2 \equiv 1(mod\ 7)$, yet $6^1 \not\equiv 1(mod\ 7)$.

Now that we are familiar with these new terms, we shall show that $k(m) = a(m)b(m)$.

For ease of notation, let $k(m) = k$, $a(m) = a$, $s(m) = s$, and $b(m) = b$.
Let G_j denote the sequence $F(mod\ m)$, starting with the j^{th} term of $F(mod\ n)$.
So, for example, $G_0 = 0, 1, 1, \ldots$, whereas $G_a = 0, s, s, \ldots$.
Essentially, we see that G_a is equivalent to G_o, but with every term multiplied by s. So, we can write $G_a = (s)G_0$.
Similarly, we can write $G_{2a} = (s)G_a = (s^2)G_0$.
We eventually arrive at the conclusion that $G_{ba} = (s^b)G_0 = G_0$, as b is the order of s. Also, since b is the order of s, it follows that $ab = k$.

If we inspect our list for $F(mod\ 7)$ again, we can better illustrate these new points.

Example 5. *Clearly $G_0 = 0, 1, 1, 2, 3\ldots$.*
Furthermore, $G_a = G_8 = 0, 6, 6, 5, 4\ldots$, which we see is equivalent to $(6)G_0 = (s)G_0$. Also, $G_{2a} = G_{16} = 0, 1, 1, 2, \ldots$, and as $36 \equiv 1(mod\ 7)$, we see that indeed, $G_{2a} = (36)G_0 = (s^2)G_0$.
Finally, since $a = 8$, $b = 2$, and $k = 16$, it is clear that $ab = k$ in this example.

8. The Lucas Numbers and L(mod m)

Similar to the Fibonaci numbers, there exists another interesting group of numbers known as the Lucas numbers. Like the Fibonacci numbers, each term of the Lucas numbers is found by computing the sum of the previous two terms. However, the Lucas numbers start with the terms $L_0 = 2$,and $L_1 = 1$, instead of $F_0 = 1$ and $F_1 = 1$.

A list of the first 30 Lucas numbers is

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, 5778, 9349, 15127, 24476, 39603, 64079, 103682, 167761, 271443, 439204, 710647, 1149851\ldots$$

Now, in a similar manner to the Fibonacci numbers, we will compute $L(mod\ m)$ for the first 30 Lucas numbers, for values of m ranging from 2 to 10.

$$L(mod\ 2) = 0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,1,0,1,0,1,1,0,1,1,0,1,1,0$$
$$L(mod\ 3) = 2,1,0,1,1,2,0,2,2,1,0,1,1,2,0,2,2,1,0,1,1,2,0,2,2,1,0,1,1,2$$
$$L(mod\ 4) = 2,1,3,0,3,3,2,1,3,0,3,3,2,1,3,0,3,3,2,1,3,0,3,3,2,1,3,0,3,3$$
$$L(mod\ 5) = 2,1,3,4,2,1,3,4,2,1,3,4,2,1,3,4,2,1,3,4,2,1,3,4,2,1,3,4,2,1$$
$$L(mod\ 6) = 2,1,3,4,1,5,0,5,5,4,3,1,4,5,3,2,5,1,0,1,1,2,3,5,2,1,3,4,1,5$$
$$L(mod\ 7) = 2,1,3,4,0,4,4,1,5,6,4,3,0,3,3,6,2,1,3,4,0,4,4,1,5,6,4,3,0,3$$
$$L(mod\ 8) = 2,1,3,4,7,3,2,5,7,4,3,7,2,1,3,4,7,3,2,5,7,4,3,7,2,1,3,4,7,2$$
$$L(mod\ 9) = 2,1,3,4,7,2,0,2,2,4,6,1,7,8,6,5,2,7,0,7,7,5,3,8,2,1,3,4,7,2$$
$$L(mod\ 10) = 2,1,3,4,7,1,8,9,7,6,3,9,2,1,3,4,7,1,8,9,7,6,3,9,2,1,3,4,7,1$$

Inspecting this list, we find that the period of $L(mod\ m)$ is not quite as easy to identify as the period of $F(mod\ m)$. Although it appears that a period still exists for the same reason as the Fibonacci numbers, we can no longer look for the simple repetition of $0,1,1,\ldots$ that was found in the sequence $F(mod\ m)$ and acted as an indicator for the repetition of the period of $F(mod\ m)$. However, we can see that for $n > 2$, the pair $2,1$ acts as an indicator for the period of $L(mod\ m)$. Also, by the same means as with the Fibonacci numbers we can still show that the sequence $L(mod\ m)$ must be periodic for all m.

Theorem 9. *The Lucas series under modular representation is always periodic.*

Proof. Let us take any term from the Lucas sequence $(mod\ m)$. There are a total of m options for what the value of this term may be. Similarly, there are exactly m options for the term directly follwing. Thus, there are m^2 possibilities for any two consecutive terms in the sequence $L(mod\ m)$. Since m^2 is obviously a finite value, we know that there are finite options for any two consecutive terms in $L(mod\ m)$. As there are finite options for any pair of terms in the sequence, we know that some pair of terms must repeat at some point. Also, since any two consecutive terms determine the rest of the Lucas sequence, we see that once a pair of terms is at some point repeated, so is the rest of the sequence. Thus, $L(mod\ m)$ must be periodic. □

We know that for all values of m, $L(mod\ m)$ is periodic. Furthermore, we can examine the above sequences to find the period of each sequence. As before with the Fibonacci numbers, let $k(m)$ denote the period of $L(mod\ m)$.

So,

$$k(2) = 3, k(3) = 8, k(4) = 6, k(5) = 4, k(6) = 24, k(7) = 16, k(8) = 12, k(9) = 24, k(10) = 12.$$

Although we are again left with a small sample size, it appears that our previous rules for $F(mod\ m)$ remain even for the sequence $L(mod\ m)$. In fact, the following rules will hold not only for the Fibonacci numbers as well as the Lucas numbers, but for any generalized Fibonacci sequence. That is, a sequence of the form $g_{m+2} = g_m + g_{m+1}$.

Theorem 10. *For $m > 2$, $k(m)$ is even.*

Theorem 11. *If $j|m$, then $k(j)|k(m)$.*

Theorem 12. *Let m have the prime factorization $m = \Pi p_i^{e_i}$. Then $k(m) = lcm[k(p_i^{e_i})]$, the least common multiple of the $k(p_i^{e_i})$.*

Theorem 13. $k[lcm(m,j)] = lcm[k(m), k(j)]$.

Theorem 14. *If t is the largest integer such that $k(p^t) = k(p)$, then $k(p^e) = p^{e-t}k(p)$ for all $e > t$.*

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