PIANO TUNING AND CONTINUED FRACTIONS

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ABSTRACT. In this paper, we first establish algorithms for creating continued fractions representing rational numbers. From there, we prove that infinitely long fraction expressions represent irrational numbers, along with methods for rationally approximating these numbers. As we analyze the effectiveness of any given approximation, we provide examples for finding these numbers. Next, we use of these fractions to evaluate how pianos are tuned and why one cannot be tuned perfectly. We focus mainly on the most common way to tune pianos in Western music, but will briefly explore alternate scales, such as the one used in Chinese music. Finally, we conclude the paper with a discussion of theoretical alternative scales to the ones in place, and why the ones that are used are the most popular.

1. INTRODUCTION

While continued fractions have been studied for over 2000 years, most of the earliest examples were not generalized into useful, universal theorems. For instance, the Indian mathematician Aryabhata recorded his use of continued fractions in 550 A.D. when solving linear equations with infinitely many solutions. However, his use of these interesting mathematical expressions was limited to the specific problems he was solving. Furthermore, ancient Greek and Arab mathematical documents are covered with instances of continued fractions. However, like the ones Aryabhata used, early mathematicians did not expand into more generalized theorems. In the sixteenth and seventeenth centuries, examples of continued fractions that resemble the ones we know of today began to arise, specifically as Rafael Bombelli discovered that the square root of 13 could be expressed as a continued fraction. Pietro Cataldi did the same thing just years later with the square root of 18. Eventually, throughout the seventeenth and eighteenth centuries, John Wallis in his work *Arithemetica Infinitorium*, Christiaan Huygens in his astronomical research, and Leonhard Euler in his work *De Fractionlous Continious* established the theorems we know about continued fractions today.[1]

Currently, continued fractions have many practical uses in mathematics. For instance, we can express any number, rational or irrational, as a finite or infinite continued fraction expression. We can also solve any Diophantine Congruence, that is any equivalence of the form $Ax = B(\mod M)$. In terms of practical applications, continued fractions tend to "suffer from poor performance"[2]. In other words, in most real-world applications of mathematics, continued fractions are rarely the most practical way to solve a given set of problems as decimal approximations are much more useful (and computers can work with decimals at a much faster rate). However, some interesting observations can still be made using continued fractions. Namely, in this project, we will be exploring how continued fractions can be used to analyze the tuning of pianos.

2. Setting up Our Problem

In the music world, it is well known that it is impossible to tune a piano perfectly. Starting with two very common scales used to tune pianos, we explore how the frequencies of the various notes in these scales are calculated, and why all calculations for these frequences have margins of error - which actually make a huge difference to the sound a piano makes.

The heart of the problem, as we discover, actually lies in basic number theory: after we calculate the frequencies of our given keys, it turns out that in order to tune a piano perfectly, there needs to exist a solution to the following equation:

$$2^x = 3^y.$$

But there is no rational solution to this equation. We can approximate an irrational solution, however, with the use of continued fractions.

So before we delve into any music theory, we first establish the existence and use of continued fractions in a general form. Namely, we can explore how the Euclidean Algorithm is used to establish continued fraction expressions for rational numbers. Eventually, we move onto infinite continued fractions and realize that these expressions actually represent irrational numbers. Next, we study the accuracy of different continued fraction approximations. Finally, we use continued fractions to analyze the frequencies to which piano keys are set.

3. The Euclidean Algorithm and Continued Fractions [3]

A tool used to find the greatest common divisor of two numbers, the Euclidean Algorithm, is also used to establish general constructions of finite continued fractions. To begin, the following two algorithms are needed. **Division Algorithm**: Let a, b be integers such that $b \neq 0$; then there exist unique integers s and t such that $a = sb + t$ where $t < |b|$ and $t \geq 0$.

Euclidean Algorithm: Given any integers u_0, u_1 such that u_0, $u_1 > 0$, we can repeatedly apply the division algorithm using our remainders and divisors in the following manner:

$$\begin{gathered} u_0 = u_1 a_0 + u_2, 0 \leq u_2 < u_1 \\ u_1 = u_2 a_1 + u_3, 0 \leq u_3 < u_2 \\ u_2 = u_3 a_2 + u_4, 0 \leq u_4 < u_3 \\ \vdots \\ u_{j-1} = u_j a_{j-1} + u_{j+1}, 0 \leq u_{j+1} < u_j \\ u_j = u_{j+1} a_j \end{gathered}$$

It has been proven that u_{j+1} is the greatest common divisor between u_0 and u_1 [3]; however, we do not use the gcd in our work with continued fractions, so the proof has been omitted.[3]

The Division Algorithm formally establishes the basic process of long division, with a representing the dividend, s representing the divisor, and t representing the

remainder. The Euclidean Algorithm is the repeated application of the Division Algorithm using the remainders found for each u_i.
In terms of establishing continued fractions, we first explore the continued fraction expression of any rational number. Taking an arbitrary rational number of the form u_i/u_{i+1}, we apply the Division Algorithm and label the remainder between u_i and u_{i+1} as u_{i+2}, as follows:

$$u_i = u_{i+1}a_i + u_{i+2}.$$

¿From here we get a sequence of u_is, and we denote any ratio u_i/u_{i+1} as Φ_i.

Next, we divide both sides by u_{i+1} in order to obtain

$$\frac{u_i}{u_{i+1}} = a_i + \frac{u_{i+2}}{u_{i+1}}.$$

Substituting using our Φ notation, we have determined that

$$\Phi_i = a_i + \frac{1}{\Phi_{i+1}}, 0 \leq i \leq j-1.$$

If we apply the Euclidean Algorithm to our expression u_i/u_{i+1} and define $u_j = u_{j+1}a_j$ where $u_{j+2} = 0$, we know that the Φ_i identity above holds for any u term in the algorithm. Finally, knowing that $u_j = u_{j+1}a_j$, we can divide both sides by u_{j+1} in order to get that $\Phi_j = a_j$.

Keeping in mind the Euclidean Algorithm expansion and Φ notation of our generic rational number, we want to expand our arbitrary rational number in Φ notation:

$$\frac{u_0}{u_1} = \Phi_0 = a_0 + \frac{1}{\Phi_1}.$$

If we evaluate our Φ expressions, we obtain:

$$\Phi_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\Phi_2}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\Phi_3}}}$$

If we expand this until we reach Φ_j, we can express u_0/u_1 as

$$a_0 + \cfrac{1}{a_1 + \cfrac{}{\ddots + \frac{1}{a_{j-1} + \frac{1}{a_j}}}} \tag{1}$$

Expression 1 is called the *continued fraction expansion* of the rational number u_0/u_1, or Φ_0. Further, the integers a_0 through a_j are called *partial quotients* (remember, in each application of the division algorithm each respective a_i is the quotient of its given expression). Finally, for notation's sake, we can then express our continued fraction expansion for Φ_0 as $[a_0, a_1, \cdots, a_{j-1}, a_j]$.

We summarize the ideas of this section into a theorem as follows.

Theorem 3.1. *Any rational number of the form* $\frac{u_0}{u_1}$ *where the gcd*$(u_0, u_1) = 1$ *can be expressed as the continued fraction denoted* $[a_0, a_1, \cdots, a_{j-1}, a_j]$ *where every* a_i *is the partial quotient from the* $(i-1)^{st}$ *step of the Euclidean Algorithm on* $\frac{u_0}{u_1}$.

4. Uniqueness [3]

Now that we have determined an algorithm for determining a continued fraction expression for any rational number, we now want to determine whether or not the expansions we are using are unique. Observe that when applying the Euclidean Algorithm to the number 51/22, we obtain the following:

$$51 = 22(2) + 7$$
$$22 = 7(3) + 1$$
$$7 = 1(7).$$

We know that the continued fraction expression of 51/22 is $[2, 3, 7]$. Just to double check our work, the continued fraction can be evaluated as follows:

$$2 + \cfrac{1}{3 + \cfrac{1}{7}} = 2 + \frac{1}{22/7} = 2 + \frac{7}{22} = \frac{51}{22}.$$

Now we examine the continued fraction expansion of $[2, 3, 6, 1]$:

$$2 + \cfrac{1}{3 + \cfrac{1}{6 + 1/1}} = 2 + \cfrac{1}{3 + \cfrac{1}{7}} = \cdots = \frac{51}{22}.$$

Although this may seem a bit trivial, as in our example, we should note that

$$[a_0, a_1, \cdots, a_{j-1}, a_j] = [a_0, a_1, \cdots, a_{j-1}, a_j - 1, 1].$$

As proof, note that the final denominator of our continued fraction expression of $[a_0, a_1, \cdots, a_{j-1}, a_j - 1, 1]$ is

$$a_j - 1 + \frac{1}{1} = a_j - 1 + 1 = a_j.$$

Theorem 4.1 provides us with a more formal statement for continued fraction uniqueness.

Theorem 4.1. *If $[a_0, a_1, \cdots, a_{j-1}, a_j] = [b_0, b_1, \cdots, b_{n-1}, b_n]$ where both finite continued fractions are simple, and if $a_j > 1$ and $b_n > 1$, then $j = n$ and $a_i = b_i$ for $i = 0, 1, \cdots, n$. Simple continued fractions are those whose terms $a_0, a_1, \cdots, a_j$ are all natural numbers.*

Proof. We keep the Φ notation for the continued fraction $[a_0, a_1, \cdots, a_j]$ and use β notation for another generic continued fraction $[b_0, b_1, \cdots, b_n]$. Further, if $\Phi_0 = [a_0, a_1, \cdots, a_j]$, note that $\Phi_1 = [a_1, a_2, \cdots, a_j], \Phi_2 = [a_2, a_3, \cdots, a_j]$, and further, for any integer i where $0 \le i \le j$, $\Phi_i = [a_i, a_{i+1}, \cdots, a_j]$. Similarly, for any integer i where $0 \le i \le n$, $\beta_i = [b_i, b_{i+1}, \cdots, b_n]$.

Notice that $\beta_i = b_i + \frac{1}{[b_{i+1}, b_{i+2}, \cdots, b_n]}$. We know then that $\beta_i > b_i$, exactly greater than, by $\frac{1}{[b_{i+1}, b_{i+2}, \cdots, b_n]}$). Further, because of our assumption that ever b is both an integer and greater than 1, we know $\beta_i > 1$ for $i = 1, 2, \cdots, n-1$. Similarly, as b_n is the last term in our b sequence, $\beta_n = b_n > 1$. Finally, note that $b_i = [\beta_i]$ as we use the $[x]$ notation to denote the integer part of x.

Remember, we assume that $\beta_0 = \Phi_0$, and we want to prove that β and Φ are of the same length and that each i^{th} integer of each fraction is the same. We use mathematical induction do complete this proof. Note that $\Phi_{i+1} > 1, i \geq 0$ (unless $a_{i+1} = 0$, where the continued fraction would have terminated already) and that $a_i = [\Phi_i]$ for $0 \leq i \leq j$. We know that $b_0 = [\beta_0] = [\Phi_0] = a_0$. Therefore, we know the following:

$$\begin{aligned}
\frac{1}{\Phi_1} &= \Phi_0 - a_0 = \beta_0 - b_0 = \frac{1}{\beta_1} \\
\Phi_1 &= \beta_1 \\
a_1 &= [\Phi_1] = [\beta_1] = b_1.
\end{aligned}$$

Next, we assume that $\Phi_i = \beta_i$ and that $a_1 = b_1$. Let us take a look at the following:

$$\frac{1}{\Phi_{i+1}} = \Phi_i - a_i = \beta_i - b_i = \frac{1}{\beta_{i+1}}.$$

We know then that

$$\Phi_{i+1} = \beta_{i+1}, a_{i+1} = [\Phi_{i+1}] = [\beta_{i+1}] = b_{i+1}.$$

We have just proved that if $\Phi_0 = \beta_0$, then $a_i = b_i$ for $i = 0, 1, \cdots, n$. By mathematical induction, the first part of our theorem holds. From here, knowing that $\Phi_0 = \beta_0$ and that each $a_i = b_i$, clearly, the last b term equals the last a term, so there are the same number of b and a terms. Hence $j = n$.

□

Next, we generalize this uniqueness for all rational numbers with Theorem 4.2.

Theorem 4.2. *Any finite simple conitinued fraction represents a rational number. Conversely any rational number can be expressed as a finite simple continued fraction in exactly two ways.*

Proof. The first part of this theorem is proven by mathematical induction on the number of integers in a continued fraction expansion. Suppose a continued fraction has just one term. Then $[a_0] = a_0$ is an integer and a rational number by definition. Now let us suppose that if we have i integers in a continued fraction expansion, then we can express the fraction as a rational number. Let us examine the case where we have $i + 1$ integers in an expansion:

$$[a_0, a_1, \cdots, a_i] = a_0 + \frac{1}{[a_1, a_2, \cdots, a_i]}.$$

We know that $[a_1, a_2, \cdots, a_i]$ has i terms in its expansion; therefore we know that it is rational, and therefore we know that $[a_0, a_1, \cdots, a_i] = a_0 + [a_1, a_2, \cdots, a_i]$ is rational.

The second part of Theorem 4.2 simply restates a use of the Euclidean Algorithm in creating continued fraction expansion combined with Theorem 4.1. It has also been determined that these are the only two ways (even trivially) to express any continued fraction. □

Therefore, we have successfully established the uniqueness of continued fractions in expressing finite rational numbers. We continue our examination of these fractions by taking a look at infinite continued fractions.

5. Infinite Continued Fractions [3]

Now that we have established how to represent rational numbers as continued fractions with a finite number of convergents, the idea of extending continued fractions to infinite convergents will now be explored.

We start with an infinite sequence of integers $a_0, a_1, \cdots$. Further, we define two sequences of integers, h, k as follows:

$$\begin{aligned} h_{-2} &= 0, & h_{-1} &= 1, & h_i &= a_i h_{i-1} + h_{i-2}, & i &\geq 0 \\ k_{-2} &= 1, & k_{-1} &= 0, & k_i &= a_i k_{i-1} + k_{i-2}, & i &\geq 0. \end{aligned}$$

Using these sequences of numbers, the following three theorems establish a definition for infinite continued fractions.

Theorem 5.1. *For any positive real number x,*

$$[a_0, a_1, \cdots, a_{j-1}, x] = \frac{xh_{j-1} + h_{j-2}}{xk_{j-1} + k_{j-2}}, j \geq 0.$$

Proof. We proceed by induction on the index j. First, we examine our base cases. If $j = 0$, then

$$[x] = x$$

$$\frac{xh_{-1} + h_{-2}}{xk_{-1} + k_{-2}} = \frac{x + 0}{0 + 1} = x.$$

Next, if $j = 1$, then

$$\frac{xh_0 + h_{-1}}{xk_0 + k_{-1}} = \frac{x(a_0 h_{-1} + h_{-2}) + h_{-1}}{x(a_0 k_{-1} + k_{-2}) + k_{-1}} = \frac{xa_0 + 1}{x} = a_0 + \frac{1}{x} = [a_0, x].$$

Assuming that the result holds true for $[a_0, a_1, \cdots, a_{n-1}, x]$, we manipulate the left hand expression:

$$[a_0, a_1, \cdots, a_n, x] = [a_0, a_1, \cdots, a_{n-1}, a_n + \frac{1}{x}]. \tag{2}$$

Equation 2 can be expressed as

$$\frac{(a_n + 1/x)h_{n-1} + h_{n-2}}{(a_n + 1/x)k_{n-1} + k_{n-2}} = \frac{x(a_n h_{n-1} + h_{n-2}) + h_{n-1}}{x(a_n k_{n-1} + k_{n-2}) + k_{n-1}} = \frac{xh_n + h_{n-1}}{xk_n + k_{n-1}}.$$

Therefore, this theorem holds for $j \geq 0$.

□

Theorem 5.1 enables us to prove the next two theorems.

Theorem 5.2. *If we define* $r_n = [a_0, a_1, \cdots, a_n]$ *for all integers* $n \geq 0$*, then* $r_n = h_n/k_n$.

Proof. We simply apply Theorem 5.1 by replacing x with a_n and we will get

$$r_n = [a_0, a_1, \cdots, a_n] = \frac{a_n h_{n-1} + h_{n-2}}{a_n k_{n-1} + k_{n-2}} = \frac{h_n}{k_n}.$$

□

Theorem 5.2 is acually an important theorem, and the fact that we can find a value for r_n will be analyzed in a later section. Before we move to that, however, we first need to establish a few more theorems.

Theorem 5.3. *The identities*

(1) $h_i k_{i-1} - h_{i-1} k_i = (-1)^{i-1}$
(2) $r_i - r_{i-1} = \frac{(-1)^{i-1}}{k_i k_{i-1}}$
(3) $h_i k_{i-2} - h_{i-2} k_i = (-1)^i a_i$
(4) $r_i - r_{i-2} = \frac{(-1)^i a_i}{k_i k_{i-2}}$

hold for $i \geq -1$.

Proof. Once again, we proceed by mathematical induction. We start with the base case for identity 1, $i = -1$:

$$h_{-1} k_{-2} - h_{-2} k_{-1}.$$

Evaluating this expression, we get $(1)(1) - (0)(0) = 1$.

Now for our induction step, we assume that $h_{i-1} k_{i-2} - h_{i-2} k_{i-1} = (-1)^{i-2}$. Using the definition of h, k:

$$h_i k_{i-1} - h_{i-1} k_i = (a_i h_{i-1} + h_{i-2}) k_{i-1} - h_{i-1}(a_i k_{i-1} + k_{i-2})$$
$$= a_i h_{i-1} k_{i-1} + h_{i-2} k_{i-1} - a_i k_{i-1} h_{i-1} - h_{i-1} k_{i-2} = h_{i-2} k_{i-1} - h_{i-1} k_{i-2}$$
$$= -(h_{i-1} k_{i-2} - h_{i-2} k_{i-1}) = -1(-1)^{i-2} = (-1)^{i-1}.$$

Thus, we have proved the first part of Theorem 5.3 by induction.

In order to obtain the second part of Theorem 5.3, we divide the first expression by $k_{i-1} k_i$. We get the following:

$$\frac{h_i k_{i-1} - h_{i-1} k_i}{k_{i-1} k_i} = \frac{(-1)^{i-1}}{k_i k_{i-1}}$$
$$\frac{h_i k_{i-1}}{k_{i-1} k_i} - \frac{h_{i-1} k_i}{k_{i-1} k_i} = \frac{h_i}{k_i} - \frac{h_{i-1}}{k_{i-1}} = r_i - r_{i-1}.$$

Once again, by the definition of h, k, we know that $h_i k_{i-2} - h_{i-2} k_i = (a_i h_{i-1} + h_{i-2}) k_{i-2} - h_{i-2}(a_i k_{i-1} + k_{i-2}) = a_i(h_{i-1} k_{i-2} - h_{i-2} k_{i-1}) = a_i(-1)^i$. Finally, as we did in the first part of the proof, we can divide this first identity by $k_{i-2} k_i$ to get our desired result. □

Next, we describe the behavior of r_n defined in Theorem 5.1.

Theorem 5.4. *The values r_j defined in Theorem 5.1 satisfy the infinite chain of inequalities $r_0 < r_2 < r_4 < r_6 <$ [every even-subscripted r] $\cdots$ [every odd-subscripted r] $< r_7 < r_5 < r_3 < r_1$. Note $j \geq 0$ Stated in words, the r_n with even subscripts form an increasing sequence, those with odd subscripts form a decreasing sequence, and every r_{2j} is less than every r_{2j-1}. Furthermore, $\lim_{n\to\infty} r_n$ exists, and for every $j \geq 0, r_{2j} < \lim_{n\to\infty} r_n < r_{2j+1}$.*

Proof. We know that $k_i > 0$ for $i \geq 0$ and $a_i > 1$ for $i \geq 1$ (any middle term in the continued fraction expression must be a positive integer unless it is zero; if it were zero, then there would be no terms following a_i); further, we are given the equations for $r_i - r_{i-1}$ and $r_i - r_{i-2}$ by Theorem 5.3. Applying the identies in the above sentence to these equations gives us that $r_{2j} < r_{2j+2}, r_{2j-1} > r_{2j+1}$, and $r_{2j} < r_{2j-1}$. From here we conclude that

$$r_0 < r_2 < r_4 < \cdots \tag{3}$$

$$r_1 > r_3 > r_5 > \cdots . \tag{4}$$

Next, we want to prove that $r_{2n} < r_{2j-1}$ for any natural numbers n, j. Using the inequalities in Expressions 3 and 4, we rewrite our inequalities as $r_{2n} < r_{2n+2j} < r_{2n+2j-1} \leq r_{2j-1}$. From here, we know that r_0 is our smallest value while r_1 is our biggest value. Therefore, we know our r terms with odd subscripts form a monotonically decreasing sequence that is bounded below by r_0, while our r terms with even subscripts form a monotonically increasing series that is bounded above by r_1. Therefore, both of these sets of r terms expressed as sequences have limits. We know that these two limits are actually equal because, by Theorem 5.3, $r_i - r_{i-1}$ tends to zero as i goes to infinity (since k_i increases as i increases). Therefore, a limit for r_n exists as n tends to infinity. Finally, because we know that all of our evenly-subscripted terms are strictly less than all of our oddly-subscripted terms and that both of these sequences of terms have the same limit (namely $\lim_{n\to\infty} r_n$), we know that the limit lies between every even term and every odd term. More formally, $r_{2j} < \lim_{n\to\infty} r_n < r_{2j+1}$ for all $j \geq 0$. □

Theorems 5.1-5.4 reveal that an infinite sequence of integers determines an infinite continued fraction (for $a_1, a_2, \cdots$). Further, these theorems suggest that the value of $\lim_{n\to\infty}[a_0, a_1, \cdots, a_n]$ is $\lim_{n\to\infty} r_n$.

6. Irrational Numbers [3]

Now that we have explored infinite continued fractions, we make the following claim:

Theorem 6.1. *The value of any infinite simple continued fraction $[a_0, a_1, \cdots]$ is irrational.*

Proof. Dentote our generic infinite continued fraction $[a_0, a_1, \cdots]$ as θ (note Φ denotes a generic finite continued fraction, θ denotes a generic infinite continued fraction). By Theorem 5.4 we know that θ lies between r_n and r_{n+1} for every $n \geq 0$. So we know that $0 < |\theta - r_n| < |r_{n+1} - r_n|$. Multiply by k_n:

$$0 < |k_n\theta - h_n| < |r_{n+1} - r_n|.$$

Using the fact from Theorem 5.3 that $|r_{n+1} - r_n| = (k_n k_{n+1})^{-1}$, we can rewrite our expression as

$$0 < |k_n\theta - h_n| < \frac{1}{k_{n+1}}.$$

We prove Theorem 6.1 by contradiction. Let us suppose then that θ is both an infinite continued fraction and a representation of a rational number. So let us suppose that $\theta = a/b$ with a, b being positive integers. We multiply our inequality by b and obtain the following:

$$0 < |k_n a - h_n b| < \frac{b}{k_{n+1}}.$$

We know by the definition of k_n that the sequence $\{r_n\}$ is bounded and strictly increasing (as per the equations proved in Theorems 5.1, 5.2). So we could pick a large enough n such that $b < k_{n+1}$. So the integer $|k_n a - h_n b|$ would have to lie between 0 and 1, which cannot happen since $|k_n a - h_n b|$ must be an integer. Therefore Theorem 6.1 is true. □

Now that we have established that infinite continued fractions represent irrational numbers, our next two theorems will prove that if two infinite continued fractions are different, they cannot converge to the same value.

Theorem 6.2. *Let $\theta = [a_0, a_1, \cdots]$ be a simple continued fraction. Then $a_0 = [\theta]$. Futhermore, if θ_1 denotes $[a_1, a_2, \cdots]$ then $\theta = a_0 + 1/\theta_1$.*

Proof. From Theorem 5.4 we know that $r_0 < \theta < r_1$; applying this inequality to θ yields $a_0 < \theta < a_0 + 1/a_1$. We know that $a_1 \geq 0$, so $a_0 < \theta < a_0 + 1/a_1 \leq a_0 + 1$. We can rewrite this expression as $a_0 < \theta < a_0 + 1$. We know that a_0 is the integer part of θ, or that $a_0 = [\theta]$.

Further, in order to evaluate the value of θ, we know from Theorem 5.4 that the value lies in the limit as n approaches infinity. So we evaluate

$$\lim_{n\to\infty} [a_0, a_1, \cdots, a_n] = \lim_{n\to\infty} (a_0 + \frac{1}{a_1, \cdots, a_n]}) = a_0 + \frac{1}{\lim_{n\to\infty} [a_1, a_2, \cdots, a_n]} = a_0 + \frac{1}{\theta_1}.$$

□

Theorem 6.3. *Two distinct infinite simple continued fractions converge to different values.*

Proof. Let us now suppose that $[a_0, a_1, a_2, \cdots]$ and $[b_0, b_1, b_2, \cdots]$ both converge to θ. By Theorem 6.2, $[\theta] = a_0 = b_0$. Further, we know that

$$\theta = a_0 + \frac{1}{[a_1, a_2, \cdots]} = b_0 + \frac{1}{[b_1, b_2, \cdots]}.$$

Therefore, we know that $[a_1, a_2, \cdots]$ must be equal to $[b_1, b_2, \cdots]$. By this same reasoning we can conclude that $a_1 = b_1$.

Now let us assume that $a_i = b_i$ for $i = 0, 1, \cdots, n-1$. We know the following

$$\theta_n = [a_n, a_{n+1}, + \cdots] = [b_n, b_{n+1}, \cdots]$$

$$\theta_n = a_n + \frac{1}{[a_{n+1}, a_{n+2}, \cdots]} = b_n + \frac{1}{[b_{n+1}, b_{n+2}, \cdots]}$$

Thefore we know that $a_{n+1} = b_{n+1}$, and it follows then that if two simple infinite continued fractions converge to the same value, then they are the same fraction. □

Niven and Zuckerman provide us with a theorem that combines the theorems of Section 6.

"Any irrational number θ is uniquely expressible, by the procedures of our two given algorithms, as an infinite simple continued fraction $[a_0, a_1, \cdots]$. Conversely any such continued fraction determined by integers a_i which are positive for all $i > 0$ represents an irrational number, θ. The finite simple continued fraction $[a_0, a_1, a_2, \cdots, a_n]$ has the rational value $h_n/k_n = r_n$, and is called the nth convergent to θ. The equations we established for h, k relate the h_i and $k_i$to the a_i. For $n = 0, 2, 4, \cdots$ these convergents form a monotonically increasing sequence with θ as a limit. Similarly, for $n = 1, 3, 5, \cdots$, the convergents form a monotonically decreasing sequence tending to θ. The denominators of k_n of the convergents are an increasing sequence of positive integers for $n > 0$."

7. Approximations to Irrational Numbers [3]

The next three theorems will allow us to make a stronger statement regarding the approximation of a continued fraction. In this section, θ is the simple, infinite, continued fraction from Theorem 6.2.

Theorem 7.1. *For any $n \geq 0$ and an infinite continued fraction θ, any h_n/k_n is guarenteed be within $1/k_n k_{n+1}$ of the fraction's value. Numerically,*

$$|\theta - \frac{h_n}{k_n}| < |\frac{1}{k_n k_{n+1}}|$$

$$|\theta k_n - h_n| < \frac{1}{k_{n+1}}.$$

Proof. To begin, by Theorem 5.3, for an irrational number θ, we are able to determine that

$$\theta - r_{n-1} = \theta - \frac{h_{n-1}}{k_{n-1}} \tag{5}$$

$$= \frac{\theta_n h_{n-1} + h_{n-2}}{\theta_n k_{n-1} + k_{n-2}} - \frac{h_{n-1}}{k_{n-1}} \tag{6}$$

$$= \frac{-(h_{n-1}k_{n-2} - h_{n-2}k_{n-1})}{k_{n-1}(\theta_n k_{n-1} + k_{n-2})} \tag{7}$$

$$= \frac{(-1)^{n-1}}{k_{n-1}(\theta_n k_{n-1} + k_{n-2})} \tag{8}$$

Further, using our simple algorithm we know that

$$a_i = [\theta_i]$$

$$\theta_{i+1} = \frac{1}{\theta_i - a_i}.$$

Putting these two facts together, we express equation 8 as

$$\frac{1}{k_n(\theta_{n+1}k_n + k_{n+1})} = |\theta - \frac{h_n}{k_n}|. \tag{9}$$

Finally, using the original definitions of h and k, we can replace $a_{n+1}k_n + k_{n-1}$ with k_{n+1} in order to obtain our first inequality. The second inequality in Theorem 7.1 is merely the first expression multiplied by k_n. □

Theorem 7.2. *: The convergents h_n/k_n are successively closer to θ, that is*

$$\left|\theta - \frac{h_n}{k_n}\right| < \left|\theta - \frac{h_{n-1}}{k_{n-1}}\right|$$

In fact, the stronger inequality $|\theta k_n - h_n| < |\theta k_{n-1} - h_{n-1}|$ holds.

Proof. : We can use the fact that $k_{n-1} \leq k_n$ in order to show that the stronger inequality implies the first:

$$\left|\theta - \frac{h_n}{k_n}\right| = \frac{1}{k_n}|\theta k_n - h_n| < \frac{1}{k_n}|\theta k_{n-1} - h_{n-1}| \leq \frac{1}{k_{n-1}}|\theta k_{n-1} - h_{n-1}| = \left|\theta - \frac{h_{n-1}}{k_{n-1}}\right|.$$

Next, in order to prove our stronger equality, notice that $a_n + 1 > \theta_n$ by the algorithm for determining continued fractions. Therefore, once again using our definitions for h and k,

$$\theta_n k_{n-1} + k_{n-2} < (a_n + 1)k_{n-1} + k_{n-2} = k_n + k_{n-1} \leq a_{n+1}k_n + k_{n-1} = k_{n+1}.$$

The above equation along with the expression for $\theta - r_{n-1}$ that we proved in Theorem 7.1 gives us that

$$\left|\theta - \frac{h_{n-1}}{k_{n-1}}\right| = \frac{1}{k_{n-1}(\theta_n k_{n-1} + k_{n-2})} > \frac{1}{k_{n-1}k_{n+1}}.$$

If we multiply by k_{n-1} and use Theorem 7.1 we obtain

$$\theta k_{n-1} - h_{n-1} > \frac{1}{k_{n+1}} > |\theta k_n - h_n|.$$

□

Theorem 7.3. *If a/b is a rational number with positive denominator such that $|\theta - a/b| < |\theta - h_n/k_n|$ for some $n \geq 1$, then $b > k_n$. In fact if $|\theta b - a| < |\theta k_n - h_n|$ for some $n \geq 0$, then $b \geq k_{n+1}$.*

Proof. First, we prove the second part of this theorem by way of contradiction. We begin by assuming that $|\theta b - a| < |\theta k_n - h_n|$ and $b < k_{n+1}$. Consider the following linear equations using the variables x and y:

$$x k_n + y k_{n+1} = b, \qquad x h_n + y h_{n+1} = a.$$

Treating our h and k terms as coefficients and using the formula 1 from Theorem 5.3, we know that the determinant of these coefficients is ± 1. Thus these equations have integer solutions. Suppose $x = 0$; then $b = y k_{n+1}$, so y is positive. Since y is a positive integer, $b \geq k_{n+1}$, which contradicts our assumption that $b < k_{n+1}$. If $y = 0$, then $a = x h_n$, $b = x k_n$; so the following holds true as x is a positive integer:

$$\begin{aligned} |\theta b - a| &= |\theta x k_n - x h_n| && (10)\\ &= |x||\theta k_n - h_n| && (11)\\ &\geq |k_n \theta - h_n| && (12) \end{aligned}$$

Equation 9 contradicts our other assumption that $|\theta b - a| < |\theta k_n - h_n|$.

We next prove that x and y have opposite signs. If $y < 0$, then $x k_n = b - y k_{n+1}$, which shows x is positive. Next, if $y > 0$, then $b < k_{n+1}$ implies that $b < y k_{n+1}$, making $x k_n$ negative, so x is negative. From Theorem 7.2, we know that $\theta k_n - h_n$ and $\theta k_{n+1} - h_{n+1}$ have opposite signs, so $x(\theta k_n - h_n)$ and $y(\theta k_{n+1} - h_{n+1})$ have the same sign. From our original equations defining x and y we get that $\theta b - a = x(\theta k_n - h_n) + y(\theta k_{n+1} - h_{n+1})$. We know that the two terms on the right side have the same sign, we know in determining the absolute value of the equation we can separate the right side as follows:

$$\begin{aligned} |\theta b - a| &= |x(\theta k_n - h_n) + y(\theta k_{n+1} - h_{n+1})|\\ &= |x(\theta k_n - h_n)| + |y(\theta k_{n+1} - h_{n+1})|\\ &> |x(\theta k_n - h_n)|\\ &= |x||\theta k_n - h_n| \geq |\theta k_n - h_n| \end{aligned}$$

Here we get our contradiction of the assumption that $|\theta b - a| < |\theta k_n - h_n|$, so we know x and y have opposite signs.

Finally, we prove that if the second part of the theorem holds true, then the first part is true as well. Suppose there exists a rational number a/b such that

$$\left|\theta - \frac{a}{b}\right| < \left|\theta - \frac{h_n}{k_n}\right|, \qquad b \leq k_n.$$

Multiplying these inequalities together yields

$$|\theta b - a| < |\theta k_n - h_n|, n \geq 0, b > 0 \tag{13}$$

□

These three theorems inform us that the convergent h_n/k_n is actually the best approximation to our irrational number among all of our approximations whose denominators are at most k_n.

8. Some Workable Examples

In order to fuller illustrate how continued fractions work, we provide some examples. First, consider the rational number $x = 649/200$. Using our algorithm, we obtain the continued fraction $[3, 4, 12, 4]$.

$$3 + \cfrac{1}{4 + \cfrac{1}{12 + \cfrac{1}{4}}} =$$

$$3 + \cfrac{1}{4 + \cfrac{4}{49}} =$$

$$3 + \cfrac{1}{\frac{200}{49}} =$$

$$3 + \frac{49}{200} = \frac{649}{200}$$

Next, let us consider the irrational number $x = \pi$. The expansion of π is given by $[3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, \cdots]$[4]. Continuing, we determine the first few convergents of this continued fraction (the i^{th} convergent denoted as r_i) as follows: [6]

- $r_0 = 3 \approx \pi - 0.141593$
- $r_1 = 22/7 = 3.142857 \approx \pi + 0.00126$
- $r_2 = 333/106 = 3.141509 \approx \pi - 0.000083$
- $r_3 = 355/113 = 3.14159292 \approx \pi + 0.000000266$
- $r_4 = 103993/33102 = 3.1415926530 \approx \pi - 0.00000000057$
- $r_5 = 104348/33215 = 3.1415926539 \approx \pi + 0.00000000033$

Before we relate these approximations back to the theorems we have proven, let us provide one more example. The golden ratio is the irrational number $x = (1 + \sqrt{5})/2 \approx 1.61803399$ represented by $[1, 1, 1, 1, \cdots]$. [7] Here are its first few convergents:

- $r_0 = 1 \approx x - 0.61803399$
- $r_1 = 2 \approx x + 0.38196601$
- $r_2 = 3/2 = 1.5 \approx x - 0.11803399$
- $r_3 = 5/3 = 1.66666667 \approx x + 0.04863271$
- $r_4 = 8/5 = 1.6 \approx x - 0.0180339887$

FIGURE 1. Keys on the Piano [5]

- $r_5 = 13/8 = 1.625 \approx x + 0.00696601$

Note that for the first six convergents in both of our samples, Theorem 5.4 holds. We notice that for both sets of r_0, r_2, and r_4, the convergents under approximate the actual value. Further, both sets of r_1, r_3, and r_5 over estimate the actual value. Finally, we can also see that our approximation moves closer and closer to the actual value of our irrational number as we calculate more convergents.

Having established a basis for continued fraction expansions, we now set some musical foundation.

9. A Little Bit of Physics and Music Theory [4]

Sound from an object is produced when the object vibrates in the air. For simplicity's sake, suppose that one string in the piano vibrates at a rate of v cycles per second. We also know that this object will continue to vibrate at all positive n integer intervals of v, namely every nv. Every interval of v is called an *overtone*.

Next, the keys of the piano run through a, b, c, d, e, f, and g. These are the standard plain white keys. The dark keys found in between pairs of white keys are called the "sharp" key of the white key that precedes this key, and called the "flat" key of the white key that supercedes it. For instance, the black key found in between c and d is called "sharp c" and "flat d." The basic set up of these keys can be found in Figure 1.

The scale we will start with be using with the c' key, the apostrophe standing for our first octave, and run through key b' (we set the frequency of c' at 1).

Putting the ideas of physics and music together, we are able to calculate the frequencies of all of the keys. We do this by starting with our frequency 1 key, c', and moving in the following musical ratios:

- Octave 2:1
- Perfect Fifth 3:2
- Perfect Fourth 4:3
- Whole Step 9:8

Note the terms octave, perfect fifth, perfect fourth, and whole step are musical terms for the numerical ratios they represent.

These are used as follows: Say you start with a frequency of 1. If we move up one octave from this key, we will have a frequency of 2 (hence a two to one ratio between the two frequencies). If we want to move up a perfect fourth, we will have to multiply the original frequency by 4/3. We can do this for any of the remaining ratios as well.

With these basic ratios, a music scale was created by a man named Pythagoras.

9.1. **Pythagorean Scale [4].** Legend has it that Pythagoras heard the pleasant sound of four different hammers weighing 12, 9, 8, and 6 pounds. These provided convenient intervals with which he was able to derive our basic intervals:

- Octave = 12:6 = 2:1
- Perfect Fifth = 12:8 = 3:2
- Perfect Fourth = 12:9 = 4:3
- Whole Step = 9:8

In terms of playing these ratios on a piano, each interval on the piano has seven keys. So playing one octave up would be playing the same key in adjacent intevals, or two keys exactly seven keys apart. Further, each interval on the piano also has five sharp and flat keys. So for those intervals that do not align perfectly within seven keys, we can use these keys as well.

From here, in order to construct musical notes, Pythagoras used only the octave and the perfect fifth in order to come up with two basic rules:

- "Doubling the frequency moves up an octave"
- "Multiplying the frequency by $\frac{3}{2}$ moves up a perfect fifth"

From these rules, we are able to construct the basic scale on a piano, albeit an "artificial comparison to those actually used by musicians." [4]

As mentioned earlier, we are going to start with key c' and set its frequency to 1. We can use rule 1 to move up to c" in the second octave, or we can apply rule 2 to c' and obtain g' (the key which corresponds to the perfect fifth above c'). The frequency of g' is then 3/2. In order to determine the frequency of the G key in the second octave, we need just apply rule 1 to g' (which would give us the key g" with a frequency of 3). Next, we can apply rule 2 to g', and obtain d", which is the D key in the second octave (and has a frequency of 9/4). Applying the inverse of rule 1, we can divide the frequency of d" by 2 in order to get to the D key in the first octave, d' (which has a frequency of 9/8). Applying rule 2 to d', we find the perfect fifth above d', a', which has a frequency of 27/16. Applying rule 2 and the inverse of rule 1 to a' brings us to key e', which has a frequency of $27/16 \times 3/4 = 81/64$. Continuing this pattern we are able to obtain the following frequencies:

- c' = 1
- g' = 3/2
- d' = 9/8
- a' = 27/16
- e' = 81/64
- b' = 243/128
- f♯' (f sharp) = 729/512
- c♯' (c sharp, or b flat) = 2187/2048
- g♯' = 6561/4096
- d♯' = 19683/16384
- a♯' = 59049/32768
- f' = 4/3 (obtained by applying inverse of rule 2 to c")

9.2. **Pythagorean Comma** [4]**.** Using the Pythagorean scale to tune a piano would give us keys that had the respective frequencies found in the section above. However, using this scale, we should be able to jump up by two perfect fifths in order to move from a♯' to c". So multiplying the frequency of a♯' by the appropriate ratios of three perfect fifths (27/8) yields a frequency of 531441/262144, which numberically comes out to 2.02728653. So moving back down to c', c' should have a frequency of 1.0136432647705078125.

But c' has a frequency of 1; clearly $1 \neq 1.0136432647705078125$. This discrepancy of 0.013643264770507815 is called the Pythagorean Comma. Basically, if a piano is tuned exactly to the Pythagorean Scale and a set of notes were played perfectly, the notes would sound harmonically off by this small factor.

9.3. **Helmholtz's Scale** [4]**.** Pythagoras only used two ratios. What if we included some of the other frequency ratios? Would this correct our problem? The perfect fourth ratio, 4/3, already comes into play if you combine rule 1 with the inverse of rule 2. However, this first scale does not use the major third ratio (5/4) nor does it use the ratio of minor thirds (6/5). A second scale, Helmholtz's Scale, puts these ratios into play.

Despite using more ratios, Helmholtz's Scale actually is more problematic than the Pythagorean Scale as the discrepancies created by this second scale actually sound significantly worse than the first scale. In order to show this discrepancy, it will suffice to examine the white keys in one octave for Helmholtz's Scale:

- c' = 1
- d' = 9/8
- e' = 5/4
- f' = 4/3
- g' = 3/2
- a' = 5/3
- b' = 15/8
- c"= 2

As c"=2 and c'=1, the problem created by the Pythagorean Scale appears to be fixed. The notes c' and g', e' and b', and f' and c" are all in 3 : 2 ratios. The notes c' and e', f' and a', and g' and b' are all in 5 : 4 ratios (major thirds). However, we have instead created another problem on the interval d'-a'. This problem lay in something called the Circle of Fifths.

9.4. **Circle of Fifths and Syntonic Comma** [4]**.** If we were to start with the C note on a piano and repetedly take perfect fifths of our notes, we would soon discover that there is a pattern among the twelve tones of a piano keyboard according to these fifths. The twelve notes actually loop around in a circle according to which fifth follows which. If we take the fifth of C, we get G. If we take the fifth of G, we get D. If we take the fifth of D, we get A. And this continues until we see the circle develop like the one in Figure 2 (Pattern of C, G, D, A, E, B, G♭, D♭, A♭, E♭, B♭, F, C, $\cdots$).

But according to Figure 2, the frequencies of d' and a' should represent a 3 : 2 ratio. But according to Helmholtz's Scale, the interval from d' to a' is represented

FIGURE 2. Circle of Fifths[8]

by $5/3 \times 8/9 = 40/27$. We can determine how far off our number is from this perfect value by calculating $3/2 \times 40/27 = 81/80$. So we are off by a factor of $1/80$, or approximately 0.0125. This factor is called the syntonic comma.[1]

9.5. **Quick Notation: Measuring Musical Tones in Cents [4].** In order to formalize our musical notation in the future, we note that musicians have conveniently developed a system for measuring intervals called "cents." As there are twelve tones in an octave, there are 1200 equal parts (or cents) in one octave. The progression from one key to the next key immediately to the right of it is called a half-step. Moving from one key to the next adjacent key moves the tone of our key by 100 cents (so each half-step, which corresponds to moving over one adjacent key on a piano, is actually divided by 100 cents). So if we let the variable I represents the ratio of two frequencies of two tones, the number of cents on the interval can be calculated as:

$$1200 \log_2 (I) .$$

We can calculate more formally how much of a fraction of a half-step our scales were off (or how much off of one key they were) by applying the formula to each of the commas. Measuring the Pythagorean Comma in cents yields

$$1200 \log_2 \left(\frac{531441}{524288} \right) \approx 23.5.$$

This represents 235 cents, or 23.5 half steps. The syntonic comma yields

$$1200 \log_2 \left(\frac{81}{80} \right) \approx 21.5.$$

This represents 215 cents, or 21.5 half steps.

10. Basic Number Theory [4]

While the Pythagorean Scale prepares a scale close to the frequencies for each key in an octave, it only uses octaves and perfect fifths. As a result, we can only multiply and divide frequencies by factors of 2 or 3. This is problematic as "the fundamental problem is in trying to equate a function based on tripling (fifths) with a function based on doubling (octaves). Phrased mathematically, we are trying to solve an equation of the type:

$$2^x = 3^y$$

where x and y are rational."[4]

11. Solutions to Problem[4]

Two older methods used to ameliorate the incorrect frequencies created by the Pythagorean Scale are the *mean-tone system* and the *well-tempered system*. Both involve an averaging of intervals; and while the latter is better than the former, both are severely outdated (one would only use these in order to sound old-fashioned).

The more modern technique used is known as *equal temperament*. With this system, "the ratio of the frequencies of any two adjacent half-steps is constant, and the only interval that is acoustically correct is the octave." [1] Unlike in the other tuning systems, equal temperament does as much as it can to spread the error out between the different keys. For instance, equally tempered fifths are only off of the true 3 : 2 ratio by 2% of a half-step. Specifically, the ideal fifth corresponds to the following number of cents:

$$1200 \log_2 \left(\frac{3}{2}\right) \approx 702.0 \text{ cents.}$$

A fifth in equal temperament yields:

$$1200 \log_2 \left(2^{7/12}\right) \approx 700.0 \text{ cents.}$$

Because the octaves are the only perfect interval in the equal temperament scale and the rest of the error is spread equally among each half step, we know that the ratio between a key and its fifth is $2^{7/12}$. In other words, the ratio between a key and and its partner one octave over is two. So moving over twelve equal ratios over 12 half steps yields 2. So if we decide to represent the ratio between two frequencies by the variable r we know that $r^{12} = 2$. So each half step represents an interval ratio of $2^{1/12}$. This is why a fifth is represented by $2^{7/12}$.

12. Relating Continued Fractions to Piano Tuning [4]

Looking back at the root of the problem with tuning pianos, searching for a solution to the equation $2^x = 3^y$, we can set $x/y = \log_2 3$. Using continued fractions, we can approximate this value for x by two rational numbers. We know by simple algebra that

$$\log_2 3 = \frac{\log 3}{\log 2}.$$

Evaluating these values in a calculator, we determine that $\log_2 3 = 1.1.58496250072116....$ Using our algorithm for determining partial fractions, we get that

$$\log_2 3 = [1, 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, 2, 1, \cdots]$$

.

From this we want to make the following observation: Using equal temperament, Western Music has adopted the fourth approximation to the to the Pythagorean scale (fourth continued fraction approximation, terminating our infinite sequence at the term a_4). Remember, based on our notation, a_0, a_2, and a_3 are all equal to 1. And $a_4 = 2$. So the first approximation would simply involve expanding our fraction to a_1, or $1 + \frac{1}{1} = 2$. Expanding this to the fourth approximation, we get the following:

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}} = 1 + \frac{7}{12} = 19/12.$$

So our irrational solution can be approximated as follows:

$$3 = 2^{\log_2 3} \approx 2^{19/12}.$$

And if we divide each side by 2 we can determine approximately how much of the interval the ratio $3/2$ represents:

$$\frac{3}{2} \approx 2^{7/12}.$$

This makes sense as we know that a perfect fifth involves seven half-steps.

13. Examining Intervals Using the Fourth Approximation [4]

Using our 12 tone scale (fourth approximation) yields the following (Remember that the intervals that a ratio represents is equal to $12\log_2(I)$):

- Perfect fifth $= 12\log_2(3/2) \approx 7.0196 \approx 7$ basic intervals
- Perfect fouth $= 12\log_2(4/3) \approx 4.9804 \approx 5$ basic intervals
- Major third $= 12\log_2(5/4) \approx 3.8631 \approx 4$ basic intervals
- Minor third $= 12\log_2(6/5) \approx 3.1564 \approx 3$ basic intervals

We know that these are the half-step moves actually used by Western musicians; however, the major third and minor third values are disproportionate from the intervals they represent. These are known as "imperfect consonances" because they are off by a factor (the ratios are imperfect). However, these are still used on the twelve-tone scale (this is where most of the error is heard). Using our cents calculation, we know that the ratio of a perfect fifth actually represents approximately 701.96 cents. And seven basic intervals (half-steps) represents 700 cents. So the perfect fifth is only off by 1.96 cents. The perfect fourth actually represents 498.04 cents, which is off by only 2.96 cents. The major third, however, actually represents 407.82 cents, which is off by 7.82 cents. The minor third represents 294.13 cents, which is off of 300 cents by 5.87 cents.

14. Taking a Guess [3]

The proofs regarding our approximations up until this point have involved analyzing numbers for which we already knew were approximations. Now we know the relationship between the denominator of an approximation and the strength of the approximation itself..

We establish a theorem that allows us the throw out "guesses" for our approximation. The following theorem will allow us to make guesses at approximations without directly calculating them.

Theorem 14.1. *Let θ denote any irrational number. If there is a rational number a/b with $b \geq 1$ such that*

$$|\theta - \frac{a}{b}| < \frac{1}{2b^2},$$

then a/b equals one of the convergents of the simple continued fraction expansion of θ.

Proof. For this proof, we are going to assume that the rational number a/b is simplified, namelely $\gcd(a,b) = 1$. Suppose that all of the convergents of θ take on the explicit form h_i/k_i and that a/b is in fact not a convergent of θ. Keeping the inequalities $k_n \leq b < k_{n+1}$ for all natural numbers n in mind (from Theorem 7.3), we know that $|\theta b - a| < |\theta k_n - h_n|$. But this inequality is impossible from Theorem 7.3. Thus, we obtain the following

$$|\theta k_n - h_n| \leq |\theta b - a| < \frac{1}{2b},$$

$$|\theta - \frac{h_n}{k_n}| < \frac{1}{2bk_n}.$$

Knowing that $a/b \neq h_i/k_i$ (remember a/b is not a convergent of θ) and $bh_n - ak_n$ is an integer, we know

$$\frac{1}{bk_n} \leq \frac{|bh_n - ak_n|}{bk_n} = |\frac{h_n}{k_n} - \frac{a}{b}| \leq |\theta - \frac{h_n}{k_n}| + |\theta - \frac{a}{b}| < \frac{1}{2bk_n} + \frac{1}{2b^2}.$$

These set of inequalities imply that $b < k_n$. Thus, Theorem 14.1 is true. □

If we want to throw out a guess in the future about an approximation of an irrational number, we can use the inequality from Theorem 14.1 in order to test whether or not our number is a legitimate guess.

15. Best Possible Approximation [3]

Logically, if irrational numbers are represented by infinite continued fractions, then there are infinitely many rational approximations to this number. In fact, we do know the closest upper bound to these approximations. Before we prove this fact, however, we must first prove the following:

Theorem 15.1. *If x is real, $x > 1$, and $x + x^{-1} < \sqrt{5}$, then $x < \frac{1}{2}(\sqrt{5}+1)$ and $x^{-1} > \frac{1}{2}(\sqrt{5}+1)$.*

Proof. This proof is pretty straightforward. For any real $x \geq 1$, note that $x + x^{-1}$ increases with x; so $x + x^{-1} = \sqrt{5}$ if $x = \frac{1}{2}(\sqrt{5}+1)$. □

Now onto the first of two theorems establishing an upper bound for our approximation (Theorem 15.2 is also known as Hurwitz's Theorem).

Theorem 15.2. *Given any irrational number θ, there exist infinitely many positive rational numbers h/k $(h, k > 0)$ such that*

$$|\theta - \frac{h}{k}| < \frac{1}{\sqrt{5}k^2}.$$

Proof. For this proof, we will be establishing that this inequality holds at least once out of every three consecutive convergents. Let $q_n = k_n/k_{n-1}, n \geq 0$. We want to first show that $q_j + q_j^{-1} < \sqrt{5}, j \geq 0$ is true if the inequality in Theorem 15.2 is false for $h/k = h_{j-1}/k_{j-1}$ and $h/k = h_j/k_j$.

Suppose that the inequality in this theorem is false for two values $h/k = h_{j-1}/k_{j-1}$ and $h/k = h_j/k_j$. We would have then

$$|\theta - \frac{h_{j-1}}{k_{j-1}}| + |\Phi - \frac{h_j}{k_j}| \geq \frac{1}{\sqrt{5}k_{j-1}^2} + \frac{1}{\sqrt{5}k_j^2}.$$

We know that θ lies in between h_{j-1}/k_{j-1} and h_j/k_j by Theorem 5.3. Now using Theorem 5.2, we know

$$|\theta - \frac{h_{j-1}}{k_{j-1}}| + |\theta - \frac{h_j}{k_j}| = |\frac{h_{j-1}}{k_{j-1}} - \frac{h_j}{k_j}| = \frac{1}{k_{j-1}k_j}.$$

Combining our results yields

$$\frac{k_j}{k_{j-1}} + \frac{k_{j-1}}{k_j} \leq \sqrt{5}.$$

We know that the left side of this equation is rational and represents $q_j + q_j^{-1}$.

Lastly, suppose that the our Hurwitz inequality is false for $h/k = h_i/k_i = n - 1, n, n+1$. Then we now have $q_j + q_j^{-1} < \sqrt{5}$ for $j = n, n+1$. By Theorem 15.1 we know that $q_n^{-1} > \frac{1}{2}(\sqrt{5} - 1)$ and $q_{n+1} < \frac{1}{2}(\sqrt{5} + 1)$. Further, using our equations for h, k, we can determine that $q_{n+1} = a_{n+1} + q_n^{-1}$. Finally, we get

$$\frac{1}{2}(\sqrt{5} + 1) > q_{n+1} = a_{n+1} = q_n^1 > a_{n+1} + \frac{1}{2}(\sqrt{5} - 1)$$
$$\geq 1 + \frac{1}{2}(\sqrt{5} - 1) = \frac{1}{2}(\sqrt{5} + 1)$$

revealing that Hurwitz's inequality must be true. □

16. Conclusion

As we have seen in our analysis of piano tuning, the equal temperament system for tuning pianos is the most practical option. If we wanted a more accurate approximation for our tuning ratios, we would need to more than triple the number of keys on the piano in order to make more accurate and usable intervals. Thus, for practical musical purposes, musicians have developed the best system.

17. Bibliography

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