CURVES OF CONSTANT WIDTH AND THEIR SHADOWS

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ABSTRACT. In this paper we will investigate curves of constant width and the shadows that they cast. We will compute shadow functions for the circle, Reuleaux Triangle, and the curves of constant width described by Stanley Rabinowitz. From these functions we will prove that you can distinguish the different curves from their shadows. A result about the perimeter and area of these curves is also presented.

Have you ever wondered why a manhole cover is in the shape of a circle? This is because the circle has the property that it has constant width. No matter which way you turn the circular lid, there is no danger of it falling into the manhole. If you use a square to make a manhole cover, you could turn the square and the lid would fall right through. The most commonly known curve of constant width is the circle, but there are actually an infinite number of these curves that can be created.

Martin Gardner, author of "The Colossal Book of Mathematics: Classical puzzles, paradoxes, and problems," defines a curve of constant width to be a convex planar shape whose width, which can be measured by the distance between two opposite parallel lines touching its boundary, is the same regardless of the direction of the lines. He also defined the width of the curve in a given direction to be the perpendicular distance between the parallel lines [2].

In the following sections we will be looking at examples of these different curves of constant width and how they are constructed. After we know how these curves are constructed, we will show how you can distinguish any two curves of constant width, even with the same width, from the shadows that they cast. This idea is

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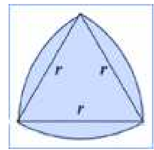

FIGURE 1. Reuleax Triangle [8]

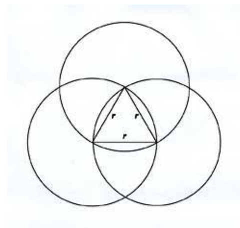

FIGURE 2. Reuleax Triangle [8]

counterintuitive. If a curve has the same width as another, shouldn't the shadows cast by the curves be the same?

1. Curves of Constant Width and Their Constructions

The Reuleaux Triangle is the most common curve of constant width, other than the circle. This curve was developed by Franz Reuleaux, an engineer who is considered to be the father of modern kinematics. The construction of this curve can be viewed in Figure 1. In order to construct this curve you start with an equilateral triangle. Then you draw three arcs each with radii equal to the side of the triangle and each centered at one of the vertices. Looking at Figure 2, we see that this curve can also be constructed as the intersection of three circles with equal width [3].

Using the Reuleaux Triangle, we are able to construct other curves of constant width. One such curve can be viewed in Figure 3. This curve is constructed if you extend the sides of the triangle an equivalent distance beyond each vertices, creating pairs of vertical 60° angles. In each of these angles draw an arc with the center at the nearest vertex. All three arcs should be drawn with the same radius. Connect these arcs with each other with circular arcs centered at the opposite vertices of the triangle [3].

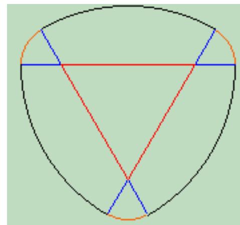

FIGURE 3. Extension of Reuleaux Triangle [3]

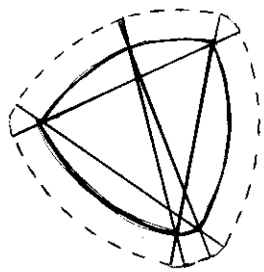

FIGURE 4. Construction of a curve of constant width using a Star Polygon (inside curve) and extension of this polygon (dotted curve).

Starting with an equilateral (equal length sides), but not necessarily an equiangular star (angle between sides not equal), another curve of constant width can be created, as seen in Figure 4. This curve is the interior curve in solid black. Because the basis of the curve is a star with equilateral sides, the curve is going to have an odd number of vertices and the star will have sides of equal length. The angles of the vertices do not however need to be equal. If we use the vertices of the star as centers and then draw circular arcs with radius equal to the side of the star, it will result in a curve of constant width. The arcs should connect pairs of adjacent vertices [3].

We can use these same stars to create curves by using the same method used to get the extension of the Reuleaux Triangle. This construction can also be viewed in Figure 4. It is the curve with the dotted line. Extend the lines out past each vertex and connect the arcs made by the vertices of the star with the arcs made from the intersections at the end of the vertices [3].

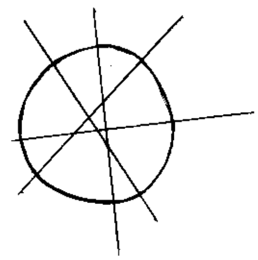

Figure 5. Curve of constant width using mutually intersecting line segments.

Martin Gardner describes another type of curve of constant width. He states that if you draw as many mutually intersecting straight line segments of equal length as you wish, and place a compass at the intersection of the two lines that bound the arc, and go around the curve connecting each arc, you will get a curve of constant width. An example of this curve can be seen in Figure 5. This construction will produce shapes of constant width as long as all sides intersect and there are an odd number of vertices. This method creates curves that aren't necessarily smooth curves in the sense that the points where two arcs join may not be continuously differentiable [2].

All of the curves that have been described thus far are constructions that do not necessarily create infinitely differentiable or smooth curves, like the circle which has an infinitely smooth parameterization. An infinitely smooth curve, in this case, is a curve whose parametric representation is infinitely differentiable. A construction of this type of curve is described by Stanley Rabinowitz. This method is described later in section 2.4.

2. Computing Shadow Functions

In Charles L. Epstein's book, "Introduction to the Mathematics of Medical Imaging," he introduces the idea of determining the outline of a convex object from its shadows. We will be considering the 2-dimensional case, or more specifically, curves of constant width. Surprisingly, these curves of constant width, even of the same width, can be distinguished by their shadows. First we must define what we mean

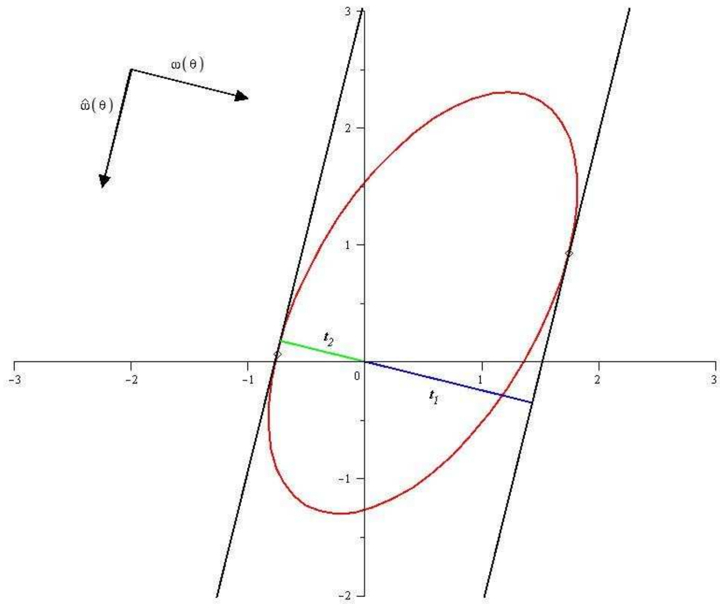

FIGURE 6. Image depicting t, $\omega(\theta)$, and $\hat{\omega}(\theta)$ [6]

by "shadow". We will assume that there is a light source far away so that all the light rays will be traveling in the same direction. We are interested in the length of the shadow that is cast by the curve. In order to find this we need to know where the shadow begins and ends, or the extent. The extent of an object's shadow can be quantified by the lines parallel to the light source that are tangent to the curve on both sides. In order to describe the entire curve, we would need to examine these lines at each angle while rotating the light source [4].

For the following calculations, we will let $\hat{\omega}(\theta) = (-\sin(\theta), \cos(\theta))$, where θ is the angle of the light source, represent the direction of light rays and the direction perpendicular to the light source be $\omega(\theta) = (\cos(\theta), \sin(\theta))$. We are interested in the distance between the two bracketing light rays, which we will call t, where $t = t_1 + t_2$. Looking at Figure 6 the bracketing light rays are those tangent to the ellipse. Because the lines are parallel, this distance is equivalent to the width of the object at that particular angle. An example using an ellipse is depicted in Figure 6[4].

We are trying to derive an equation that will give us t_1, which is the distance from the center of the object to the tangent line. In order to find the length of t_1 we first need to determine a parametric representation of the boundary of the object casting the shadow, which we will denote

$$\vec{r}(\theta) = (x(\theta), y(\theta)).$$

Let the tangent vectors be

$$\vec{r}_T(\theta) = (x'(\theta), y'(\theta)).$$

Next, we need to find the equation for t_1. Note that if we project $\vec{r}(\theta)$ onto $\omega(\theta)$ we will get the desired result. Because $\omega(\theta)$ is a unit vector the following will be the equation for $t_1(\theta)$.

$$t_1(\theta) = \omega(\theta) \cdot \vec{r}(\theta)$$

Substituting in the values for $\vec{r}(\theta)$ and $\omega(\theta)$ yields

$$t_1(\theta) = x(\theta)\cos(\theta) + y(\theta)\sin(\theta). \tag{1}$$

We will call t_1 the shadow function for the object described by $\vec{r}(\theta)$.

2.1. **Shadow Function For the Circle.** To determine the shadow function for the circle we first determine the parameterization. Consider the circle centered at the origin with width $\sqrt{3}$. The parameterization is

$$\vec{r}(\theta) = \big(\frac{\sqrt{3}}{2}\cos(\theta), \frac{\sqrt{3}}{2}\sin(\theta)\big).$$

Plugging this into Equation 1 we will get the following

$$\begin{aligned} t_1(\theta) &= \frac{\sqrt{3}}{2}\cos^2(\theta) + \frac{\sqrt{3}}{2}\sin^2(\theta) \\ &= \frac{\sqrt{3}}{2}(\cos(\theta)^2 + \sin(\theta)^2) \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

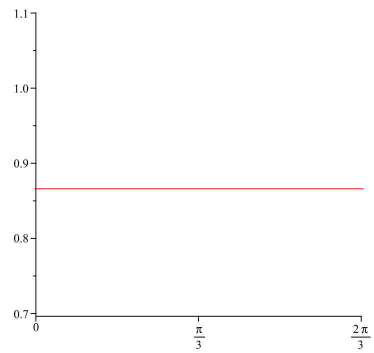

FIGURE 7. Shadow Function of Circle with Radius $\sqrt{3}/2$.

To visualize this see Figure 7 which shows a plot of the shadow function of the circle. Here we note that the shadow function is constant which makes sense for the circle because no matter which angle the light source is coming from, the width from the origin is going to be the same.

2.2. **Parametric Representation of the Reuleaux Triangle.** Let us consider the shadow function of a Reuleaux Triangle of the same width, in particular $\sqrt{3}$. First we need to recall how the triangle is constructed in order to create a parameterization to be used later to determine the shadow function for the curve. In order to make this curve, start with an equilateral triangle. Then draw three arcs with radius equal to the side of the triangle and each centered at one of the vertices. In order to parameterize it, place the equilateral triangle that is the basis for the triangle within a unit circle. The vertices are placed at $A = (1, 0), B = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and $C = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. The arc that will connect each of the vertices will have width equal to $\sqrt{3}$. The following will therefore be a parametric representation of the Reuleaux Triangle that has $\theta \in [0, 2\pi)$, which is the angle of the light source. The parametric representation is also accounting for the time that the light source is rotating around the cusps of the curve. Figure 8 depicts this curve.

$$x(\theta) = \begin{cases} 1, & \frac{-\pi}{6} \leq \theta < \frac{\pi}{6} \\ \sqrt{3}\cos(\theta) - \frac{1}{2}, & \frac{\pi}{6} \leq \theta < \frac{\pi}{2} \\ \frac{-1}{2}, & \frac{\pi}{2} \leq \theta < \frac{5\pi}{6} \\ \sqrt{3}\cos(\theta) + 1, & \frac{5\pi}{6} \leq \theta < \frac{7\pi}{6} \\ \frac{-1}{2}, & \frac{7\pi}{6} \leq \theta < \frac{3\pi}{2} \\ \sqrt{3}\cos(\theta) - \frac{1}{2}, & \frac{3\pi}{2} \leq \theta < \frac{11\pi}{6} \end{cases}$$

and

$$y(\theta) = \begin{cases} 0, & \frac{-\pi}{6} \leq \theta < \frac{\pi}{6} \\ \sqrt{3}\sin(\theta) - \frac{\sqrt{3}}{2}, & \frac{\pi}{6} \leq \theta < \frac{\pi}{2} \\ \frac{\sqrt{3}}{2}, & \frac{\pi}{2} \leq \theta < \frac{5\pi}{6} \\ \sqrt{3}\sin(\theta), & \frac{5\pi}{6} \leq \theta < \frac{7\pi}{6} \\ \frac{-\sqrt{3}}{2}, & \frac{7\pi}{6} \leq \theta < \frac{3\pi}{2} \\ \sqrt{3}\sin(\theta) + \frac{\sqrt{3}}{2}, & \frac{3\pi}{2} \leq \theta < \frac{11\pi}{6} \end{cases}$$

2.3. **Shadow Function For The Reuleaux Triangle.** If we recall from above, the following equation yields the shadow function for a curve with a parametric representation given by $\vec{r}(\theta) = (x(\theta), y(\theta))$,

$$t_1(\theta) = x(\theta)\cos(\theta) + y(\theta)\sin(\theta).$$

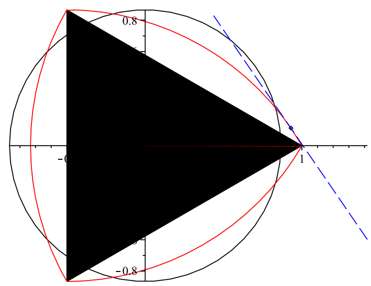

FIGURE 8. Plot of the Reuleaux Triangle inside a circle with a tangent line at $\theta = \frac{\pi}{20}$.

Therefore, for the first portion of the curve where $\vec{r}(\theta) = (1, 0)$ and $\frac{-\pi}{6} \leq \theta < \frac{\pi}{6}$, the shadow function will be

$$t_1(\theta) = 1 \cdot \cos(\theta) + 0 \cdot \sin(\theta)$$

We can carry these calculations out for each portion of the triangle. The following is the piecewise shadow function for the Reuleaux Triangle as θ goes from 0 to 2π.

$$t_1(\theta) = \begin{cases} \cos(\theta), & \frac{-\pi}{6} \leq \theta < \frac{\pi}{6} \\ \sqrt{3} - \frac{1}{2}\cos(\theta) - \frac{\sqrt{3}}{2}\sin(\theta), & \frac{\pi}{6} \leq \theta < \frac{\pi}{2} \\ \frac{-1}{2}\cos(\theta) + \frac{\sqrt{3}}{2}\sin(\theta), & \frac{\pi}{2} \leq \theta < \frac{5\pi}{6} \\ \sqrt{3} + \cos(\theta), & \frac{5\pi}{6} \leq \theta < \frac{7\pi}{6} \\ \frac{-1}{2}\cos(\theta) - \frac{\sqrt{3}}{2}\sin(\theta), & \frac{7\pi}{6} \leq \theta < \frac{3\pi}{2} \\ \sqrt{3} - \frac{1}{2}\cos(\theta) - \frac{\sqrt{3}}{2}\sin(\theta), & \frac{3\pi}{2} \leq \theta < \frac{11\pi}{6} \end{cases}$$

Figure 9 shows the plot of this curve. A closer examination of this plot shows that it's consistent with the geometric notion of the shadow function. If we look at

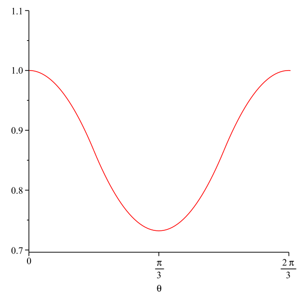

FIGURE 9. Plot of the shadow function for the Reuleaux Triangle.

Figure 8 at the point $(0, 1)$ we see quite clearly that the length of t_1 is equal to 1. Plugging this point into the shadow function we get, $\cos(0) = 1$, which verifies this result. And at $\pi/3$ there is a minimum, which makes sense because this is directly opposite where there is a maximum on the other side of the curve. By comparing the results from the shadow functions for both the circle and the Reuleaux Triangle, we see that although both curves have the same width they have very different shadow functions. Though in both cases the width $t_1 + t_2$ is $\sqrt{3}$, the variation in t_1 for the Reuleaux Triangle shows the shadow sliding back and forth as the angle of the light source changes. We will see another example of this in the following section.

2.4. **Another Construction for Smooth Curves of Constant Width.** Stanley Rabinowitz, in an article titled "A Polynomial Curve of Constant Width", describes a method for constructing curves of constant width that result in a smooth curve, in fact represented by a polynomial. In his paper he defines the curve of interest, C, to be a smooth convex curve that contains the origin, O. The following summarizes what he states in this paper. The parametric representation of this curve is represented in the following equations,

$$x = p(\theta)\cos(\theta) + p'(\theta)\sin(\theta) \tag{2}$$

$$y = p(\theta)\sin(\theta) - p'(\theta)\cos(\theta) \tag{3}$$

where the width of the curve in the direction of θ can be given by $q(\theta) = p(\theta) + p(\theta + \pi)$. The derivation of these equations is shown later in Sec 3. We therefore need to find a function for $p(\theta)$ such that $p(\theta) + p(\theta + \pi)$ is equal to a constant. He gives us $p(\theta) = a\cos^2(\frac{k\theta}{2}) + b$ where k is any positive odd integer and $a, b \geq 0$. Next we can make changes to k, a and b in order to produce various curves of constant width. Changing k produces curves with k "vertices", while making changes to a and b will change the width. We already know the width to be $q(\theta) = p(\theta) + p(\theta + \pi)$. If we substitute in the above value for $p(\theta)$ we get

$$q(\theta) = a\cos^2(\frac{k\theta}{2}) + b + a\cos^2(\frac{k(\theta + \pi)}{2}) + b$$

and using the half angle formula, $\cos^2(\frac{\theta}{2}) = \frac{[1+\cos(\theta)]}{2}$, and using the fact that $\cos(\theta + \pi) = -\cos(\theta)$ we get

$$q(\theta) = \frac{a}{2}(1 + \cos(k\theta)) + 2b + \frac{a}{2}(1 - \cos(k\theta)).$$

Thus we find that the width of the curve is $q = a + 2b$. We can therefore create a curve with a certain width using this method. We do this by substituting values for k, a and b into our equation for $p(\theta)$ and then substituting this back into Equations 2 and 3. Figure 10 shows that for $k = 1, a = 2, b = 2$ we get back just a circle. Another example, with $k = 3, a = 2, b = 8$, we get an image that looks a lot like the extension of the Reuleaux Triangle that we constructed from a previous method, but this however is a smooth curve with a parametric representation of

$$x = 9\cos(\theta) + 2\cos(2\theta) - \cos(4\theta)$$

$$y = 9\sin(\theta) - 2\sin(2\theta) - \sin(4\theta)$$

and therefore is not the same curve (much simplification is required to reduce to these equations for x and y.) Figure 11 shows a plot of this curve.

Figure 12 shows a plot of this construction with $k = 5, a = 1, b = 12$. Making alterations with this method, it appears that we are getting the curves that we

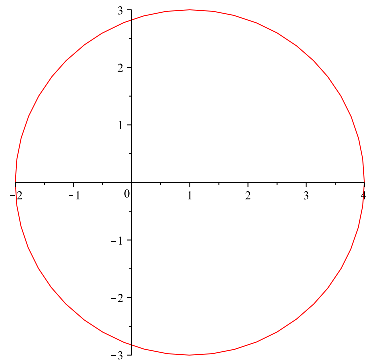

FIGURE 10. Graph of new construction with $k = 1, a = 2, b = 2$.

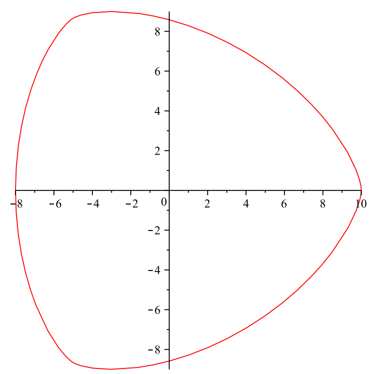

FIGURE 11. Plot of new construction with $k = 3, a = 2, b = 8$.

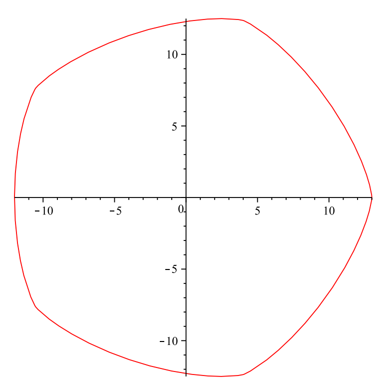

FIGURE 12. Plot of new construction with $k = 5, a = 1, b = 12$.

had previously already known to be curves of constant width, however these curves have different representation and therefore cannot be the same curves.

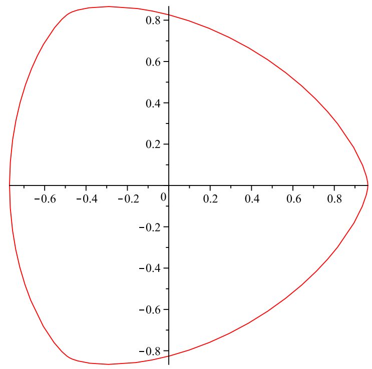

FIGURE 13. Plot of the curve with $k = 3, a = \frac{\sqrt{3}}{9}, b = \frac{4\sqrt{3}}{9}$.

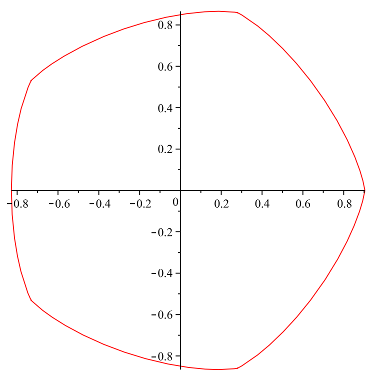

FIGURE 14. Plot of the curve with $k = 5, a = \frac{\sqrt{3}}{27}, b = \frac{13\sqrt{3}}{27}$.

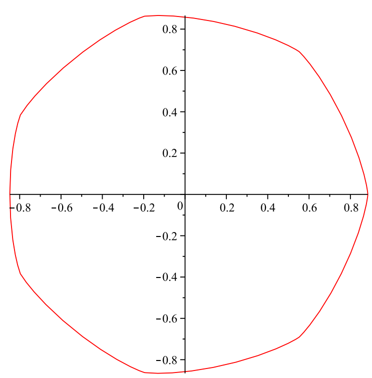

FIGURE 15. Plot of the curve with $k = 7, a = \frac{\sqrt{3}}{47}, b = \frac{23\sqrt{3}}{47}$.

2.5. **Shadow Functions for Rabinowitz Curves.** In order to compare the Rabinowitz curves with the Reuleaux and the circle, I chose an a and a b that gave me

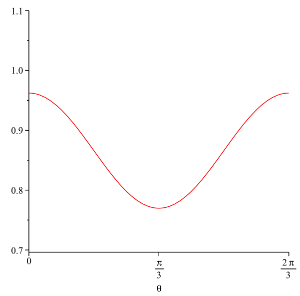

FIGURE 16. Shadow Function for Rabinowitz Curve with $k = 3$.

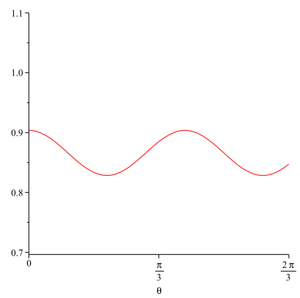

FIGURE 17. Shadow Function for Rabinowitz Curve with $k = 5$.

curves with width equal to $\sqrt{3}$. These curves can be viewed in Figures 13, 14 and 15. Because the Rabinowitz curves are infinitely smooth with a simple parametric representation, we can plug the parametric representations into Equation 1. Figure 16 shows the plot of the shadow function for the Rabinowitz curve with $k = 3$, Figure 17 with $k = 5$ and Figure 18 with $k = 7$, all with width equal to $\sqrt{3}$. A plot of these functions on the same graph along with the shadow function for the circle can be viewed in Figure 19. Again here we can see graphically that there are indeed differences in the shadow functions for these curves. In the following section we will show this mathematically.

3. Distinguishing Curves of Constant Width via Shadow Functions

In this section we are going to show that curves of constant width can be distinguished by their shadow functions. The following theorem outlines this idea.

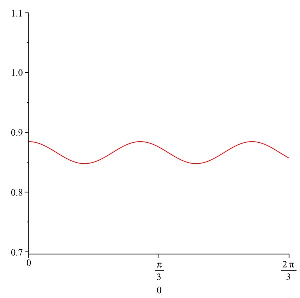

FIGURE 18. Shadow Function for Rabinowitz Curve with $k = 7$.

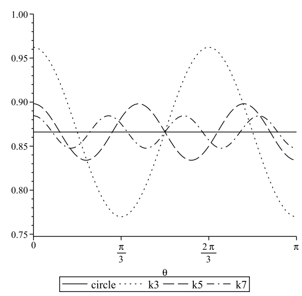

FIGURE 19. Combination of the shadow functions for the Rabinowitz curves and circle.

Theorem 1. If you know the shadow function for a differentiable curve you can determine the parametric representation of that curve.

Proof. Let $\omega(\theta) = (\cos(\theta), \sin(\theta))$ and $\hat{\omega}(\theta) = (-\sin(\theta), \cos(\theta))$ and our parameterization for our curve is given by $\vec{r}(\theta)$. It follows that

$$\vec{r}(\theta) = t_1(\theta)\omega(\theta) + g(\theta)\hat{\omega}(\theta)$$

for some function $g(\theta)$. If we express

$$\vec{r}(\theta) = (x(\theta), y(\theta)).$$

and take the dot product of both sides with $\omega(\theta)$, we get

$$\begin{aligned}
x(\theta)\cos(\theta) + y(\theta)\sin(\theta) &= \vec{r}(\theta)\cdot\omega(\theta) \\
&= (t_1(\theta)\omega(\theta) + g(\theta)\hat{\omega}(\theta))\cdot\omega(\theta) \\
&= t_1(\theta)\cdot 1 + g(\theta)\cdot 0 \\
&= t_1(\theta)
\end{aligned}$$

Where $t_1(\theta) = x(\theta)\cos(\theta) + y(\theta)\sin(\theta)$, is the equation we know already for the shadow function. Note that $\omega''(\theta) = -\omega(\theta)$ and taking the derivative of $\vec{r}(\theta)$ we get,

$$\begin{aligned}
\vec{r}_T(\theta) &= t_1'(\theta)\omega(\theta) + t_1(\theta)\hat{\omega}(\theta)) + g'(\theta)\hat{\omega}(\theta)) + g(\theta)\omega''(\theta) \\
&= (t_1'(\theta) - g(\theta))\omega(\theta) + (t_1(\theta) + g'(\theta))\hat{\omega}(\theta)
\end{aligned}$$

We know that $\vec{r}_T(\theta)$ is perpendicular to $\omega(\theta)$ and for this to be true for this representation of $\vec{r}_T(\theta)$ then,

$$t_1'(\theta) - g(\theta) = 0$$

or $t_1'(\theta) = g(\theta)$ for all values of θ. Therefore we now know that the following is true

$$\vec{r}(\theta) = t_1(\theta)\omega(\theta) + t_1'(\theta)\hat{\omega}(\theta).$$

Which implies that $\vec{r}(\theta)$ is determined by $t_1(\theta)$ for differentiable curves. If we substitute the values for each of the functions of θ into this equation we can express it in terms of $x(\theta)$ and $y(\theta)$.

$$(x(\theta), y(\theta)) = (t_1(\theta)\cos(\theta), t_1(\theta)\sin(\theta)) + (t_1'(\theta)\sin(\theta), -t_1'(\theta)(\theta)\cos(\theta))$$

or

$$x(\theta) = t_1(\theta)\cos(\theta) + t_1'(\theta)\sin(\theta)$$

$$y(\theta) = t_1(\theta)\sin(\theta) - t_1'(\theta)\cos(\theta)$$

From this we can see that the parametric representation is determined by the shadow function.

□

From this theorem we see that curves of constant width, q, can be distinguished by their shadows. One interesting property to consider when looking at curves of constant width is the perimeter. There is no way of distinguishing a curve of width q by its perimeter. We are going to show that a curve with width q will have perimeter $q\pi$. We already know the perimeter formula for one curve of constant width, the circle. The perimeter of a circle is given by $2\pi r$, where the width q is equal to $2r$. We see that this is consistent with $q\pi$. Consider next the Reuleaux Triangle.

Reuleaux Triangle. We can show that this holds for the Reuleaux Triangle. Recall that an arc of angle θ of a circle with radius r has length $r\theta$. Using this on the first region of the Reuleaux triangle yields $S = \frac{\sqrt{3}\pi}{3}$. Because there are three regions in the triangle with the same arclength, we can conclude that the arclength of the triangle is $\sqrt{3}\pi$.

Surprisingly, all curves of constant width, q, have perimeter πq. The following theorem provides a way to determine the perimeter of a curve of constant width, q.

Theorem 2. If the region C is bounded by a two times differentiable curve of constant width q, then the perimeter S of C is equal to πq.

Proof. For this proof we need a two times continuously differentiable parameterization. This proof does not work for the Reuleaux Triangle even though we know from above that it is in fact true for the Reuleaux Triangle. An alternate proof for curves that are not differentiable can be viewed in Steven Lay's book "Convex Sets and their Applications," [7]. From above we have that $t_1''(\theta) = g'(\theta)$ which means that

$$\vec{r}_T(\theta) = (t_1(\theta + t_1''(\theta))\hat{\omega}(\theta).$$

Since $||\hat{\omega}(\theta)|| = 1$ then when you take the magnitude of $\vec{r}_T(\theta)$ you get

$$||\vec{r}_T(\theta)|| = ||(t_1(\theta) + t_1''(\theta))\hat{\omega}(\theta)|| = \sqrt{(t_1(\theta) + t_1''(\theta))^2} = t_1(\theta) + t_1''(\theta).$$

Here we chose the positive root because we have a counterclockwise parameterization. This means that the curve will have length equal to

$$S = \int_0^{2\pi} t_1(\theta) + t_1''(\theta)\, d\theta.$$

Because

$$\int_0^{2\pi} t_1''(\theta)\, d\theta = t_1'(2\pi) - t_1'(0) = 0,$$

we are left with the following

$$S = \int_0^{2\pi} t_1(\theta)\, d\theta.$$

If we split this integral the equation will be

$$S = \int_0^{\pi} t_1(\theta)\, d\theta + \int_{\pi}^{2\pi} t_1(\theta)\, d\theta.$$

If we shift $t_1(\theta)$ by π in the second half of the equation using $\phi = \theta - \pi$ as a substitution we get the following,

$$\int_{\pi}^{2\pi} t_1(\theta) = \int_0^{\pi} t_1(\phi + \pi)\, d\phi.$$

Therefore we can write the perimeter as

$$S = \int_0^{\pi} t_1(\theta)\, d\theta + \int_0^{\pi} t_1(\theta + \pi)\, d\theta.$$

Combining the integral we get

$$S = \int_0^{\pi} t_1(\theta) + t_1(\theta + \pi) d\theta.$$

Because the width of the curve is given by $q = t_1(\theta) + t_1(\theta + \pi)$ and is a constant, then the integral is

$$S = \int_0^\pi q d\theta = q\pi.$$

[5]

$\square$

3.1. **Area Result.** Another interesting property of curves of constant width can be found in Steven Lay's book. Here he proves that of the various curves of constant width, q, the circle has the largest area while the Reuleaux Triangle has the smallest [7].

4. Conclusion

In this paper we explored the construction of various curves of constant width. These curves were used to investigate the question of whether or not we could distinguish two curves with the same width by looking at their shadows. Before we did this we derived an equation for the shadow function. We then used the shadow functions for a circle, the Reuleaux Triangle, and some examples of curves of constant width proposed by Stanley Rabinowitz (all with width $q = \sqrt{3}$) to graphically see that you can in fact determine differences. Next a proof was used to show that if you know the shadow function for a differentiable curve you can determine the parametric representation of that curve.

Finally, we showed that although it is possible to determine a curve from its shadow function, it is not possible to distinguish two curves of the same width from their perimeter. More specifically, we showed that if the region C is bounded by a two times differentiable curve of constant width q, then the perimeter S of C is equal to πq.

These findings could be furthered by considering these same properties, shadow functions and the perimeter result, for infinitely smooth curves of constant width.

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