Senior Written Exam SAMPLE

Do all 16 problems, each problem on a separate sheet of the provided blank paper. All problems are equally weighted. Turn in this exam sheet and your work. Be sure to write your name and the relevant problem number on every sheet of paper.

- Show your work and justify your answers.
- Be neat and clear.
- Use complete English sentences to explain your work and to indicate the final answer to your problem.
- You may not use a calculator.

1. Function Theory and Differential Calculus

(1) This problem involves the Mean Value Theorem (MVT):
   (a) Carefully state the MVT.
   (b) Let $f(x) = (x+1)/(x-1)$. Show that there is no value of c that satisfies the conclusion of the MVT on the interval $[0, 2]$. Why does this not contradict the MVT?

(2) The angle of elevation of the sun is decreasing at a rate of 0.25 radian/hour. How fast is the shadow cast by a 400-ft-tall building increasing when the angle of elevation of the sun is $\pi/6$? Be certain to include the units of your answer.

(3) Use calculus to find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.

(4) Given a differentiable function f and a point x_0 , let x_t be the x-intercept of the tangent line to f at x_0. The subtangent of f at x_0 is defined to be $x_t - x_0$. Find a formula for the subtangent of f at x_0. Apply your formula to the function $f(x) = x^3$.

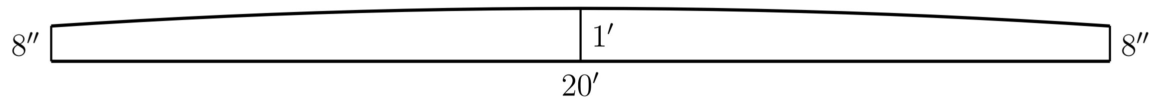
2. Integral Calculus, Sequences and Series

(1) Evaluate the indefinite integral
$$\int \frac{t-9}{t^2+3t-10}\, dt.$$

(2) The curves $y = \cos x$ and $y = \cos x \sin x$ intersect twice between $x = 0$ and $x = 2\pi$. Find the two points of intersection and compute the area bounded by the two curves between these points.

(3) Compute the Maclaurin series for the specific function $f(x) = \ln(x+1)$. Give both the radius of convergence and the interval of convergence of the series. Also, be careful to describe the convergence properties at the endpoints of the interval of convergence.

(4) Compute the number of cubic yards of crushed rock necessary to make a roadbed one mile long with cross section shown below.



Assume that the crown of the pavement is a parabola. (Note: 5,280 ft = 1,760 yd = 1 mi, you need not compute tedious products)

3. Vector Calculus

(1) Let
$$f(x,y) = \frac{y}{x+y}.$$
Find a unit vector, $\mathbf{u}$, for which $D_{\mathbf{u}}f(2,3) = 0$.

(2) Find the work done if a particle moves from $(-2,4)$ to $(1,1)$ along the parabola, $y = x^2$, while subject to the force $\mathbf{F}(x,y) = x^3y\mathbf{i} + (x-y)\mathbf{j}$.

(3) Use a double integral to compute the volume of the solid bounded by the cylinder, $x^2 + y^2 = 9$, and the planes $z = 0$ and $z = 3 - x$.

(4) Explain the origins of the standard equation for a plane in $\mathbb{R}^3$. What information is needed to find the equation of a plane? Illustrate some of these ideas by finding an equation for the plane that passes through three points, say $(0,0,1)$, $(0,1,0)$, and $(2,0,0)$.

4. Linear Algebra

(1) By examining the determinant, show that the following system has a nontrivial solution if and only if the parameters α and β are equal. Explain your reasoning.
$$\begin{bmatrix} 1 & 1 & \alpha \\ 1 & 1 & \beta \\ \alpha & \beta & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(2) Consider the matrix
$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

   (a) Show that if $0 < \theta < \pi$ then A has no real eigenvalues and consequently no real eigenvectors.

   (b) Geometrically, the action of A on a vector is to rotate the vector through an angle θ. Use this fact to give a geometric explanation of the result in Part (a).

(3) Consider the vectors $\mathbf{v}_1 = (0,3,1,-1)$, $\mathbf{v}_2 = (6,0,5,1)$, and $\mathbf{v}_3 = (4,-7,1,3)$ in $\mathbb{R}^4$.

   (a) Express $\mathbf{v}_1$ as a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$.

   (b) Refer to the definition of linear independence to explain why the solution to Part (a) shows that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly *dependent*.

(4) Let D be the vector space of all differentiable functions and let P_2 be the vector space of all real polynomials of degree less than or equal to 2. Define $T : D \to P_2$ by $T(f) = f'(0)x + f(0)$. Prove that T is a linear transformation. How is the function $T(f)$ related to the function f?