On Proofs Without Words

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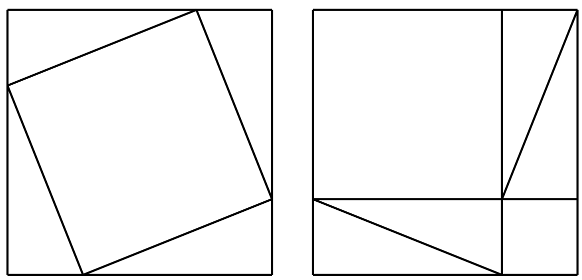

Behold!

1 Introduction

Most mathematicians will be familiar with the above picture. This diagram, credited to the Ancient Chinese mathematical text *Zhou Bi Suan Jing*, is a charmingly simple visual proof of the Pythagorean Theorem, one of mathematics' most fundamental results. It would be hard to argue that this proof is not convincing. In fact, most standard proofs of the Pythagorean Theorem still use this picture, or variations of it. In this paper, we will examine pictures, such as this one, which claim to prove mathematical theorems and have come to be known as "Proofs Without Words", or PWWs.

This paper will have two main parts. In the first part, we will present collection of PWWs accompanied by explanations. Essentially we will be putting the words back into these "Proofs Without Words" by explicitly stating what our brains are seeing, and how we are supposed to reach the intended conclusions given only the visual clues contained in the figure. In addition, where appropriate, we will include "parallel proofs", which are more traditional proofs of the same results portrayed by the PWWs. The aim of this is to see the differences between formal logical structure, and the logic that our brains will follow given visual information.

The second part will examine what it actually means to "prove" a result, whether Proofs Without Words satisfy this definition, and if they do not, what value they have for mathematics. Any textbook will tell you that a proof is a series of statements that show how a new statement is true by using logic and statements that are already known to

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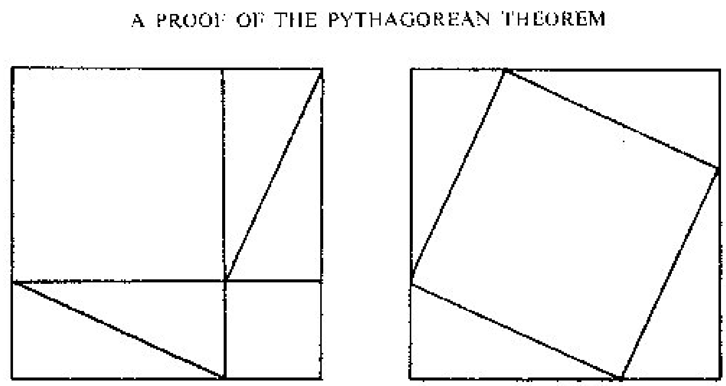


Figure 1: The Pythagorean Theorem from *Mathematics Magazine, Sep. 1975* [4]

be true. However, proofs have other characteristics which leave this definition somewhat open to interpretation. Proofs are necessarily irrefutable, generalizable, and convincing to the intended audience. If a series of statements matches the accepted definition, but does not meet one of these criteria, is it not a proof? If a picture, which by nature is not a series of steps, meets all three criteria, can it be a proof? Is it even possible for a picture to meet these criteria? These are just several of the questions which can arise. With this paper, we will present as many facets as possible of a complicated philosophical debate while exploring the undeniable value and contributions of visual proofs.

2 A Brief History of Proofs Without Words

Even ancient mathematicians expressed mathematical ideas with pictures. The proof of the Pythagorean Theorem presented in the introduction (See Figure 1) is an example of how ancient mathematicians found evidence of mathematical relationships by drawing pictures. In addition to the ancient Chinese mathematical text, *Zhou Bi Suan Jing* (c. 200 BCE), variations of this geometric proof have been credited to Pythagoras himself (c. 600 BCE), and to the Hindu mathematician Bhaskara (c. 200 CE) [19]. Even Euclid included a now famous proof of the theorem in *The Elements* (See Figure 2)[6].

Despite their ancient roots, visual proofs are still utilized by modern mathematicians. However, they did not garner official recognition (and the title "Proofs Without Words") until the Mathematics Association of America began publishing them regularly in *Mathematics Magazine* and *The College Mathematics Journal* staring in the mid 1970s. In September 1975, Rufus Issacs published an article entitled "Two Mathematical Papers Without Words" in *Mathematics Magazine*[4]. This short "paper" appeared at the end of a longer article and included two figures. One was of the Pythagorean Theorem proof discussed above (see Figure 1), and the other was of a hypothetical device designed to trisect an angle, a task deemed impossible by ancient Greek mathematicians[4]. While neither of these was specifically designated as proofs, they were clearly intended to convey a mathematical idea in a purely visual manner.

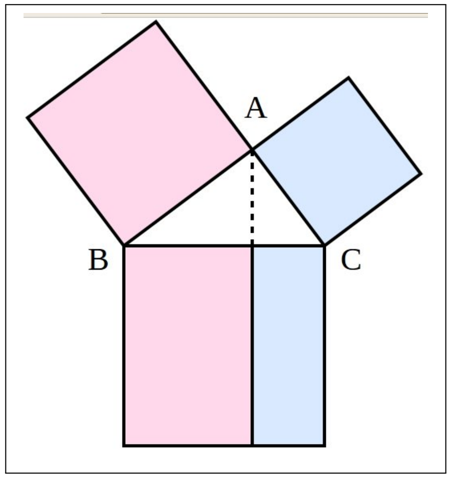


Figure 2: Euclid's Pythagorean Theorem [19]

In January 1976, two months after these figures were published, *Mathematics Magazine* came under the direction of two new co-editors, J. Arthur Seebach, and Lynn Arthur Steen. With the change in editorial leadership also came a slight change in format, which included a new "News and Letters" section, which replaced the old "Notes and Comments" section, and was designed to streamline the process of reader feedback, allowing comments on published articles to be printed within months of the original publication date[4]. As a result, several readers submitted comments regarding articles published in the September 1975 issue. Interestingly, the majority of the comments submitted were regarding "Two Mathematical Papers Without Words." In the same "News and Letters" section, the new co-editors included the following statement:

> "Editor's Note: We would like to encourage further contributions of proofs without words for the reasons mentioned by Rufus Isaacs and one other: we are looking for interesting visual material to illustrate the pages of the *Magazine* and to use as end-of-article fillers. What could be better for this purpose than a pleasing illustration that made an important mathematical point?" [5]

Following the publication of this request, figures meeting this description began appearing in the *Magazine* under the heading "Proof Without Words" at a rate of approximately one or two per year. By 1987, that rate had increased to five or six per year, averaging to about two per issue. Needless to say, mathematicians began to take notice of these intriguing mathematical gems. Dr. Roger Nelsen, professor of mathematics at Lewis and Clark College in Portland, Oregon was no exception. In June 1987, after several attempted submissions, he published his own PWW entitled "The Harmonic Mean-Geometric Mean-Arithmetic Mean-Root Mean Square Inequality."[8] [20] In the spirit of the peer-reviewed publication, the MAA then asked him to referee other PWW submissions. Over the years, he began collecting any PWWs that came to him for feedback. Eventually, he had enough to make a collection so he published *Proofs Without Words: Exercises in Visual Thinking*[8]. Many of the PWWs that appear in this paper have been taken from this

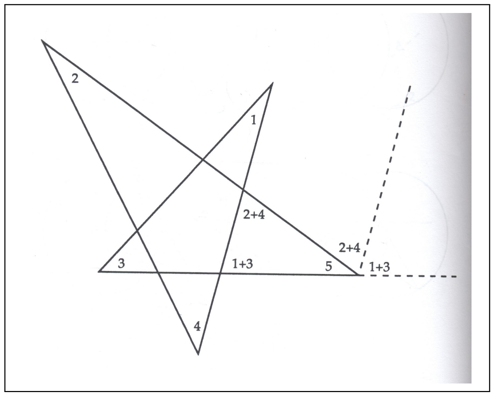


Figure 3: The Vertices of a Star Sum to 180° [6, p.14]

collection and its sequel, *Proofs Without Words II: More Exercises in Visual Thinking.*

3 Examples and Explanations

We will explore three overarching categories of PWW for this paper: Geometry, Calculus, and Integer Sums. In this section, we present a collection of Proofs Without Words from each of these categories. For most of these, we will include more traditional proofs alongside their corresponding PWWs (or "parallel proofs") as a source comparison. For the sake of attempting to understand how PWWs convey information we will also, where possible, include explanations of how the intended result follows from the picture.

3.1 Geometry

Because geometry by nature deals with figures in space, results in this branch of mathematics tend to lend themselves easily to Proofs Without Words.

As a starting example, consider Figure 3. This is a Proof Without Words originally created by Fouad Nakhli and included in Nelsen's first collection which proves the property that the angle measures of the five vertices of a star sum to 180°. In order provide a traditional proof of this result, we will state without proof the following two familiar theorems from Euclidean Geometry.

Theorem 1. *If two parallel lines are intersected by a transversal, then alternate interior angles are congruent.*

Theorem 2. *If two parallel lines are intersected by a transversal, then corresponding angles are congruent.*

Now, we are ready to prove the theorem itself. The proof will refer to Figure 4.

Theorem 3. *The vertex angles of a star sum to* 180°.

Proof. Construct lines EE_2 and CC_2 parallel to line AD and line DD_1 parallel to line CE. By Theorem 1, $\angle ACC_2 \cong \angle 2$. Then it also follows from Theorem 1 that $\angle C_2CG =$

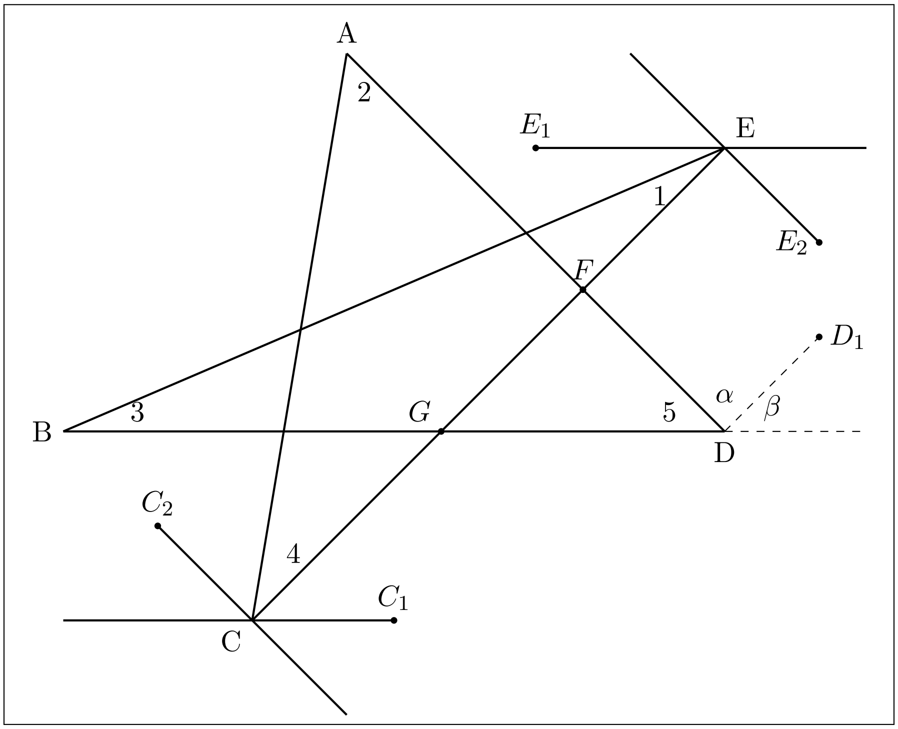


Figure 4

$\angle 2+\angle 4 \cong \angle FEE_2$. Then by Theorem 2, $\angle 2+\angle 4 \cong \angle GFD$. By Theorem 1, $\angle\alpha \cong \angle 2+\angle 4$. Now construct lines E_1E and C_1C parallel to line BD. By Theorem 1 $\angle E_1EB \cong \angle 3$. Then it also follows from Theorem 1 that $\angle E_1EC = \angle 1 + \angle 3 \cong \angle GCC_1$. Then by Theorem 2, $\angle FGD \cong \angle 1 + \angle 3$. Theorem 2 also establishes that $\angle\beta \cong \angle 1 + \angle 3$. Since line BD is straight, it follows that $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 \cong 180°$. □

As with many geometric proofs, the traditional proof of this result requires a picture (Figure 4) for clarification of the angle names and line segments to which the proof refers. However, this visual aid is meant only as an instrument, not a proof in itself. What makes Nakhli's diagram (Figure 3) worthy of the title "Proof Without Words" is the way the angles are labeled. Labeling the angles in the inner triangle $1 + 3$ and $2 + 4$ gives the reader a reason to look for a relationship between these angles. A reader who is presumably familiar with Theorems 1 and 2, will then make the same connections spelled out in the traditional proof, thereby coming to the same conclusion. The assumption that there are certain expectations of the reader in a PWW that are not expected of a traditional reader is the key to the success of many PWWs.

Another example of a geometric result represented with a PWW is Viviani's Theorem. As with the vertices of a star, the traditional proof [9] of this result also uses a non-PWW figure as an aid (See Figure 5).

Theorem 4. *The perpendiculars p_i to the sides from a point P on the boundary or within an equilateral triangle add up to the height of the triangle.*

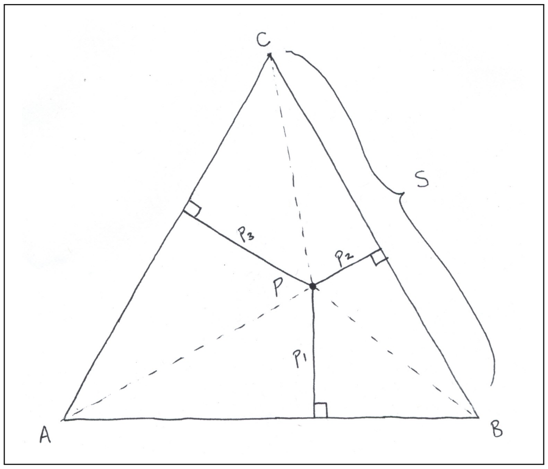


Figure 5

Proof. Consider an equilateral triangle with side length s, height h, and vertices A, B, and C, such as the one in Figure 5. Pick an arbitrary point P within the triangle or on the boundary, and extend perpendicular p_1 to side $\overline{AB}$, perpendicular p_2 to $\overline{BC}$, and perpendicular p_3 to $\overline{AC}$. Note that the area of $\triangle ABC$ is equal to $\triangle PAB + \triangle PBC + \triangle PAC$. Since each of the perpendiculars, p_i, is the height for one of these triangles, it follows that

$$\frac{1}{2}hs = \frac{1}{2}sp_1 + \frac{1}{2}sp_2 + \frac{1}{2}sp_3$$

and thus,

$$h = p_1 + p_2 + p_3.$$

□

Clearly, the figure used in the traditional proof plays a crucial role in conveying understanding, but only in the sense that it clarifies the meanings of the variables involved. Without the equations in the proof, the pictures says very little about the validity of Viviani's theorem. In contrast, the PWW (See Figure 6) is meant to make the validity of the statement clear without the need for words or equations. However, some explanation may be needed to see exactly why this picture proves Viviani's Theorem.

Referring to Figure 6, we see that $\triangle ABC$ is the original triangle, and point P has been chosen as the arbitrary point. Triangle $\triangle ABC$ has been translated to the left to form $\triangle A'B'C'$ such that point P lies on the edge of $\triangle A'B'C'$.

The goal of this PWW is to show that $\overline{PF} + \overline{PD} + \overline{PG} = h$ where h is the height of $\triangle ABC$. This can be done by showing that $\overline{PF} = \overline{GQ}$, and that $\overline{PG} = \overline{C'G}$. It should then be clear from the picture that $\overline{C'G} + \overline{GQ} + \overline{PD} = h$.

First, we must first show that $\triangle C'HP$ is an equilateral triangle. To do this, it is sufficient to show that the three angles of this triangle are equal. Since $\overline{HP}$ and $\overline{A'B'}$ are parallel, it follows by the Corresponding Angles Theorem that $\angle C'PH = \angle A'B'C'$ and

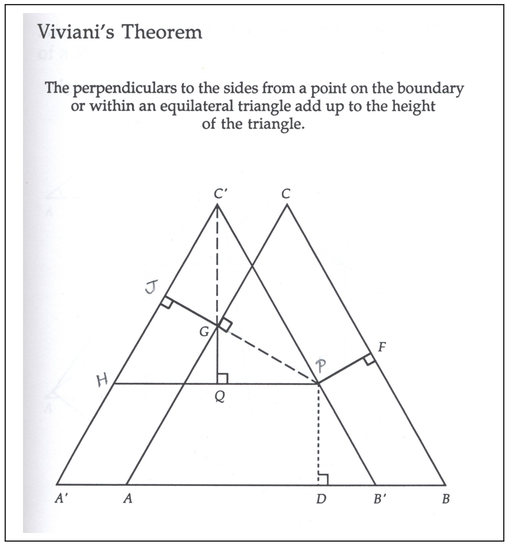


Figure 6: Viviani's Theorem PWW [6, p.15]

$\angle C'HP = \angle B'A'C'$. Since $\triangle A'B'C'$ is equilateral, $\angle A'C'B' = \angle A'B'C' = \angle B'A'C'$, and thus $\angle A'C'B' = \angle C'HP = \angle C'PH$, It follows that $\triangle C'HP$ is also equilateral.

Next, we want to show that $\triangle C'GJ \cong \triangle GPQ$. Note that since $\triangle C'HP$ is equilateral, and $\overline{C'Q}$ and $\overline{JP}$ are perpendicular to the edges $\overline{PH}$ and $\overline{C'H}$ respectively, $\overline{C'Q}$ and $\overline{JP}$ bisect $\angle C'$ and $\angle P$, respectively. The perpendicular bisector of an angle of an equilateral triangle bisects not only the angle, but also the line segment which it meets. And since each angle and each side in the triangle are equal, their halves will also be equal to each other. Thus $\angle JC'G = \angle GPQ$ and $\overline{PQ} = \overline{C'J}$. Since $\angle GQP = \angle C'JG = 90°$, it follows by angle-side-angle that $\triangle C'GJ \cong \triangle GPQ$. This step has shown us that $\overline{PG} = \overline{C'G}$ and that $\overline{GJ} = \overline{GQ}$.

The final step is to show that $\overline{PF} = \overline{GJ}$ meaning that $\overline{PF} = \overline{GQ}$. This follows easily from the fact that $\triangle A'B'C'$ was formed by translating $\triangle ABC$ to the left by a fixed amount. Since both line segments are perpendicular to the edges of the triangle, $\overline{PF}$ represents the fixed distance between parallel lines $\overline{BC}$, and $\overline{B'C'}$ and $\overline{GJ}$ represents the fixed distance between $\overline{AC}$ and $\overline{A'C'}$. Since this distance must be the same on both sides, $\overline{PF} = \overline{GJ}$.

The reader may notice that this explanation is quite involved and that the PWW likely had to be studied carefully before the result became apparent. However, another PWW of this result (See Figure 7) has since been published, which requires less explanation. The reader will notice that by simply rotating the inner triangles, the relationship between the perpendiculars and the height of the triangle becomes easily apparent, without the need to invoke theorems about corresponding triangles.

3.2 Calculus

Although pictures are often used in calculus courses to help students understand unfamiliar concepts such as integration and differentiation, proofs of standard calculus results rarely

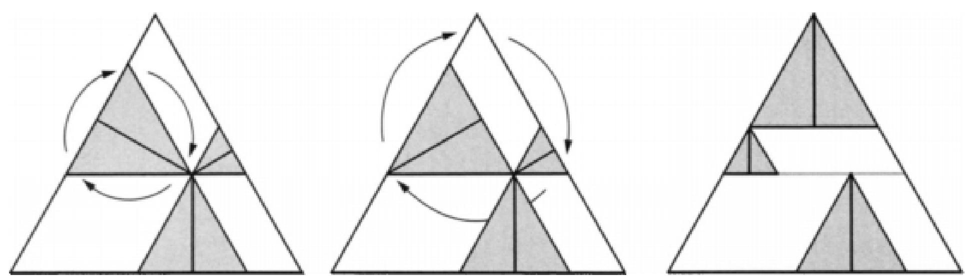


Figure 7: A new PWW of Viviani's Theorem [22]

depend on pictures, since the majority of calculus involves manipulating functions. Unlike geometric objects, functions can be represented by pictures, but they can also exist and be manipulated without the aid of a picture. Since functions are not inherently visual, they are somewhat less suited to PWWs. Nevertheless, PWWs related to calculus do exist, and we will explore a few of them here.

One of the most well-known results in calculus is Integration By Parts. We will first present the traditional proof, followed by the PWW and an accompanying explanation.

The following is the proof of Integration by Parts as found in Stewart's *Calculus* [15].

Theorem 5. *If f and g are differentiable functions, then*

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

This formula extends to definite integrals as

$$\int_a^b f(x)g'(x)dx = [f(b)g(b) - f(a)g(a)] - \int_a^b g(x)f'(x)dx. \tag{1}$$

Proof. Since f and g are differentiable functions, then The Product Rule states that

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x). \tag{2}$$

Integrating both sides of Equation 2 yields

$$\int [f(x)g'(x) + g(x)f'(x)]dx = f(x)g(x)$$

or

$$\int f(x)g'(x)dx + \int g(x)f'(x) = f(x)g(x). \tag{3}$$

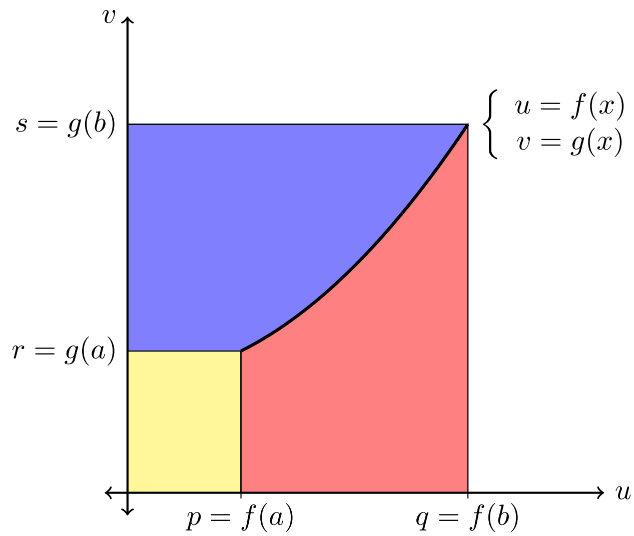


Figure 8: Integration By Parts [6, p.42]

Then rearranging Equation 3 then gives us

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx. \tag{4}$$

$\square$

The formula expressed in Equation 4 is more commonly written as

$$\int udv = uv - \int vdu$$

where $u = f(x)$, $du = f'(x)dx$, $v = g(x)$, and $dv = g'(x)dx$.

The Proof Without Words of Integration By Parts is found in Nelsen's first collection [6, p.42], and is reproduced in Figure 8. This picture graphically represents the quantities $f(b)g(b)$ and $f(a)g(a)$ as rectangles formed in a coordinate system, where the function $u = f(x)$ is expressed on the horizontal axis, and $v = g(x)$ is expressed on the vertical axis. By convention, we assume that $a < b$. The blue area, which represents $\int_a^b f(x)g'(x)dx$ (or $\int_{f(a)}^{f(b)} udv$), can be found in three steps:

1. Notice that the area of the whole rectangle (RED + BLUE + YELLOW) is $f(b)g(b)$.
2. Then from that subtract the area of the YELLOW rectangle, which is $f(a)g(a)$.
3. Then subtract the RED area, which is mathematically represented by $\int_a^b g(x)f'(x)dx$ (or $\int_{g(a)}^{g(b)} vdu$).

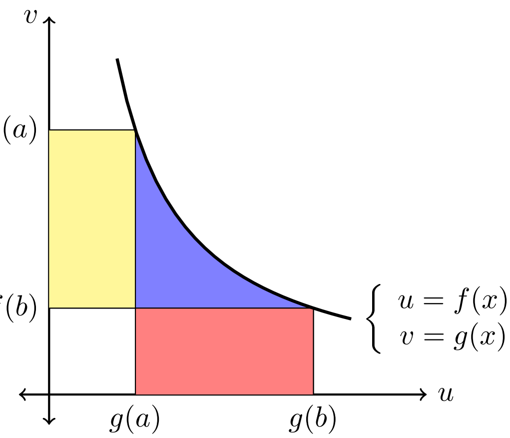


Figure 9: Integration By Parts for Decreasing Functions

This gives us that BLUE = [total - YELLOW] - RED, or mathematically,

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)dx,$$

which is equivalent to Equation 1.

It is important to note that the proof in Figure 8 is only a representation of specific functions, with specific properties. In this particular PWW, the functions in question, $v = f(x)$ and $u = g(x)$, are both increasing, since $a < b$ and $f(a) < f(b)$ and $g(a) < g(b)$. This proof says nothing about the case where one function is increasing, and the other is decreasing.

To remedy this, we present an original PWW, which is a modification of Figure 8 that demonstrates the validity of Integration by Parts for functions whose slopes have opposite signs. Consider Figure 9. In this picture, we again have two parametrically defined functions, $u = f(x)$ and $v = g(x)$. But notice that while u is still increasing, since $a < b$ and $f(a) < f(b)$, the function v is now decreasing since $g(b) < g(a)$.

We can make the following observation about the figure:

$$\Big(BLUE + RED\Big) - RED - WHITE = \Big(BLUE + YELLOW\Big) - YELLOW - WHITE.$$

Then, representing these colored areas as integrals, as we did above, this equality becomes

$$\int_{g(a)}^{g(b)} vdu - f(b)g(b) = \int_{f(b)}^{f(a)} udv - f(a)g(a). \tag{5}$$

Rearranging the terms in Equation 5 then gives us

$$-\int_{f(b)}^{f(a)} udv = f(b)g(b) - f(a)g(a) - \int_{g(a)}^{g(b)} vdu. \tag{6}$$

We note that the bounds on $\int_{f(b)}^{f(a)} udv$ are the reverse of those found on the corresponding integral in the previous proof (the blue area in Figure 8). This is due to fact that $u = f(x)$ is decreasing, so when the parameter x is equal to b, the function $f(x)$ is at its smallest value, rather than its largest value.

Taking this into consideration, we then multiply the left side of Equation 6 by -1, which gives us

$$\int_{f(a)}^{f(b)} udv = f(b)g(b) - f(a)g(a) - \int_{g(a)}^{g(b)} vdu. \tag{7}$$

Expressed in terms of the parameter x, this becomes

$$\int_{a}^{b} f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_{a}^{b} g(x)f'(x)dx. \tag{8}$$

We have just shown using a PWW that Integration By Parts is valid for functions of the form $f(x)g'(x)$ where f and g are both monotonic but with slopes of opposite sign. Since f and g will always either have slopes of the same sign, as in Figure 8, or opposite sign as in Figure 9, these two PWWs are sufficient to verify Integration by Parts for any function of the form $f(x)g'(x)$, regardless of the signs of the slopes of its component functions. However, even with this additional figure, this PWW can only be a representation of one specific combination of functions at a time. We will address this lack of generality in Section 4.3.2.

Another result commonly found in first or second semester calculus courses is the proof that the number e can be defined using a limit. As with the proof of Integration By Parts, we will first present a traditional proof, similar to one that would appear in a typical calculus textbook. We will then present the PWW of the same result, accompanied by a short explanation.

Theorem 6. *The number e can be expressed as*

$$\lim_{x\to 0} (1+x)^{1/x}.$$

The following proof can be found in Stewart's Calculus, 5th Edition, and uses the definition of a derivative[15]. It is assumed that the reader is familiar with this definition.

Proof. Suppose $f(x) = \ln(x)$. Then it can be shown independently of the limit definition of e that $f'(x) = 1/x$ and thus, $f'(1) = 1$ (See Stewart [15]). Then from the definition of

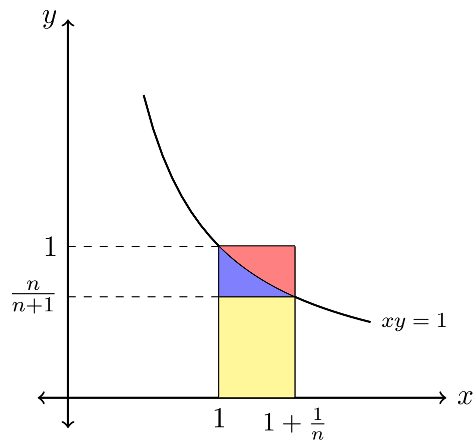


Figure 10: A Familiar Limit for e [7]

the derivative, we have

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{x \to 0} \frac{f(1+x) - f(1)}{x} \\ &= \lim_{x \to 0} \frac{\ln(1+x) - \ln(1)}{x} \\ &= \lim_{x \to 0} \frac{1}{x} \ln(1+x) \\ &= \lim_{x \to 0} \ln(1+x)^{1/x}. \end{aligned}$$

Since $f'(1) = 1$, it follows that

$$\lim_{x \to 0} \ln(1+x)^{1/x} = 1.$$

Then, by the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{1/x}} = \lim_{x \to 0} e^{\ln(1+x)^{1/x}} = \lim_{x \to 0} (1+x)^{1/x}.$$

Thus, $e = \lim\limits_{x \to 0} (1+x)^{1/x}$. $\square$

Note that this result can also be expressed as

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

The PWW of this result can be found in Figure 10 and it also uses the properties of the natural log function, specifically the property that $\ln(x)$ is an antiderivative of $1/x$.

Along with the picture, the published PWW included the following three mathematical statements:

$$\frac{1}{n} \cdot \frac{n}{n+1} \leq \ln\left(1+\frac{1}{n}\right) \leq \frac{1}{n} \cdot 1. \tag{9}$$

$$\frac{n}{n+1} \leq n \cdot \ln\left(1+\frac{1}{n}\right) \leq 1. \tag{10}$$

$$\therefore \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = 1. \tag{11}$$

The key to this PWW is the relationship between the colored areas. The Yellow region and the Yellow + Blue + Red region are both rectangles, so their areas are simply represented as $\frac{1}{n} \cdot \frac{n}{n+1}$ and $\frac{1}{n} \cdot 1$, respectively. In order to get the Yellow + Blue region, we need to find the area under the curve between 1 and $1 + \frac{1}{n}$. Notice that $xy = 1$ can also be expressed as $y = \frac{1}{x}$. Then

$$\int_1^{1+\frac{1}{n}} \frac{1}{x} = \ln\left(1+\frac{1}{n}\right) - \ln(1) = \ln\left(1+\frac{1}{n}\right).$$

Now that we know the areas of the three regions, we note the relationship between them. Visually, we can see the area of the Yellow region is less than the area of the (Yellow + Blue) region, which in turn is less than the area of the (Yellow + Blue + Red) region. This relationship is represented by Equation 9. Multiplying everything by n then gives us Equation 10.

Then by the Squeeze Theorem, since $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n+1}{n} = 1$, it follows that

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = 1. \tag{12}$$

By similar logic to the proof presented earlier,

$$e = e^1 = e^{\lim_{n\to\infty} \ln(1+\frac{1}{n})^n} = \lim_{n\to\infty} e^{\ln(1+\frac{1}{n})^n} = \lim_{n\to\infty} (1+\frac{1}{n})^n. \tag{13}$$

Essentially, this graph is a visual way of showing that $\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = 1$, which was proven using the definition of the derivative in the formal proof.

3.3 Integer Relationships

Many of the most intuitive and beautiful PWWs come from integer relationships. Because integers are used for counting things, they are easily represented by dots or other discrete objects, and when arranged thoughtfully, one can easily see the patterns that arise.

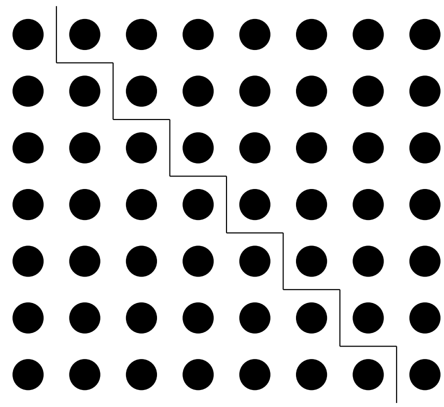


Figure 11: Sum of the first n integers

One of the most fundamental integer formulas is the formula for the sum of the first n integers, which is stated as follows:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \tag{14}$$

We can easily represent this formula with a PWW (See Figure 11). As one of the most basic and beautifully simple PWWs, this picture effortlessly reveals that the sum of the first n integers is a triangle, which composes one half the area of an $n \times (n+1)$ rectangle, which gives us the formula in Equation 14.

Another interesting way of looking at integer relationships is through triangular numbers. The definition of a triangular number is as follows:

Definition 7. *A triangular number, T_n, is the sum of the first n integers. That is,*

$$T_n = 1 + 2 + 3 + \cdots + n.$$

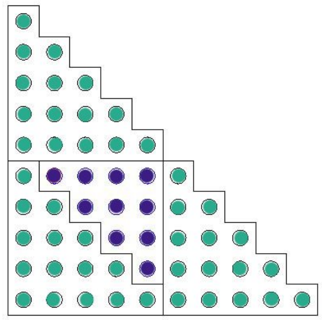
The following are two identities regarding triangular numbers. I will first present traditional induction proofs of these two theorems, followed by their PWWs.

Theorem 8. *If T_n is the n^{th} triangular number, then $3T_n + T_{n-1} = T_{2n}$.*

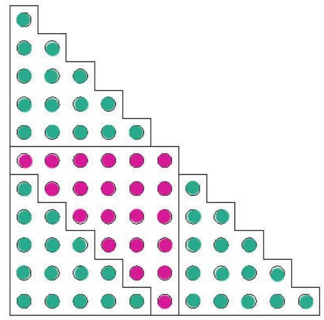
Proof. We will proceed by Mathematical Induction. Consider the case where $n = 1$. Since $3T_1 + T_0 = 3 + 0 = 3 = T_2$, the identity is true for $n = 1$. Now, suppose the identity is true for $n = k$. That is, suppose $3T_k + T_{k-1} = T_{2k}$. Now consider the case where $n = k + 1$.

$$\begin{aligned} 3T_{k+1} + T_k &= 3[T_k + (k+1)] + T_{k-1} + k \\ &= 3T_k + T_{k-1} + 4k + 3 \\ &= T_{2k} + (2k+1) + (2k+2) \\ &= T_{2(k+1)} \end{aligned} \tag{15}$$

Thus, by the Principle of Mathematical Induction, the identity holds for all positive integers. □



(a) $3T_n + T_{n-1} = T_{2n}$



(b) $3T_n + T_{n+1} = T_{2n+1}$

Figure 12: Triangular Number Identities [6]

Theorem 9. *If T_n is the n^{th} triangular number, then $3T_n + T_{n+1} = T_{2n+1}$.*

Proof. We will again proceed by Mathematical Induction. Consider the case where $n = 1$. Since $3T_1 + T_2 = 3 + 3 = 6 = T_3$, the identity is true for $n = 1$. Now, suppose the identity is true for $n = k$. That is, suppose $3T_k + T_{k+1} = T_{2k+1}$. Now consider the case where $n = k + 1$.

$$\begin{aligned} 3T_{k+1} + T_{k+2} &= 3[T_k + (k+1)] + T_{k+1} + (k+2) \\ &= 3T_k + T_{k+1} + 4k + 5 \\ &= T_{2k+1} + (2k+2) + (2k+3) \\ &= T_{2(k+1)+1} \end{aligned} \tag{16}$$

Thus, by the Principle of Mathematical Induction, the identity holds for all positive integers. □

The Proof Without Words of these two identities is found in Figure 12. The concept behind the PWW is really quite simple. Each triangular number is represented by a collection of dots arranged in, as the name would suggest, a triangle. For ease of understanding, we use contrasting colors to differentiate between different triangular numbers. Then by clever arrangement of the shapes, the identities are revealed.

As with PWWs that we have already investigated, this proof only addresses a specific case, specifically the case where $n = 5$. However, the generalization to all integers is believable. We will discuss the reasons for this in Section 4.3.2.

As a last example of integer relationships, we will consider the Fibonacci sequence. The definition of these familiar numbers is as follows:

Definition 10. *If $F_1 = F_2 = 1$, then for $n \geq 3$, the nth Fibonacci number, F_n, is given by*

$$F_n = F_{n-1} + F_{n-2}.$$

The following theorem is a simple identity involving Fibonacci numbers. This identity can easily be proven by mathematical induction (See Guichard [18]), but for the sake of brevity, we will omit the details of that proof.

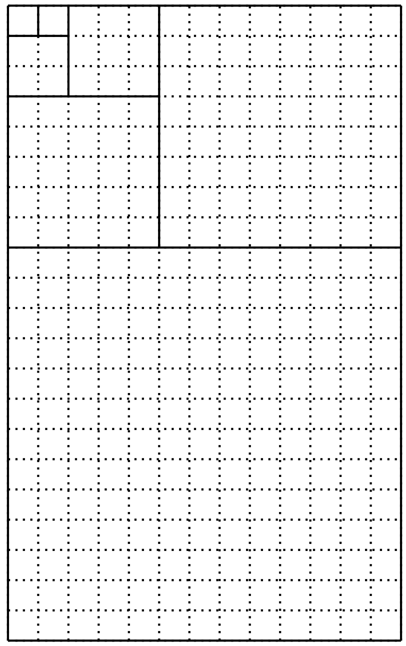


Figure 13: $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$ [6]

Theorem 11. $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$.

A visual representation of this result is given by the PWW shown in Figure 13. This figure is composed of squares, each with its side length equal to a particular Fibonacci number. By aligning n of these squares, it becomes clear that the sum of their areas is equal to a rectangle with side lengths equal to F_n and F_{n+1}, thus verifying that the sums of the first n Fibonacci numbers is equal to $F_n F_{n+1}$.

Theorem 12 presents a more complex Fibonacci Identity. This identity can also be proven by induction, but again, we will omit the details.

Theorem 12. $F_{n+1}^2 = 4F_n^2 - 4F_{n-1}F_{n-2} - 3F_{n-2}^2$.

The PWW of this result (See Figure 14), is a bit more complicated than the sum of squares identity. The entire square represents the quantity F_{n+1}^2 and it is composed of 4 light pink squares, each with area F_{n-1}^2, 4 dark pink rectangles, each with area $F_{n-1}F_{n-2}$, and 1 red square with area F_{n-2}^2. It should be clear that the sums of the side lengths of these rectangles are consistent with the corresponding Fibonacci relationships. For example, when a light pink square and a dark pink rectangle are put next to each other $(F_{n-1} + F_{n-2})$, the sum of their side lengths is is F_n. In turn, when F_n and F_{n-1} are put together, the resulting length is F_{n+1}, which is the side length of the whole square.

Now consider the quantity F_n^2. In terms of our diagram, this quantity would be represented by 1 light pink square, 2 dark pink rectangles, and 1 red square. Multiplying F_n^2 by 4 would then give us 4 light pink squares, 8 dark pink rectangles, and 4 red squares. Given this collection of squares and rectangles, in order to get the area of the whole square,

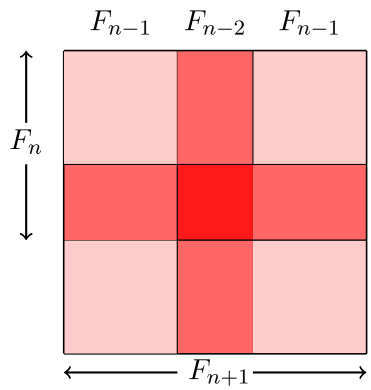


Figure 14: $F_{n+1}^2 = 4F_n^2 - 4F_{n-1}F_{n-2} - 3F_{n-2}^2$ [7]

we must subtract 4 dark pink rectangles and 3 red squares, leaving us with 4 light pink squares, 4 dark pink rectangles, and 1 red square. Given what we know about the areas of these shapes, we have just shown that $F_{n+1}^2 = 4F_n^2 - 4F_{n-1}F_{n-2} - 3F_{n-2}^2$.

4 Philosophy of Proofs Without Words

Now that we have explored a collection of various types of PWW, we turn to the philosophy behind them. Before we get into the debate about whether PWWs are really proofs, it is necessary to discuss just what exactly we mean by "proof". The definition of mathematical proof, while often taken for granted, can be nuanced and subtle and often varies depending on which mathematician you ask. These differences are discussed in the following section.

4.1 Definitions

The natural place to begin the discussion about the definition of proof is with just that: definitions. The mission of most "Introduction to Higher Mathematics"-style textbooks is to teach beginning mathematicians how to write proofs, and in order to do that, many start with a formal definition of proof. Here are a few examples:

1. "In mathematics, a proof is a demonstration that if some fundamental statements (axioms) are assumed to be true, then some mathematical statement is necessarily true." [19]
2. "The demonstration that a theorem does indeed follow from axioms and previously established theorems is known as a proof."[17]
3. "A proof is a sequence of statements. These statements come in two forms: givens and deductions." [18]
4. "To prove a statement is to proceed logically from premises to conclusions." [10]
5. "A proof is a series of statements, each of whose validity is based on an axiom or a previously proved theorem." [11]

6. "A proof is a sequence of irrefutable, logical steps that proceed from axioms and previously proved statements" [12]

The above definitions all seem to agree that a proof must be a series of statements or steps, each of which is a logical deduction from an axiom or theorem. The sixth definition even takes this a step further by insisting that the logical steps imply irrefutability. In contrast, the following definitions have a more "fuzzy" interpretation of what a proof should be.

7. "A proof is the demonstration of validity of some precise mathematical statement. The demonstration should contain sufficient detail to convince the intended audience of its validity." [16]

8. "A proof is a chain of statements leading, implicitly or explicitly from the axioms to a statement under consideration compelling us to declare that *that* statement, too, is true." [13]

These two definitions emphasize that proofs should first and foremost be convincing. Definition 7 makes no indication that axioms, or theorems, or even logic are necessary, and Definition 8 suggests that logical steps can be implicitly assumed, rather than explicitly stated.

From this sample of textbook definitions, it seems as though there are two ways of thinking about proof. On one side, proofs are strings of logical statements that irrefutably establish the truth of a given statement using axioms and previously proven theorems. On the other side, proofs can take almost any form, as long as they reasonably convince an educated audience that a statement is true. Given these two contrasting interpretations, do proofs necessarily need to exemplify both? And then how far can we conceivably push the boundaries of what can be considered "proof"?

The standard model of proof has already been challenged by the topological four-color problem, which we discuss in the following section.

4.2 The Four Color Problem: Verification as Proof

The premise of the four-color problem is fairly straightforward: is it possible to draw a map on a sheet of paper and color it with only four colors such that countries who share a common border are shaded with different colors?

Starting in 1852, prominent mathematicians, including Augustus DeMorgan, Arthur Cayley, and Arthur Bray attempted to prove the four-color problem using the axioms of topology[14]. They discovered that the calculations necessary were far too unwieldy and thus they were unsuccessful in proving the four-color problem by traditional means.

However, with the advent of computer technology, the unreasonable calculations which hindered early mathematicians became possible, and in 1976, Kenneth Appel and Wolfgang Haken used a specialized computer program to prove the four-color problem. Essentially, the computer checked every possible map configuration using four colors to verify the premise of the problem.

Although it was clear that the results generated by the computer in 1976 verified the four-color conjecture, it raised the philosophical question of whether verification is equivalent to proof[3]. Performing every possible calculation and determining that every one of them is valid certainly shows the truth of the statement, but it is not based on a series of logical deductions from a set of axioms.

We can conduct a similar analysis on a Proof Without Words. Consider again the PWW of the sum of the first n integers (See Figure 11). A mathematician looking at this picture would agree not only that this confirms the formula in question, but also that changing the number of dots would preserve the relationship, meaning that the formula remains true for all integers. This is undeniably convincing, and yet the picture itself does not in any way resemble a list of logical axiomatic statements, so it does not satisfy both of the characteristics of proof mentioned earlier. So then that brings us to ask, which characteristics of proof are necessary, and which are not? What makes something a proof or not a proof, and where do PWW's fit on that spectrum?

The aim of this paper is not to give a definitive PROOF or NOT PROOF verdict in response to these questions. Instead, we will discuss both sides of the debate, giving the reader the opportunity to decide for themselves.

4.3 Proofs Without Words: Proof or Not Proof?

Philosophically speaking, there are quite a few mathematical schools of thought, including realism, logicism, intuitionism, and structuralism, among others[19]. For the purposes of this paper, we will limit our discussion to the two that offer the most insight into the nature of Proofs Without Words: formalism and platonism.

4.3.1 Formalism

The main assumption of the formalist school of thought is that mathematical truth may only be established through formal logical deduction from existing axioms. Under formalism, proofs are divided into three categories [2]:

1. pre-formal proofs
2. formal proofs
3. post-formal proofs.

For the purposes of this paper, we will only cover pre-formal(informal) proofs and formal proofs. We will begin with a discussion of pre-formal proofs.

According to Imre Lakatos, in his article "What Does a Mathematical Proof Prove", an informal proof is one in which the axioms and postulates are not formally connected by logical steps, but instead allowed to speak for themselves[2]. It is possible to think of this type of proof as an outline of sorts, or a list of the theorems and axioms that are necessary to prove a statement without explicitly stating how each one follows logically from the others. As an example of an informal (specifically pre-formal) proof, Lakatos offers a famous thought experiment. The problem is to find two points P and Q on the surface or border of a triangle that are the farthest possible distance apart. In order to find the solution to this thought experiment, consider the following three possibilities:

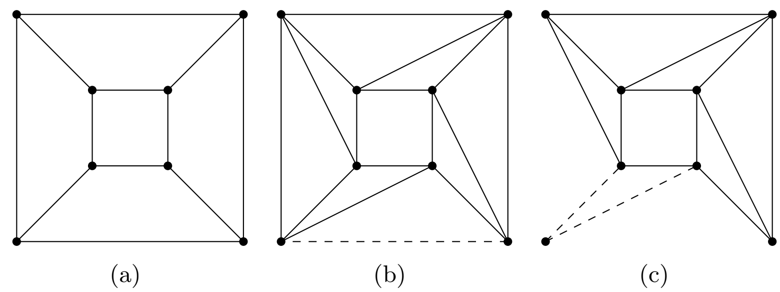


Figure 15: Euler's Theorem

- One of the points, Q, lies on the surface and the other lies on a border. Obviously, this is not the maximum distance since extending Q to one of the borders would produce a longer distance.

- Both points lie on the boundary but Q is not at a vertex. Then extending Q to a vertex would also produce a longer distance.

- Both points are vertices. Obviously, this is where the maximum must occur. Specifically, at the vertices of the longest side of the triangle

Thus, by systematic intuition, we have just shown that the points P and Q located on a triangle with the greatest distance between them are the vertices of the longest side.

This is a convincing argument, but it clearly does not utilize any formal postulates or logic, and there does not seem to be any "feasible way of formalizing the reasoning." So is it proof? Lakatos says 'no'. This thought experiment was a method of "intuitively showing that the theorem was true," but the absence of "well-defined underlying logic" makes it merely persuasive argumentation, not proof[2].

As a similar example, consider the following proof of Euler's theorem on simple polyhedra. The theorem states that, for a polyhedron with V vertices, E edges, and F faces, the equation

$$V - E + F = 2 \tag{17}$$

will always hold. The proof, first introduced by Cauchy in 1811 [19] and summarized by Lakatos [2], proceeds as follows:

Imagine a simple polyhedron (one without "holes"). For simplicity, we will depict it as a cube. Now take off one face and flatten the cube so it looks like Figure 15(a). We will call this our "network". Let $X = V - E + F$ for this original network. Since we just removed one face, we are now trying to show that $X = 1$.

Now draw diagonals in all polygons in our network which are not triangles. Notice that doing this increases both the number of edges and the number of faces by the same amount, so $V - E + F$ for this new network is still equal to X. Do this until the entire network is made up of triangles, as shown by Figure 15(b).

Next, repeat the following two steps as indicated:

1. If a triangle has one edge on the boundary of the network (see Figure 15(b)), remove that triangle from the network. Then both E and F decrease by 1 and V remains the same, so again, $V - E + F = X$.

2. If a triangle has 2 edges on the boundary (see Figure 15(c)), remove it from the network. Then V decreases by 1, E decreases by 2, and F decreases by 1, thus $V - E + F = X$ once again.

Repeat these steps until all that remains in the network is one triangle. Note that since none of our actions have changed the value of $V - E + F$, for this final network, $V - E + F = X = 1$. Thus, for the original polyhedron, $V - E + F = 2$.

This proof, like the first thought experiment, does not assume a formal logical system of postulates. It is merely intuition. So what makes a formal proof different from this "sweepingly convincing"[2] demonstration? Let's explore the philosophy of formal proof.

A formal proof is one which is given within the framework of a formal axiomatized theory[2]. By that, we mean a set of sentences in formal language from which a "deductive system" is understood [19]. Essentially, a mathematical theory is a set of understood axioms from which theorems may be derived. There are hundreds of mathematical theories in existence (some familiar ones may be Probability Theory, Set Theory, Group Theory, Measure Theory, or the Theory of Real Functions). If a proof is comprised of theorems derived from axioms within a particular formal theory, then it is a formal proof[2].

By the nature of formal theories, if a statement is proved within that theory, the result is infallible. Proceeding logically from axioms is a "foolproof verification procedure," thus there can be no counterexamples to the statement within that particular theory. This is in contrast to informal proofs, in which there could be hidden falsifications. In Cauchy's proof of Euler's Theorem, if we had failed to assert that the polyhedron must be simple (no holes), then the intuition of the proof would still have made sense, but we could have easily found a counterexample (by choosing a non-simple polyhedron) that failed to have the correct property. This is an example of an assumption of false generality and it is common to informal proofs. Without formal structure, there is no way to guarantee that the logic is valid[2].

Clearly, the nature of PWWs excludes them from being called "proofs" under the formalist view. As illuminating as they may be, without formal structure and logical semantics, they cannot guarantee any sort of truth. James Robert Brown summarizes the viewpoint of those who do not believe PWWs are proof.

> "Mathematicians, like the rest of us, cherish clever ideas; in particular they delight in an ingenious picture. But this appreciation does not overwhelm a prevailing skepticism. After all, a diagram is–at best–just a special case and so can't establish a general theorem. Even worse, it can be downright misleading. Though not universal, the prevailing attitude is that pictures are really no more than heuristic devices; they are psychologically suggestive and pedagogically important–but they *prove* nothing." [1]

Essentially, a picture can only represent a special case. So even if that picture appears to be convincing, it has no systematic way of eliminating doubt about the general case. For this reason, many mathematicians do not consider PWWs to be true proofs.

It is important to note though that formal proofs are only infallible in the context of the axiomatic system to which they subscribe. Because Number Theory has a different set of axioms than Measure Theory (for example), a statement that is true in one context may not be true in another. This is one of the reasons it is important to keep an open mind. Even under formalism, infallibility is a matter of perspective.

4.3.2 Platonism

This brings us to Platonism, which is friendlier to the notion that PWWs are proofs. The principle of Platonism is that mathematical truths exist independently of semantics and our task is to discover them by whatever means necessary or effective[1]. This leaves open the possibility that if PWWs display sufficient evidence of a mathematical truth, they could be considered proof. To see why, we consider the notion of *isomorphisms* and *homomorphisms*.

Two structures are *isomorphic* if they satisfy the following two conditions: (1) they have the same number of objects or elements, and (2) the relationships between objects in one group follow the same pattern as the relationships between objects in the second group[1]. Alternatively, two structures are *homomorphic* if they satisfy the second condition, but not the first condition.

For example, consider the PWW of the sum of the first n integers presented in Figure 11. The picture itself is a representation of the $n = 7$ case. Thus, we can claim it to be isomorphic to some structure with the same cardinality. But what about the rest of the integers? Our intention by presenting this figure as a PWW is to extend the relationship to ALL integers. That is, we intend to claim that the relationship between the first 7 integers represented by the picture is homomorphic to the set of positive integers. Despite the fact that a picture representing only the $n = 7$ case is certainly not a homomorphism, this diagram is successful in establishing the general relationship[1]. Why is this?

To answer this question, James Robert Brown makes the following bold suggestion: "Some pictures are not really pictures, but rather are windows into Plato's heaven." [1] By Plato's heaven, he is referring to the innate existence of mathematical ideas and our need to discover them. A picture, although it may only represent a specific case, provides our brains with the necessary instrument to discover the general truth for ourselves. In the case of the integer sum, upon seeing the specific case, our brain also sees the "possibility of reiteration"[1]. The diagram allows our brain to extend the relationship to any possible integer, and that is why the picture is successful. Since Platonism allows for mathematical truth to be discovered by the mind itself, without requiring formal logical statements, Proofs Without Words that suggest the possibility of reiteration can be considered valid proof.

5 Conclusion

Clearly, we have established that the answer to the question "Are PWWs really proofs?" is far from black and white. Because the definition of proof varies depending on which mathematical philosophy we adhere to or which textbook we consult, it then becomes difficult to determine what meets the criteria, and what does not, or even what those criteria are.

However, proof or not proof, PWWs are valuable tools in mathematics, especially in teaching. Take, for example, the formula for the sum of the first n integers. Most students who have taken an introductory higher mathematics class could prove with their eyes closed that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

using Mathematical Induction. But an induction proof only proves that the formula is true. It does not show *why* the formula is true.

This is where PWWs become useful. Consider the PWW of the same integer sum formula (See Figure 11). Using the picture, it becomes obvious that the sum of $1+2+\cdots+n$ is half the area of the rectangle with side lengths n and $n+1$, or $n(n+1)/2$. All of a sudden, the formula ceases to be random and instead has concrete visual meaning. We can see *why* the formula is true instead of just proving that it is true.

As noted by Professor Lynn Arthur Steen, co-editor of *Mathematics Magazine* when PWWs began appearing,

> For most people, visual memory is more powerful than linear memory of steps in a proof. Morever, the various relationships embedded on a good diagram represent real mathematics awaiting recognition and verbalization. So as a device to help students learn and remember mathematics, proofs without words are often more accurate than (mis-remembered) proofs with words[21].

Our ability to perceive concepts visually is incredibly powerful and we would be foolish not to utilize it to maximize the extent of our understanding. In general, PWWs can, and should, be used in addition to formal proofs. Even if they do not logically prove a theorem from given axioms, they can add clarity and concrete understanding to an otherwise opaque series of mathematical deductions. And if nothing else, PWWs serve to stimulate mathematical thought and curiosity, which are as vital to mathematical progress as the results themselves.

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