# PROOFS YOU CAN COUNT ON

by

Helen K. Jenne

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*Certificate of Approval*

This is to certify that the accompanying thesis by Helen K. Jenne has been accepted in partial fulfillment of the requirements for graduation with Honors in Mathematics.

Barry Balof, Ph.D.

Whitman College
May 8, 2013

ABSTRACT

PROOFS YOU CAN COUNT ON

Benjamin and Quinn's (2003) proofs by direct counting reduce the proof of a mathematical result to a counting problem. In comparison to other proof techniques such as proof by induction, proofs by direct counting are concrete, satisfying, and accessible to an audience with a variety of mathematical backgrounds. This paper presents proofs by direct counting of identities involving the Fibonacci numbers, the Lucas numbers, continued fractions, and harmonic numbers. We use the Fibonacci numbers and Lucas numbers primarily to introduce proofs by direct counting. We then present Benjamin and Quinn's combinatorial interpretation of continued fractions, which allows us to reduce identities involving continued fractions to counting problems. We apply the combinatorial interpretation to infinite continued fractions, and ultimately present a combinatorial interpretation of the continued fraction expansion of $e$. We conclude this paper by discussing Benjamin and Quinn's combinatorial interpretations of harmonic numbers and a generalization of the harmonic numbers called the hyperharmonic numbers.

Helen Jenne
Whitman College
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# 1 Introduction

Many undergraduate math students who have taken an introductory combinatorics course have seen proofs that use direct counting of identities involving the binomial coefficient. The definition of $\binom{n}{k}$, the number of ways to choose $k$ elements from an $n$ element set, gives us all the machinery we need to prove many binomial identities. For example, to prove

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

we count the same set - the number of ways to pick a $k$ person team from $n$ people, where one of those $k$ people is a captain - in two different ways. We can count the number of ways to pick a $k$ person team with a captain from $n$ people by first picking the team, and then choosing one of those $k$ people to be captain. Alternatively, we can count the number of ways to pick a captain from $n$ people, and then pick $k-1$ people from the remaining $n-1$ people to get the rest of the team. Therefore, the number of ways to first pick the team, and then choose a captain from the team members (the left side of the equation) is the same as the number of ways to first pick a captain, and then pick the rest of the team (the right side of the equation). Consequently, proving this useful identity amounts to simply considering a real world example. The resulting proof is much more satisfying and accessible than algebraic manipulations of the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

It turns out that direct counting can also be used to prove identities involving the Fibonacci numbers, the Lucas numbers, continued fractions, and harmonic numbers. In their 2003 book, *Proofs that Really Count: The Art of Combinatorial Proof*, A. T. Benjamin and J. J. Quinn present combinatorial interpretations of these sequences and prove hundreds of identities using only direct counting. In the entire book they use just two methods: defining a set and counting the quantity in two different ways (as we did in the above example) or proving a correspondence between two different sets. One of the nice aspects of proofs that use direct counting is how easy they are to understand. After all, anyone can count! This paper is written so that any interested undergraduate math major can understand it.

The primary purpose of this paper is to explain and apply Benjamin and Quinn's combinatorial method of proof in several scenarios. In section 2, we introduce proofs by direct counting using the Fibonacci and Lucas numbers. The purpose of this section is to familiarize the reader with these proofs. The Fibonacci numbers and Lucas numbers are a good place to start because these sequences are likely familiar to the reader. Some readers have probably seen proofs of Fibonacci identities that use induction. While a proof by induction may accomplish its purpose and be logically correct, there is a lack of elegance in that it does not give the reader intuition as to why the identity is true. In comparison, the proofs by direct counting of the Fibonacci numbers are much more concrete and satisfying.

The subject of section 3 is continued fractions. We begin with the significance and mathematical properties of continued fractions. Then, we explain Benjamin and Quinn's combinatorial interpretation of continued fractions. After examples of how to apply this combinatorial interpretation, we turn to infinite continued fractions. We continue our discussion of continued fractions in section 4 by presenting a combinatorial interpretation of the continued fraction expansion of $e$.

We conclude the paper by presenting the combinatorial interpretation of the harmonic numbers, a sequence that will look familiar to many readers. Explaining the combinatorial interpretation requires an introduction to Stirling numbers of the first kind. Section 5 closes by extending this combinatorial intepretation to a generalization of the harmonic numbers.

# 2 Fibonacci Numbers

## 2.1 Introduction and Background

The *Fibonacci numbers* date back to Leonardo of Pisa, who posed the following question in his book *Liber Abaci*:

*Starting with a single pair of rabbits, how many pairs of rabbits will we have in the nth month, if every month each mature pair of rabbits gives birth to a new pair, and it takes rabbits two months to mature?* [8]

Let $F_n$ denote number of the pairs of rabbits in the $n$th month. By convention, $F_0 = 0$. Since we start with a single pair of rabbits, which we will call pair A, $F_1 = 1$. Pair A does not reproduce the second month because it takes rabbits two months to mature, so $F_2 = 1$. The next month, pair A reproduces, so now we have two pairs of rabbits, pair A and pair B, and $F_3 = 2$. In the fourth month, pair A reproduces again, giving birth to pair C, but pair B is not mature yet, so $F_4 = 3$. In the fifth month, pair A and pair B reproduce, giving birth to pairs D and E, but pair C is not mature yet. Consequently, $F_5 = 5$. Continuing this reasoning we get the Fibonacci sequence (which appears in Sloane's Online Encyclopedia of Integer Sequences),

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots \text{ (A000045)}$$

In general, to find the number of pairs of rabbits in the $n$th month, we count the number of pairs of rabbits in the $(n-1)$st month, because all of these rabbits are still alive in the $n$th month, and add the number of rabbits in the $(n-2)$nd month, since all of these rabbits are now mature and reproduced in the $n$th month. We formalize this below.

**Definition 2.1.** The *Fibonacci numbers* are defined by $F_0 = 0$, $F_1 = 1$, and

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

## 2.2 Tiling Interpretation

The Fibonacci numbers can be interpreted combinatorially as the number of ways to tile a board of length $n$ and height 1 using only squares (length 1, height 1) and dominoes (length 2, height 1). For an example, see Figure 1.

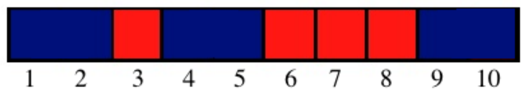


Figure 1: A 10-board tiled with squares (red) and dominoes (blue).

**Theorem 2.2** (Benjamin & Quinn, 2003)**.** *Let $f_n$ be the number of ways to tile a board of length $n$ using just squares and dominoes. Then $f_n = F_{n+1}$ for $n \geq -1$.*

*Proof.* (Benjamin & Quinn, 2003).

To prove Theorem 2.2, we use the fact that two sequences are the same if they satisfy the same initial conditions and the same recursion relation.

Let $f_0 = 1$ count the tiling of a 0-board and define $f_{-1} = 0$. Then $f_{-1} = F_0$ and $f_0 = F_1$. Next, we observe that the only way to tile a board of length 1 is with 1 square, so $f_1 = 1 = F_2$.

To see that $f_n$ satisfies the Fibonacci recursion relation (Definition 2.1), we consider the last tile of the $n$-board. The last tile is either a square or a domino.

1. If the board ends in a square, by definition there are $f_{n-1}$ ways to tile the first $n-1$ tiles of the board.

2. If the board ends in a domino, there are $f_{n-2}$ ways to tile the first $n-2$ tiles of the board.

In order to calculate the total number of ways to tile an $n$-board, we sum over these two cases. That is,

$$f_n = f_{n-1} + f_{n-2}$$

Since $\{f_n\}$ satisfies the same initial conditions and recursion relation as $\{F_{n+1}\}$, $f_n = F_{n+1}$ for all $n \geq -1$. $\square$

We conclude that $\{f_n\}$ is the Fibonacci sequence shifted by 1 term. Thinking of the Fibonacci numbers as tilings of an $n$-board allows us to prove many useful Fibonacci identities that otherwise require proof techniques such as induction or algebraic manipulation that don't reveal why the identity is true.

## 2.3 Proofs by Direct Counting of Fibonacci Identities

The goal of this section is to give examples of proofs by direct counting. There are two methods that Benjamin and Quinn (2003) use to prove an identity by direct counting:

1. Counting a quantity in two different ways.

2. Proving a correspondence between two sets.

We will begin with examples of the first method, but first we need the following definition.

**Definition 2.3.** (Benjamin & Quinn, 2003). A tiling of a $n$-board is *breakable* at tile $k$ if it can be split into two tilings, one covering tiles 1 through $k$, and one covering tiles $k+1$ through $n$.

In other words, a tiling is breakable at tile $k$ as long as there isn't a domino covering tiles $k$ and $k+1$. It follows that a tiling is always breakable at either tile $k$ or tile $k-1$. This definition allows us to separate tilings into two cases: when the board is breakable at tile $k$, and when there is a domino covering tiles $k$ and $k+1$.

**Example 2.4.** (Benjamin & Quinn, 2003).

For $m, n \geq 0$, $f_{m+n} = f_m f_n + f_{m-1} f_{n-1}$

*Proof.* We count the tilings of an $(m+n)$-board in two different ways.

**Method 1:** By definition, the number of ways to tile an $(m+n)$-board using squares and dominoes is $f_{m+n}$.

**Method 2:** We separate the tilings of an $(m+n)$-board into two cases based on whether or not the tiling is breakable at tile $m$. If the tiling is breakable at tile $m$, we break it into two tilings, one of length $m$ and one of length $n$ (see Figure 2). By definition, there are $f_m$ ways to tile the first board, and $f_n$ ways to tile the second board. By the multiplication rule, there are $f_m f_n$ ways to tile an $(m+n)$-board that is breakable at tile $m$.

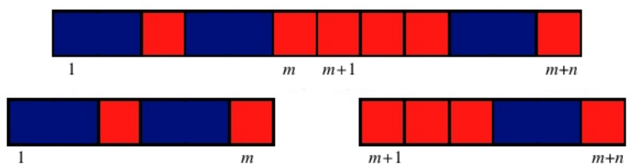


Figure 2: When the tiling is breakable at tile $m$, we break it into two tilings, one of length $m$ and one of length $n$. (Note that squares are red and dominoes are blue.)

If the tiling is not breakable at tile $m$, there is a domino covering tiles $m$ and $m+1$, and the tiling is breakable at tiles $m-1$ and $m+1$ (see Figure 3). We remove the domino, which leaves us with two tilings, one starting at tile 1 and ending at tile $m-1$, and one starting at tile $m+2$ and ending at tile $m+n$. By the multiplication rule, there are $f_{m-1}f_{n-1}$ ways to to tile an $(m+n)$-board that is not breakable at tile $m$.

Summing over both cases, we conclude that there are $f_m f_n + f_{m-1}f_{n-1}$ ways to tile an $(m+n)$-board.

Since both Methods 1 and 2 both count the number of tilings of an $(m+n)$-board, we have shown that

$$f_{m+n} = f_m f_n + f_{m-1}f_{n-1}.$$

□

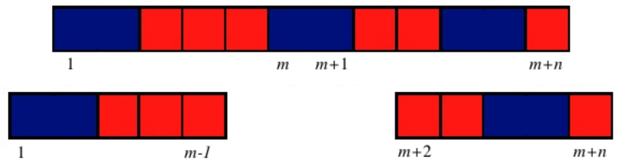


Figure 3: When the tiling of an $(m + n)$-board is not breakable at tile $m$, we remove the domino, resulting in a board of length $m - 1$ and a board of length $n - 1$.

Another technique that is useful in proving Fibonacci number identities is called *tail swapping* [3]. Tail swapping is helpful because it allows us to construct one $(n+1)$-board and one $(n - 1)$-board from two $n$-boards. The first step in tail swapping is to place two tiled $n$-boards offset, so that the second tiling begins one tile to the right of the first tiling (see Figure 4). Then, we look at where each tilings is breakable. Specifically, we want to know where these breaks line up.

**Definition 2.5.** (Benjamin & Quinn, 2003). A pair of offset tilings have a *fault* if the offset tilings have the same vertical break. We say a pair of offset tilings has a fault at tile $i$, for $1 \leq i \leq n$, if the first tiling is breakable at tile $i$ and the second tiling is breakable at tile $i - 1$ (see Figure 4).

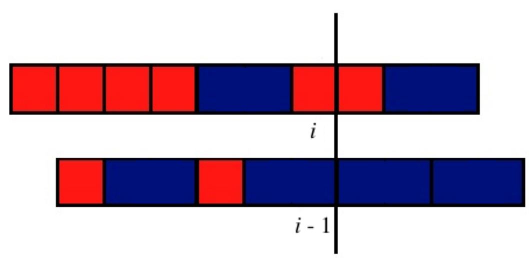


Figure 4: Two tilings placed offset, so that the second tiling begins one tile to the right of the first tiling. The black line indicates the last fault (the rightmost tile where both tilings are breakable).

Given a pair of offset tilings, we are not always guaranteed a fault. There is only one way to prevent a fault from occuring: tiling both $n$-boards with all dominoes.

As long as one of the boards has at least one square, a fault will exist. To see this, consider two tilings, A and B, placed offset as in Figure 4, and suppose that tiling A has a square covering tile $i$. Then, by definition, tiling A is breakable at tile $i$ and tile $i-1$. Now consider tiling B. There are two cases: either tiling B is breakable at tile $i$, or tiling B is not breakable at tile $i$. If B is not breakable at tile $i$, it has a domino covering tiles $i$ and $i+1$ and thus is breakable at tile $i-1$. In the first case, we have a fault at tile $i$, and in the second we have a fault at tile $i-1$.

If a fault exists, we consider the *tails* of the boards.

**Definition 2.6.** (Benjamin & Quinn, 2003). The *tails* of a tiling pair are the tiles that occur after the last fault.

After we have identified the tails of the boards, it remains to swap them, creating an $(n+1)$-board and an $(n-1)$-board. We illustrate this idea with an example.

**Example 2.7.** (Benjamin & Quinn, 2003).

For $n \geq 0, f_n^2 = f_{n+1}f_{n-1} + (-1)^n$

*Proof.* We will count the number of tilings of two $n$-boards.

**Method 1:** There are $f_n$ ways to tile the first $n$-board, and $f_n$ ways to tile the second $n$-board, since these tilings are independent of each other. By the multiplication rule, the number of tilings of two $n$-boards is $f_n^2$.

**Method 2:** Place the two tiled $n$-boards offset as in Figure 4. We will consider two cases: when $n$ is even, and when $n$ is odd.

$n$ is even: When both tilings are all dominoes, there is no fault. In every other case, at least one of the boards has at least one square and so we are guaranteed a fault. After the last fault, switch the tails of the $n$-boards. Now we have a tiled $(n+1)$-board and a tiled $(n-1)$-board. There are $f_{n+1}f_{n-1}$ ways to tile an $(n+1)$-board and an $(n-1)$-board, so, adding the case when both $n$-boards are tiled using all dominoes, there are $f_{n+1}f_{n-1} + 1$ tilings of two $n$-boards when $n$ is even.

$n$ is odd: When $n$ is odd, each board has at least one square, so there is at least one fault. Switch the tails of the $n$-boards after the last fault. We get an $(n+1)$-board and an $(n-1)$-board. There are $f_{n+1}f_{n-1}$ ways to tile an $(n+1)$-board and an $(n-1)$-board, so, subtracting the case where both the $(n+1)$-board and the $(n-1)$-board are tiled using all dominoes, there are $f_{n+1}f_{n-1} - 1$ tilings of two $n$-boards when $n$ is odd.

Since Methods 1 and 2 count the same quantity, we have shown

$$f_n^2 = f_{n+1}f_{n-1} + (-1)^n$$

□

## 2.4 Lucas Numbers

The *Lucas numbers* are closely related to the Fibonacci numbers: they follow the same recursion relation, but have different initial conditions. While $F_0 = 0$, $L_0 = 2$. More formally,

**Definition 2.8.** The *Lucas numbers* are defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

The first few Lucas numbers are

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199 \ldots \text{ (A000032)}$$

The Lucas numbers have a very similar combinatorial interpretation to that of the Fibonacci numbers: $L_n$ counts the number of tilings of an $n$-bracelet (a circular $n$-board) using squares and dominoes. For a proof of this fact, see [3]. The key difference between an $n$-bracelet and an $n$-board is that an $n$-bracelet may have a domino covering tiles $n$ and 1, and thus may not be breakable at tile $n$ (see Figure 5).

**Definition 2.9.** (Benjamin & Quinn, 2003). When a domino covers tiles $n$ and 1, we call the $n$-bracelet *out-of-phase* [3]. Otherwise, the bracelet is *in-phase* (see Figure 5).

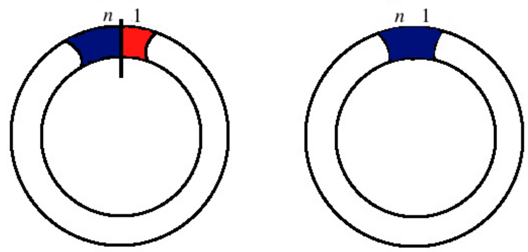


Figure 5: An in-phase $n$-bracelet (left) and an out-of-phase $n$-bracelet (right). Dominoes are blue, squares are red, and white can be either.

There are many identities that relate the Fibonacci numbers and the Lucas numbers. The following examples illustrate the second method of proof by direct counting: proving a one-to-one correspondence between two sets. In these proofs, we first present a correspondence, and then, to show that the correspondence is one-to-one, we argue that the correspondence is reversible.

**Example 2.10.** (Benjamin & Quinn, 2003).

For $n \geq 2$, $L_n = f_{n-1} + 2f_{n-2}$.

*Proof.* We will prove a one-to-one correspondence between the following two sets:

**Set 1:** The set of all tilings of an $n$-bracelet. There are $L_n$ such tilings.

**Set 2:** The set of all tilings of an $(n-1)$-board, or two $(n-2)$-boards. There are $f_{n-1} + 2f_{n-2}$ such tilings.

**Correspondence:** Consider an $n$-bracelet from Set 1. There are two cases to consider: when the $n$-bracelet is in-phase, and when it is out-of-phase.

1. If the $n$-bracelet is in-phase, break the tiling between tiles $n$ and 1. There are two such cases to consider: when the tiling ends in a square, and when it ends in a domino.

   (a) If the tiling ends in a square, remove the square. We now have a tiled $(n-1)$-board.

   (b) If the tiling ends in a domino, remove the domino. We now have a tiled $(n-2)$-board.

2. If the bracelet is out-of-phase, remove the domino covering tiles $n$ and 1. We now have a tiled $(n-2)$-board.

We have shown that given an $n$-bracelet, we can create $f_{n-1} + 2f_{n-2}$ tilings. To see that this correspondence is reversible, we observe that

1. We can create an $n$-bracelet from an $(n-1)$-board by adding a square to the end of the board and gluing the $n$th tile to the first tile.

2. There are two ways to create an $n$-bracelet from an $(n-2)$-board:

   (a) We add a domino to the end of the $(n-2)$-board and glue the $n$th tile to the first tile, creating an in-phase $n$-bracelet.

   (b) We add a domino to the end of the $(n-2)$-board, glue the $n$th tile to the first tile, and then rotate the bracelet clockwise one tile, creating an out-of-phase $n$-bracelet.

We have thus proven a one-to-one correspondence between Set 1 and Set 2. □

**Example 2.11.** (Benjamin & Quinn, 2003).

For $n \geq 0$, $f_{2n-1} = L_n f_{n-1}$

Algebraically, we can see that this identity follows from Examples 2.4 and 2.9:

$$\begin{aligned} f_{2n-1} = f_{n+(n-1)} &= f_n f_{n-1} + f_{n-1} f_{n-2} \\ &= f_{n-1}(f_n + f_{n-2}) \\ &= f_{n-1}(f_{n-1} + f_{n-2} + f_{n-2}) \\ &= f_{n-1}(f_{n-1} + 2f_{n-2}) \\ &= f_{n-1} L_n \end{aligned}$$

Alternatively, we can prove $f_{2n-1} = L_n f_{n-1}$ using direct counting.

*Proof.* We prove a correspondence between the following two sets:

**Set 1:** The set of all tilings of a $(2n-1)$-board. There are $f_{2n-1}$ such tilings.

**Set 2:** The set of all pairs of an $n$-bracelet and an $(n-1)$-board. This set has size $L_n f_{n-1}$.

**Correspondence:** Take a tiled $(2n-1)$-board from Set 1. Condition based on whether or not the tiling is breakable at the $n$th tile.

If the tiling of the $(2n-1)$-board is breakable at the $n$th tile, break it at the $n$th tile and glue the right side of the $n$th tile to the left side of the first tile, resulting in an in-phase $n$-bracelet. There is a tiled $(n-1)$-board remaining.

If the tiling of the $(2n-1)$-board is not breakable at the $n$th tile, it is breakable at the $(n-1)$st tile. Break the board at the $(n-1)$st tile. Now we have a tiled $(n-1)$-board and a tiled $n$-board that begins with a domino. Glue the $n$th tile of the $n$-board to the first tile of the $n$-board and shift it so that we get an out-of-phase $n$-bracelet.

We can reverse this correspondence by conditioning based on whether the $n$-bracelet is in-phase or out-of-phase. Thus we have a one-to-one correspondence between tilings of a $(2n-1)$-board and pairs of an $n$-bracelet and an $(n-1)$-board. □

Now that we have introduced how to prove identities by direct counting, we are ready to apply these techniques to an interesting property of the Fibonacci numbers.

## 2.5 Zeckendorf's Theorem

This section is concerned with identities of the form

$$mF_n = F_{i_1} + F_{i_2} + \cdots + F_{i_k},$$

where $m$ is an integer and $F_{i_1}, F_{i_2}, \ldots, F_{i_k}$ are distinct nonconsecutive Fibonacci numbers. A fascinating property is that any $mF_n$ can be represented uniquely as a sum of nonconsecutive Fibonacci numbers. This result is a simple consequence of Zeckendorf's Theorem.

**Theorem 2.12** (Zeckendorf's Theorem)**.** *Every positive integer can be uniquely represented as a sum of distinct nonconsecutive Fibonacci numbers [3].*

Note that there is one exception to the uniqueness claim in Zeckendorf's Theorem. Since $F_1 = F_2 = 1$, any representation that includes $F_2$ could instead include $F_1$. As a result, when we say the representation is unique, we mean there is not another representation besides the one achieved by replacing $F_2$ with $F_1$ (or by replacing $F_1$ with $F_2$).

*Proof.* We will use the Principle of Strong Induction to show that every positive integer can be uniquely represented as a sum of distinct nonconsecutive Fibonacci numbers.

We observe that $n = 1, 2, 3$ are Fibonacci numbers and that we cannot write $n = 1, 2, 3$ as sums of smaller Fibonacci numbers without repeating Fibonacci numbers or using consecutive Fibonacci numbers. Consequently, the theorem holds for $n = 1, 2, 3$. For $n = 4$, we see that $4 = F_4 + F_2$. Again, it is easy to check that this representation is unique since if we try to write 4 as a sum of other Fibonacci numbers, we either have to use consecutive Fibonacci numbers or repeat Fibonacci numbers.

Suppose that for $n < k$, we can write $n$ as a sum of distinct nonconsecutive Fibonacci numbers, and that this representation is unique. Then for $n = k$, we take the maximum integer $j$ such that $F_j \leq k$. Then

$$k = F_j + r$$

If $r = 0$, then we are done. If $r \neq 0$, by the induction hypothesis we can write $r$ as a sum of distinct nonconsecutive Fibonacci numbers. We claim that each Fibonacci number $f_i$ in the representation of $r$ is strictly less than $F_{j-1}$. To see this, suppose the representation of $r$ as distinct nonconsecutive Fibonacci numbers contained a Fibonacci number $F_l \geq F_{j-1}$. Then by the recursion relation for Fibonacci numbers, we could write $F_l$ as a sum that contained $F_{j-1}$. Then since $F_{j-1} + F_j = F_{j+1}$, we have that $F_{j+1} \leq k$, a contradiction to the fact that we took the maximum integer $j$ such that $F_j \leq k$.

Since there is only one maximum integer $j$ such that $F_j \leq k$ and the representation for $r$ is unique by the induction hypothesis, we have written $k$ uniquely as a sum of distinct nonconsecutive Fibonacci numbers. By the Principle of Strong Induction, every positive integer can be uniquely represented as a sum of distinct nonconsecutive Fibonacci numbers.

□

Since $mF_n$, $m \in \mathbb{N}$, is a positive integer, it has a unique representation as a sum of distinct nonconsecutive Fibonacci numbers by Zeckendorf's Theorem.

This proof of Zeckendorf's theorem does not show why identities of the form

$$mF_n = F_{i_1} + F_{i_2} + \cdots + F_{i_k},$$

are true. The purpose of this section is to prove examples of these identities using direct counting. Recall that $f_n = F_{n+1}$, so for the remainder of this section we write these identities in terms of $f_n$. To begin, we will consider the relatively simple case of $m = 2$. The reader may realize that the next example follows quickly from the recurrence relation for Fibonacci numbers, but we prove it combinatorially as an introductory example of how to prove correspondences that are not one-to-one.

**Example 2.13.**

$$2f_n = f_{n+1} + f_{n-2} \tag{1}$$

*Proof.* To prove this identity, we will find a 2-to-1 correspondence between the following two sets:

**Set 1:** The set of all tiled $n$-boards. There are $f_n$ such boards.

**Set 2:** The set of all tiled $(n+1)$-boards and $(n-2)$-boards. There are $f_{n+1} + f_{n-2}$ such boards.

**Correspondence:** Given an arbitrary $n$-board, there are several possible actions we can take, with the goal of creating either an $(n+1)$-board or an $(n-2)$-board. We can:

1. Add a square to an $n$-board to get an $(n+1)$-board ending in a square.

2. Condition on whether our $n$-board ends in a square or a domino.

   (a) If the board ends in a square, we remove the square and add a domino to get an $(n+1)$-board ending in a domino (see Figure 6).

   (b) If the board ends in a domino, we remove the domino to get an $(n-2)$-board.

We have used two $n$-boards to create all $(n+1)$ and $(n-2)$-boards, and thus we have found a 2-to-1 correspondence between Set 1 and Set 2. It remains to check that this correspondence is onto. To do this, we must verify that the correspondence creates all possible $(n-2)$-boards and $(n+1)$-boards. Since we start with an arbitrary $n$-board, item 2(b) of the correspondence creates all possible $(n-2)$-boards. An

$(n+1)$-board ends in either a square or a domino. Item 1 of the correspondence creates all $(n+1)$-boards that end in a square, and item 2(a) creates all $(n+1)$-boards that end in a domino. We conclude that $2f_n = f_{n+1} + f_{n-2}$. □

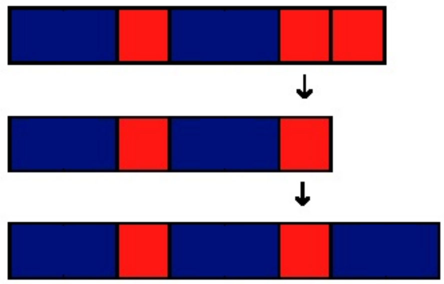

Figure 6: If the $n$-board ends in a square, we remove the square (red) and add a domino (blue) to get an $(n+1)$-board.

We next look at a more difficult example.

**Example 2.14.**

$$5f_n = f_{n+3} + f_{n-1} + f_{n-4} \tag{2}$$

*Proof.* To prove this, we will find a 5-to-1 correspondence between the following two sets:

**Set 1:** The set of all tiled $n$-boards. There are $f_n$ such boards.

**Set 2:** The set of all tiled $(n+3)$-boards, $(n-1)$-boards, and $(n-4)$-boards. There are $f_{n+3} + f_{n-1} + f_{n-4}$ such boards.

**Correspondence:** Given an $n$-board, we can

1. Add a domino followed by a square to get an $(n+3)$-board.

2. Add a square followed by a domino to get an $(n+3)$-board.

3. Add three squares to get an $(n+3)$-board. At this point, it may be helpful to refer to Figure 7 to see the possible endings of 3-boards.

4. Condition on whether the board ends in a square or a domino.

(a) If the board ends in a square:

   i. Remove the square and add two dominoes to get $(n+3)$-board.

   ii. Remove the square and add a domino followed by two squares to get an $(n+3)$-board.

(b) If the board ends in a domino we remove the domino, at which point we have an $(n-2)$-board where we can:

   i. Add a square to get an $(n-1)$-board.

   ii. Depends on whether the $(n-2)$-board ends in a square or a domino.

      A. If the board ends in a square, remove the square and add a domino to get an $(n-1)$-board.

      B. If the board ends in a domino, remove the domino to get an $(n-4)$-board.

   Notice that in 4(b) we have shown that $2f_{n-2} = f_{n-1} + f_{n-4}$, which is equivalent to equation (1) in Example 2.13.

In each of the first three items of the correspondence, we used one $n$-board to create an $(n+3)$-board. In item 4, we use two $n$-boards to create the remaining $(n+3)$-boards and all $(n-1)$-boards and $(n-4)$-boards. Thus we have 5-to-1 correspondence between the set of all $n$-boards and the set of all $(n+3)$-boards, $(n-1)$-boards, and $(n-4)$-boards.

We must show that the correspondence is onto. To do this, we will check that our correspondence creates all possible $(n-4)$-boards, $(n-1)$-boards, and $(n+3)$-boards. Clearly all possible $(n-4)$-boards are created by the correspondence since we start with an arbitrary $n$-board ending in two dominoes and just remove the two dominoes. Since item 4.b.i. creates all $(n-1)$-boards that end in a square and item 4.b.i.A. creates all $(n-1)$-boards that end in a domino, all possible $(n-1)$-boards are created by the correspondence.

It remains to check that our correspondence creates all possible $(n+3)$-boards (see Figure 7). An $(n+3)$-board ends with a domino followed by a square (item 1), a square followed by a domino (item 2), three squares (item 3), two dominoes (item 4.a.i.), or a domino followed by two squares (item 4.a.ii.). □

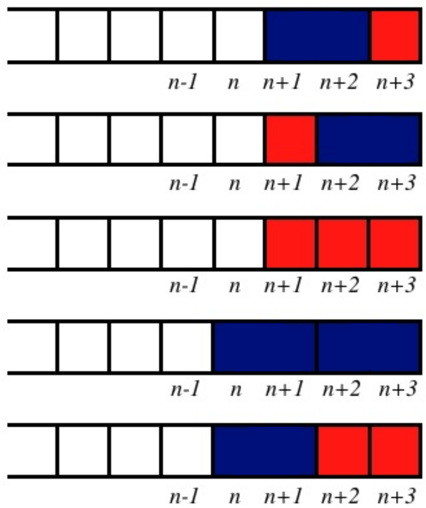


Figure 7: The five possible board endings of an $(n+3)$-board, where dominoes are blue, squares are red, and white squares can be either.

We present one final example to make the structure of these proofs clear.

**Example 2.15.**

$$6f_n = f_{n+3} + f_{n+1} + f_{n-4}$$

*Proof.* To prove this, we will find a 6-to-1 correspondence between the following two sets:

**Set 1:** The set of all tiled $n$-boards. There are $f_n$ such boards.

**Set 2:** The set of all tiled $(n+3)$-boards, $(n+1)$-boards, and $(n-4)$-boards. There are $f_{n+3} + f_{n+1} + f_{n-4}$ such boards.

**Correspondence:** Given an $n$-board, we can

1. Add a domino followed by a square to get an $(n+3)$-board.

2. Add a square followed by a domino to get an $(n+3)$-board.

3. Add three squares to get an $(n+3)$-board.

4. Add one square to get an $(n+1)$-board.

5. Condition on whether the board ends in a square or a domino.

   (a) If the board ends in a square:

      i. Remove the square and add two dominoes to get an $(n+3)$-board.

      ii. Remove the square and add a domino followed by two squares to get an $(n+3)$-board.

   (b) If the board ends in a domino we remove the domino, at which point we have an $(n-2)$-board where we can:

      i. Add a square and a domino to get an $(n+1)$-board.

      ii. Condition on whether the $(n-2)$-board ends in a square or a domino.

         A. If the board ends in a square, remove the square and add two dominoes to get an $(n+1)$-board.

         B. If the board ends in a domino, remove the domino to get an $(n-4)$-board.

In each of the first three items of the correspondence, we used one $n$-board to create an $(n+3)$-board. In item 4, we use one $n$-board to create an $(n+1)$-board. In item 5, we use two $n$-boards to create the remaining $(n+3)$-boards and $(n+1)$-boards, and all possible $(n-4)$-boards. Thus we have a 6-to-1 correspondence between Set 1 and Set 2. It remains to show that this correspondence is onto.

Checking that the correspondence creates all possible $(n-4)$-boards and $(n+3)$-boards is the same as in Example 2.14. To check that the correspondence creates all

$(n+1)$-boards, we observe that an $(n+1)$-board can end in a square (item 4) or a domino. If an $(n+1)$-board ends in a domino, it can end in either a square followed by a domino (item 5.b.i) or two dominoes (item 5.b.ii.A).

$\square$

It seems that we can prove any identity where $mf_n$ is written as the sum of nonconsecutive Fibonacci numbers using a similar method to that of examples 2.13-2.15. We first add combinations of squares and dominoes to a board of length $n$ to get a board of length $n+l$, for some positive integer $l$, and then condition on whether or not the $n$-board ends in a domino or square to get the remaining boards. Of course this method only works if we are given the identity; it doesn't help us to find the representation of $mf_n$. Furthermore, this is only a conjecture. Finding a unifying combinatorial approach to these identities is currently an open problem [3].

To conclude this section, we make one final observation. Consider equation (2) from Example 2.14:

$$5f_n = f_{n+3} + f_{n-1} + f_{n-4}$$

Since in equation (2), $5f_n$ is written as the sum of $f_{n+3}$, $f_{n-1}$, and $f_{n-4}$, we say that the coefficients of $f_n$ in equation (2) are 3, -1, and -4. When we write 5 as powers of the golden ratio, $\Phi = \frac{1+\sqrt{5}}{2}$, it turns out that the coefficients of the golden ratio are the same as the coefficients of $f_n$ in equation (2). To see this, we use the fact that

$$\Phi = \lim_{n\to\infty} \frac{f_{n+1}}{f_n}.$$

Dividing both sides of equation (2) by $f_n$, we have

$$\begin{aligned}
5 &= \frac{f_{n+3}}{f_n} + \frac{f_{n-1}}{f_n} + \frac{f_{n-4}}{f_n} \\
&= f_{n+3} \cdot \frac{f_{n+2}}{f_{n+2}} \cdot \frac{f_{n+1}}{f_{n+1}} \cdot \frac{1}{f_n} + \frac{f_{n-1}}{f_n} + f_{n-4} \cdot \frac{f_{n-3}}{f_{n-3}} \cdot \frac{f_{n-2}}{f_{n-2}} \cdot \frac{f_{n-1}}{f_{n-1}} \cdot \frac{1}{f_n} \\
&= \frac{f_{n+3}}{f_{n+2}} \cdot \frac{f_{n+2}}{f_{n+1}} \cdot \frac{f_{n+1}}{f_n} + \frac{f_{n-1}}{f_n} + \frac{f_{n-4}}{f_{n-3}} \cdot \frac{f_{n-3}}{f_{n-2}} \cdot \frac{f_{n-2}}{f_{n-1}} \cdot \frac{f_{n-1}}{f_n}
\end{aligned}$$

We see that

$$\lim_{n\to\infty} 5 = \lim_{n\to\infty} \frac{f_{n+3}}{f_{n+2}} \cdot \frac{f_{n+2}}{f_{n+1}} \cdot \frac{f_{n+1}}{f_n} + \frac{f_{n-1}}{f_n} + \frac{f_{n-4}}{f_{n-3}} \cdot \frac{f_{n-3}}{f_{n-2}} \cdot \frac{f_{n-2}}{f_{n-1}} \cdot \frac{f_{n-1}}{f_n} = \Phi^3 + \Phi^{-1} + \Phi^{-4}$$

so

$$5 = \Phi^3 + \Phi^{-1} + \Phi^{-4}.$$

We conjecture that we can apply the above process to any identity $mf_n = f_{i_1} + f_{i_2} + \cdots + f_{i_k}$ to write $m$ as nonconsecutive integer powers of $\Phi$ [3]. Again, finding a unifying combinatorial approach requires further study.

# 3 Continued Fractions

## 3.1 Introduction

Continued fractions are another object with an elegant combinatorial interpretation that, as we will see, has natural ties to number sequences such as the Fibonacci numbers and the Lucas numbers.

The goal of the present section is to define continued fractions, give examples, and highlight just how often we see continued fractions in mathematics before we explain their combinatorial interpretation.

**Definition 3.1.** (Benjamin & Quinn, 2003). Let $a_0$ be a nonnegative integer and let each $a_i$ and $b_i$ be a positive integer. Then $[a_0, (b_1, a_1), (b_2, a_2) \ldots, (b_n, a_n)]$ denotes the finite continued fraction

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{\ddots + \cfrac{b_n}{a_n}}}}$$

Infinite continued fractions are defined similarly.

**Definition 3.2.** For a nonnegative integer $a_0$ and positive integers $a_i$ and $b_i$, an infinite continued fraction is a fraction of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{\ddots}}}$$

To save space, we will often write continued fractions in the more compact form

$$a_0 + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots$$

### 3.1.1 Motivation

An elementary method for representing real numbers is the decimal expansion. Here, we introduce continued fractions as another way to represent real numbers with integers. It can be shown that every real number has a continued fraction expansion [9].

Continued fractions arise quite frequently in mathematics, particularly in number theory. One way that continued fractions arise is through "repeated divisions" of rational numbers, a process akin to the Euclidean Algorithm. The following example clarifies what we mean by this.

**Example 3.3.** Consider the rational number $\frac{355}{113}$. Dividing 113 into 355, we see that

$$\frac{355}{113} = 3 + \frac{16}{113}$$

Inverting the fraction $\frac{16}{113}$ and then dividing 16 into 113, we have

$$\begin{aligned} \frac{355}{113} &= 3 + \frac{1}{\frac{113}{16}} \\ &= 3 + \cfrac{1}{7 + \cfrac{1}{16}} \end{aligned}$$

Since 16 is a whole number, the process of repeated division stops. We have written $\frac{355}{113}$ as a finite continued fraction. Observe that since $16 = 15 + \frac{1}{1}$, we can also write

$$\frac{355}{113} = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}$$

In general, each rational number has exactly two continued fraction representations: one in which the last partial quotient is 1, as in the second case, and one in which the last partial quotient is not 1, as in the first case [6]. □

Notice that in the above example, each numerator $b_i$ of the continued fraction expansion was equal to 1. When each $b_i = 1$, we call the continued fraction expansion *simple*. When at least one $b_i \neq 1$, the continued fraction expansion is *nonsimple* [3]. We use a slightly modifed notation for simple continued fractions. Instead of denoting the continued fraction by $[a_0, (b_1, a_1), (b_2, a_2) \ldots, (b_n, a_n)]$, we denote it by $[a_0, a_1, a_2 \ldots, a_n]$.

Continued fractions also result from solving polynomial equations, as we will see in the following example.

**Example 3.4.** (Loya, 2006). Consider the equation

$$x^2 - 2x - 3 = 0.$$

Suppose we want to find the positive solution $x$ to this equation. On one hand, factoring the equation reveals that $x = 3$ is the only positive solution. On the other hand, we can write $x^2 - 2x - 3 = 0$ as $x^2 = 2x + 3$. Dividing by $x$, we have

$$x = 2 + \frac{3}{x}.$$

Next, we replace $x$ in the denominator of $\frac{3}{x}$ with $x = 2 + \frac{3}{x}$. We then have

$$x = 2 + \cfrac{3}{2 + \cfrac{3}{x}}$$

Replacing $x$ in the denominator repeatedly, we obtain the infinite continued fraction

$$x = 2 + \cfrac{3}{2 + \cfrac{3}{2 + \cfrac{3}{2 + \ddots}}}$$

Since we saw that $x = 3$, we have an infinite continued fraction representation for 3:

$$3 = 2 + \frac{3}{2+}\,\frac{3}{2+}\,\frac{3}{2+}\cdots$$

□

One interesting application of continued fractions is that they can be used to approximate irrational numbers [10]. As we mentioned earlier, every real number has a continued fraction representation. Every rational number can be written as a finite continued fraction by the method in Example 3.3, but all irrational numbers have infinite continued fraction expansions [9]. If we terminate the infinite continued fraction expansion of an irrational number at $a_n$ for each value of $n$, it turns out that the resulting sequence of numbers (called "convergents") is a sequence of best approximations to that irrational number. While we will not go into the specific definition of "best" approximation in this paper (for details, see [10]) it is worth mentioning since this is a useful property of continued fractions.

Now that we understand the definitions and some basic examples of continued fractions, we are ready to proceed on to the combinatorial interpretation of finite continued fractions.

## 3.2 Combinatorial Interpretation of Finite Continued Fractions

In this section, we will present the combinatorial interpretation of the finite nonsimple continued fraction

$$a_0 + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots + \frac{b_n}{a_n}.$$

We have a tiling interpretation very similar to what we saw in Section 2.

To interpret the continued fraction combinatorially, we must consider the numerator and denominator separately. Given the finite nonsimple continued fraction $[a_0, (b_1, a_1), \ldots, (b_n, a_n)]$, we let $p_n$ and $q_n$ denote the numerator and denominator of the continued fraction, respectively [3]:

$$a_0 + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots + \frac{b_n}{a_n} = \frac{p_n}{q_n}.$$

Here, $p_n$ and $q_n$ are are the numerator and denominator we get from algebraically simplifying the continued fraction without reducing it.

It turns out that $p_n$ and $q_n$ satisfy the following recursion relations, called the Wallis-Euler recursion relations [10]. These recursion relations will be essential in proving the combinatorial interpretation of continued fractions.

**Theorem 3.5.** *For $n \geq 1$,*

$$p_n = a_n p_{n-1} + b_n p_{n-2} \tag{3}$$

$$q_n = a_n q_{n-1} + b_n q_{n-2}, \tag{4}$$

*where $p_{-1} = 0$, $p_0 = a_0$, $q_{-1} = 0$, and $q_0 = 1$.*

We will not prove Theorem 3.5, as it is a straightforward proof by induction (see [10] for details).

As we previously mentioned, the combinatorial interpretation of continued fractions is similar to that of the Fibonacci numbers. Recall that we interpreted Fibonacci numbers as tilings of an $n$-board using squares and dominoes. To extend our tiling interpretations to continued fractions, we introduce tilings where we are allowed to stack squares and dominoes. Each tile has a limit of how many squares or dominoes can be stacked on top of it, called a *height condition*.

**Definition 3.6.** In the context of tiling a board of length $n$ using squares and dominoes, the *height condition* $a_i$ is the number of squares we may stack on the $i$th tile, and the *height condition* $b_i$ is the number of dominoes we may stack on tiles $(i-1, i)$. See Figures 8 and 9.

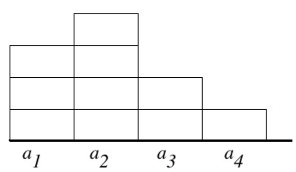


Figure 8: An example of height conditions for squares. Note that $a_1$ is the number of squares we can stack on the first tile, $a_2$ is the number of squares we can stack on the second tile, and so on.

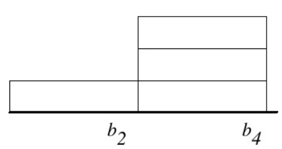


Figure 9: An example of height conditions for dominoes. Note that $b_2$ is the number of dominoes we can stack on the first and second tiles and $b_4$ is the number of dominoes we can stack on the third and fourth tiles. To make the picture clearer, we omitted $b_3$.

**Example 3.7.** For example, the list of height conditions $1, (3, 2), (4, 3), (5, 4)$ indicates that the corresponding 4-board can be tiled using stacks of squares and dominoes with the following restrictions:

**Squares:** We can stack 1 square on the first tile, up to 2 squares on the second tile, up to 3 squares on the third tile, and up to 4 squares on the fourth tile.

**Dominoes:** We can stack up to 3 dominoes on the first and second tile, up to 4 dominoes on the second and third tile, or up to 5 dominoes on the third and fourth tile.

Therefore, one conceivable tiling of this 4-board would be a single square, followed by a stack of two dominoes, followed by a stack of 4 squares. ☐

**Theorem 3.8** (Benjamin & Quinn, 2003)**.** *Consider the finite nonsimple continued fraction*

$$a_0 + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots \frac{b_n}{+ a_n},$$

*with numerator $p_n$ and denominator $q_n$. For $n \geq 0$, the numerator $p_n$ is equal to the number ways to tile an $(n+1)$-board of tiles numbered 0 through $n$ using squares and dominoes, where the tiles have height conditions $a_0, (b_1, a_1), (b_2, a_2), \ldots, (b_n, a_n)$. The denominator $q_n$ is equal to the number of ways to tile an $n$-board of tiles numbered 1 through $n$ using squares and dominoes, where the tiles have height conditions $a_1, (b_2, a_2), (b_3, a_3), \ldots, (b_n, a_n)$.*

*Proof.* Let $s_n$ be the number of ways to tile an $(n+1)$-board with height conditions $a_0, (b_1, a_1), (b_2, a_2), \ldots, (b_n, a_n)$. Let $t_n$ be the number of ways to tile an $n$-board with height conditions $a_1, (b_2, a_2), (b_3, a_3), \ldots, (b_n, a_n)$.

We will first show that $s_n$ satisfies the same initial conditions and recursion relation as $p_n$. We noted above that $p_{-1} = 1$ and $p_0 = a_0$. Previously, we defined that there is one way to tile a 0-board, so $s_{-1} = 1$. Next, we observe that a 1-board with height condition $a_0$ can be tiled $a_0$ ways since we can stack up to $a_0$ squares. Thus $s_0 = a_0$, and we have shown that $s_{-1} = p_{-1}$ and $s_0 = p_0$.

Recall that

$$p_n = a_n p_{n-1} + b_n p_{n-2}.$$

We will prove the same recursion for $s_n$. To count the number of ways to tile an $(n+1)$-board with height conditions $a_0, (b_1, a_1), (b_2, a_2), \ldots, (b_n, a_n)$, we condition on whether the last tile is a stack of squares or a stack of dominoes. If the last tile is a stack of squares, we have $a_n$ ways to choose how many squares to stack, and then $s_{n-1}$ ways to tile the first $n-1$ tiles of the board, for a total of $a_n s_{n-1}$ ways to tile the board. If the last tile is a stack of dominoes, we have $b_n$ ways to choose how many dominoes to stack, and then $s_{n-2}$ ways to tile the first $n-2$ tiles of the board, for a total of $b_n s_{n-2}$ ways to tile the board. It follows that

$$s_n = a_n s_{n-1} + b_n s_{n-2}.$$

Since $s_n$ has the same initial condition and follows the same recursion relation as $p_n$, $s_n = p_n$.

It remains to show that $t_n$ satisfies the same initial conditions and recursion relation as $q_n$. This proof will proceed very similarly. We noted above that $q_{-1} = 0$ and $q_0 = 1$. Since we cannot have a board of length $-1$, $t_{-1} = 0$. Since there is one way to tile a 0-board, $t_0 = 1$. Thus $t_{-1} = q_{-1}$ and $t_0 = q_0$.

Recall that

$$q_n = a_n q_{n-1} + b_n q_{n-2}.$$

To count the number of ways to tile an $n$-board with height conditions $a_1, (b_2, a_2), \ldots, (b_n, a_n)$, we condition on whether the last tile is a stack of squares or a stack of dominoes. If the last tile is a stack of squares we have $a_n t_{n-1}$ ways to tile the board. If the last tile is a stack of dominoes, we have $b_n t_{n-2}$ ways to tile the board. It follows that

$$t_n = a_n t_{n-1} + b_n t_{n-2}.$$

Since $t_n$ has the same initial condition and follows the same recursion relation as $q_n$, $t_n = q_n$. $\square$

We have proven that the numerator and denominator of a finite continued fraction are equal to the number of tilings of an $(n+1)$-board and an $n$-board, respectively, where we are allowed to stack squares and dominoes. From now on, we will refer to these tilings as *stackable tilings*. We will refer to our tilings from section 2 (when we were not allowed to stack squares and dominoes) as *square-domino tilings*.

Recall that in a simple continued fraction expansion, each $b_i = 1$. This means that each domino has a height condition of 1. In this case, we only list the height conditions for the squares: $a_0, a_1, a_2, \ldots a_n$.

### 3.2.1 Examples

The purpose of this section is to practice applying the combinatorial interpretation of continued fractions.

**Example 3.9.** (Benjamin & Quinn, 2003). In Example 3.3, we showed algebraically that

$$\frac{355}{113} = 3 + \frac{1}{7+}\frac{1}{15+}\frac{1}{1}$$

We will now prove that the continued fraction $[3, 7, 15, 1]$ is equal to $\frac{355}{133}$ using Theorem 3.8.

**Numerator.** The board of length 4 with height conditions $3, 7, 15, 1$ (see Figure 10) can be tiled using:

- All squares. Since we can stack up to 3 squares on the first tile, up to 7 squares on the second tile, up to 15 squares on the third tile, and 1 square on the last tile, there are $3 \cdot 7 \cdot 15 \cdot 1 = 315$ tilings that use all squares.
- Two stacks of squares followed by a domino ($3 \cdot 7 = 21$ tilings)
- A domino in between two stacks of squares ($3 \cdot 1 = 3$ tilings)
- A domino followed by two stacks of squares ($15 \cdot 1 = 15$ tilings)

- Two dominoes (1 tiling)

In total, there are $315 + 21 + 3 + 15 + 1 = 355$ stackable tilings. By Theorem 3.8, the numerator of $[3, 7, 15, 1]$ is 355.

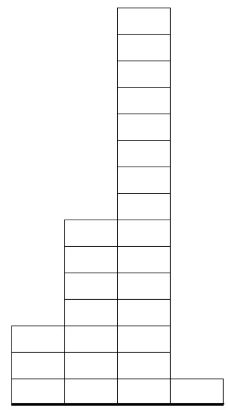

Figure 10: A 4-board with height conditions 3, 7, 15, 1.

**Denominator.** The board with height conditions $7, 15, 1$ can be tiled using

- All squares ($7 \cdot 15 \cdot 1 = 105$ tilings)
- A square followed by a domino (7 tilings)
- A domino followed by a square (1 tiling)

In total, there are $105+7+1 = 113$ stackable tilings. This means that the denominator of $[3, 7, 15, 1]$ is 113.

We conclude that $3 + \frac{1}{7+}\frac{1}{15+}\frac{1}{1} = \frac{355}{113}$. $\square$

We observe that $\frac{355}{113} \approx 3.1415929$. This is because 3, 7, 15, 1 are the first few terms of the simple infinite continued fraction expansion for $\pi$. It is quite remarkable that taking just the first 4 terms of the continued fraction expansion for $\pi$ gets us an approximation accurate to the sixth decimal place!

The next example requires both the combinatorial interpretation of continued fractions and the combinatorial interpretation of the Lucas numbers.

**Example 3.10.** (Benjamin & Quinn, 2003).

For $n \geq 1$, $[1, 1, \ldots, 1, 3] = \frac{L_{n+2}}{L_{n+1}}$.

To prove this identity, we will find a one-to-one correspondence between the denominator of both sides of the equation and the numerator of both sides of the equation.

**Denominator Set 1:** The set of all stackable tilings of an $n$-board, where we cannot stack dominoes, and we can stack squares only on the last tile, which can be a stack of up to three squares or a domino.

**Denominator Set 2:** The set of all square-domino tilings of an $(n+1)$-bracelet. This set has size $L_{n+1}$.

**Correspondence:** Suppose we have a stackable tiling of an $n$-board. There are several cases to consider:

1. If the $n$th tile is a domino or a single square, add a square to the board so that the square is glued in between tiles $n$ and 1, resulting in an $(n+1)$-bracelet that starts with a square.

2. If the last tile is two squares, unfold the squares and glue them to tile 1, resulting in an in-phase $(n+1)$-bracelet that starts with a domino.

3. If the last tile is three squares, rotate the last bracelet we made counterclockwise to create an out-of-phase $(n+1)$-bracelet.

This correspondence is easily reversed. Suppose we have a square-domino tiling of an $(n+1)$-bracelet. Then

1. If the $(n+1)$-bracelet starts in a square, remove the square, breaking the bracelet. Depending on whether the bracelet ended in a domino or a square, we now have a tiled $n$-board that ends in a domino or a single square.

2. If the $(n+1)$-bracelet starts with an in-phase domino, fold the domino to create a tiled $n$-board that ends in a stack of two squares.

3. If the $(n+1)$-bracelet starts with an out-of-phase domino, create a tiled $n$-board that ends in a stack of three squares.

We have proven a one-to-one correspondence between Denominator Set 1 and Denominator Set 2.

**Numerator Set 1:** The set of all stackable tilings of an $(n+1)$-board, where we cannot stack dominoes, and we can stack squares only on the last tile, which can be a stack of up to three squares or a domino.

**Numerator Set 2:** The set of all square-domino tilings of an $(n+2)$-bracelet. This set has size $L_{n+2}$.

**Correspondence:** The correspondence is the same as the correspondence between the denominator sets, each set just has one more tile. □

In our final example of this section, we use Theorem 3.8 to prove an interesting fact about simple continued fractions.

**Example 3.11.** A simple finite continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

and its reversal

$$a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2} + \cfrac{1}{\ddots + \cfrac{1}{a_0}}}}$$

have the same numerator [3].

To see this, simply observe that the number of ways to tile an $(n+1)$-board with height conditions $a_0, a_1, \ldots, a_n$ is the same as the number of ways to tile an $(n+1)$-board with height conditions $a_n, a_{n-1}, \ldots, a_0$. □

## 3.3 Infinite Continued Fractions

The subject of this section is infinite continued fractions

$$a_0 + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots$$

We will start by giving examples how to derive infinite continued fraction expansions for interesting numbers, including $\frac{6}{\pi^2}$.

Next, we discuss how to interpret infinite continued fractions combinatorially. To do this, we have to introduce *convergents* of continued fractions.

We conclude this section by deriving the continued fraction expansion of the golden ratio, $\Phi = \frac{1+\sqrt{5}}{2}$. This example highlights the combinatorial connection between Fibonacci numbers and continued fractions and provides a nice transition into the next goal of this paper: to present a combinatorial interpretation of the continued fraction expansion of $e$.

### 3.3.1 Deriving Infinite Continued Fraction Expansions

Let $\alpha_1$ and $\alpha_2$ be nonzero real numbers. Loya (2006) observes that

$$\begin{aligned} \frac{1}{\alpha_1} - \frac{1}{\alpha_2} &= \frac{\alpha_2 - \alpha_1}{\alpha_1 \alpha_2} \\ &= \frac{\alpha_2 - \alpha_1}{\alpha_1(\alpha_2 - \alpha_1) + \alpha_1^2} \\ &= \frac{1}{\frac{\alpha_1(\alpha_2 - \alpha_1) + \alpha_1^2}{\alpha_2 - \alpha_1}} \\ &= \frac{1}{\alpha_1 + \frac{\alpha_1^2}{\alpha_2 - \alpha_1}} \end{aligned}$$

This observation leads to the following theorem.

**Theorem 3.12** (Loya, 2006)**.** *For nonzero real numbers* $\alpha_1, \alpha_2, \alpha_3, \ldots,$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\alpha_k} = \cfrac{1}{\alpha_1 + \cfrac{\alpha_1^2}{\alpha_2 - \alpha_1 + \cfrac{\alpha_2^2}{\alpha_3 - \alpha_2 + \cfrac{\alpha_3^2}{\ddots}}}}$$

Theorem 3.12 can be proved by induction (see [10] for details). This result will be essential in proving infinite continued fraction expansions.

**Example 3.13.** (Loya, 2006). It turns out that many irrational numbers have surprisingly beautiful continued fraction expansions. For example,

$$\frac{6}{\pi^2} = 0^2 + 1^2 - \cfrac{1^4}{1^2 + 2^2 - \cfrac{2^4}{2^2 + 3^2 - \cfrac{3^4}{3^2 + 4^2 - \cfrac{4^4}{4^2 + 5^2 - \ddots}}}} \tag{5}$$

To prove this expansion, we use Euler's sum:

$$\begin{aligned} \frac{\pi^2}{6} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \\ &= \frac{1}{1^2} + \frac{-1}{-2^2} + \frac{1}{3^2} + \frac{-1}{-4^2} + \cdots \end{aligned}$$

Applying Theorem 3.12 to Euler's sum, we have

$$\frac{\pi^2}{6} = \cfrac{1}{1^2 + \cfrac{(1^2)^2}{(-2^2) - (1^2) + \cfrac{(-2^2)^2}{(3^2) - (-2^2) + \cfrac{(3^2)^2}{(-4^2) - (3^2) + \cfrac{(-4^2)^2}{\ddots}}}}} \tag{6}$$

$$= \cfrac{1}{1^2 + \cfrac{1^4}{-(1^2 + 2^2) + \cfrac{2^4}{2^2 + 3^2 + \cfrac{3^4}{-(3^2 + 4^2) + \cfrac{4^4}{\ddots}}}}} \tag{7}$$

$$= \cfrac{1}{1^2 - \cfrac{1^4}{1^2 + 2^2 - \cfrac{2^4}{2^2 + 3^2 - \cfrac{3^4}{3^2 + 4^2 - \cfrac{4^4}{\ddots}}}}} \tag{8}$$

The last step to get the expansion in equation (5) is to invert both sides of equation (8). In general, inverting a continued fraction is a useful simplification technique when $a_0 = 0$ and $b_1 = 1$. To see this, let $x = a_1 + \frac{b_2}{a_2 +} \frac{b_3}{a_3 + \cdots}$. When $a_0 = 0$ and $b_1 = 1$, our continued fraction expansion is equal to $\frac{1}{x}$, so after inverting both sides of the equation our expansion is equal to $x$. Applying this to equation (6), we have

$$\frac{6}{\pi^2} = 0^2 + 1^2 - \frac{1^4}{1^2 + 2^2 -} \frac{2^4}{2^2 + 3^2 -} \frac{3^4}{3^2 + 4^2 -} \cdots$$

$\square$

Example 3.13 deviates from our previous examples of continued fractions that have involved only positive integers. Clearly combinatorially interpreting continued fraction expansions with negative integers would be more challenging, and we will not do it in this paper. Even so, we included Example 3.13 because it is a great example of a continued fraction expansion of an irrational number that has an elegant pattern.

**Example 3.14.** (Loya, 2006). We can use Theorem 3.12 not only to derive the continued fraction representations of numbers, but also to derive the continued fraction representations of functions of $x$. For example, we can use Theorem 3.12 to find the continued fraction representation of $\log(1+x)$. To do this, we need the fact that

$$\log(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \tag{9}$$

First, we write equation (9) so that we can use Theorem 3.12,

$$\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\frac{n+1}{x^{n+1}}}$$

Then, applying Theorem 3.12, we get

$$\log(1+x) = \cfrac{1}{\frac{1}{x} + \cfrac{(\frac{1}{x})^2}{\frac{2}{x^2} - \frac{1}{x} + \cfrac{(\frac{2}{x^2})^2}{\frac{3}{x^3} - \frac{2}{x^2} + \cfrac{(\frac{3}{x^3})^2}{\ddots}}}} \tag{10}$$

Next, we use the fact that multiplying the numerator and denominator of a fraction by $x$ is equivalent to multiplying by 1 [10]. We multiply the first numerator and denominator of equation (10) by $x$, the second numerator and denominator by $x^2$, the third numerator and denominator by $x^3$, and so on, so that

$$\log(1+x) = \cfrac{x \cdot 1}{x \cdot \frac{1}{x} + \cfrac{x^2 \cdot x \cdot (\frac{1}{x})^2}{x^2 \cdot \frac{2}{x^2} - x^2 \cdot \frac{1}{x} + \cfrac{x^3 \cdot x^2 \cdot (\frac{2}{x^2})^2}{x^3 \cdot \frac{3}{x^3} - x^3 \cdot \frac{2}{x^2} + \cfrac{x^4 \cdot x^3 \cdot (\frac{3}{x^3})^2}{\ddots}}}}$$

This expansion cleans up nicely

$$\log(1+x) = \frac{x}{1+}\frac{1^2x}{2-x+}\frac{2^2x}{3-2x+}\frac{3^2x}{4-3x+}\cdots \tag{11}$$

and when we substitute $x = 1$ into equation (9), we have the beautiful continued fraction expansion

$$\log 2 = \frac{1}{1+}\frac{1^2}{1+}\frac{2^2}{1+}\frac{3^2}{1+}\cdots$$

Now that we have seen several examples of infinite continued fraction expansions, we will explain how to interpret such expansions.

### 3.3.2 Combinatorial Interpretation

We can't simply apply the combinatorial interpretation of finite continued fractions to infinite continued fractions because we can't count stackable tilings of an infinitely long board. We can, however, apply the combinatorial interpretation to the *convergents* of an infinite continued fraction.

**Definition 3.15.** The $n$th-order *convergent* of a continued fraction is

$$c_n = a_0 + \frac{b_1}{a_1+}\frac{b_2}{a_2+}\cdots+\frac{b_n}{a_n}$$

By Theorem 3.8, the numerator of $c_n$ is equal to the number of ways to tile an $(n+1)$-board with height conditions $a_0, (b_1, a_1), (b_2, a_2), \ldots, (b_n, a_n)$. The denominator of $c_n$ is equal to the number of ways to tile an $n$-board with height conditions $a_1, (b_2, a_2), \ldots, (b_n, a_n)$.

These ideas are clarified in the following examples.

**Example 3.16.** Recall the continued fraction expansion that we derived in Example 3.14:

$$\log 2 = \frac{1}{1+}\frac{1^2}{1+}\frac{2^2}{1+}\frac{3^2}{1}\cdots$$

We can directly compute the fourth convergent of $\log 2$:

$$c_4 = \cfrac{1}{1+\cfrac{1^2}{1+\cfrac{2^2}{1+\cfrac{3^2}{1}}}} = \cfrac{1}{1+\frac{1^2}{\frac{14}{10}}} = \frac{14}{24}$$

Alternatively, we can compute $c_4$ using Theorem 3.8.

By Theorem 3.8, the numerator of $c_4$ is equal to the number of tilings of a 5-board with height conditions $0, (1,1), (1^2,1), (2^2,1), (3^2,1)$ and the denominator of $c_4$ is equal to the number of tilings of a 4-board with height conditions $1, (1^2,1), (2^2,1), (3^2,1)$.

**Numerator.** We can tile a 5-board with the height conditions
$0, (1,1), (1^2,1), (2^2,1), (3^2,1)$ using:

- A single domino followed by 1 square (1 tiling)
- A single domino followed by a stack of up to 4 dominoes followed by 1 square (4 tilings)
- A single domino followed by 1 square followed by a stack of up to 9 dominoes (9 tilings)

Note that this board must begin with a domino because of the height condition $a_0 = 0$. In total, there are $1 + 4 + 9 = 14$ stackable tilings of this 5-board.

**Denominator.** We can tile a 4-board with the height conditions
$1, (1^2,1), (2^2,1), (3^2,1)$ using

- 4 squares (1 tiling)
- 1 domino followed by 2 squares (1 tiling)
- 1 domino followed by a stack of up to 9 dominoes (9 tilings)
- 1 square followed by a stack of up to 4 dominoes followed by 1 square (4 tilings)

- Two squares followed by a stack of up to 9 dominoes (9 tilings)

In total, there are $1 + 1 + 9 + 4 + 9 = 24$ stackable tilings of this 4-board.

Consistent with our earlier computation, our combinatorial interpretation indicates that $c_4 = \frac{14}{24}$. $\square$

**Example 3.17.** Consider the infinite continued fraction

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

The first few convergents are

$$c_1 = \tfrac{2}{1},\ c_2 = \tfrac{3}{2},\ c_3 = \tfrac{5}{3},\ c_4 = \tfrac{8}{5}.$$

It appears that both the numerator and denominator are following the Fibonacci sequence. To investigate this further, we look at the $n$th-order convergent

$$c_n = 1 + \frac{1}{1+}\frac{1}{1+}\cdots+\frac{1}{1}$$

By Theorem 3.8, the numerator of $c_n$ is equal to the number of ways to tile an $(n+1)$-board with height conditions all equal to 1, and the denominator of $c_n$ is the number of ways to tile an $n$-board with height conditions all equal to 1. Recall that these are just the ordinary square-domino tilings that we examined in Section 2, so

$$\begin{aligned} c_n &= \frac{f_{n+1}}{f_n} \\ &= \frac{F_{n+2}}{F_{n+1}} \end{aligned}$$

Taking the limit of both sides, we see that

$$\lim_{n\to\infty} c_n = \Phi,$$

where $\Phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Alternatively, we can derive the infinite continued fraction expansion

$$\Phi = 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+\cdots}$$

by recalling that $\Phi = \frac{1+\sqrt{5}}{2}$ is the solution to the equation

$$x^2 - x - 1 = 0.$$

We will use the same method as in Example 3.4. Writing $x^2 = x + 1$ and dividing both sides of this equation by $x$, we have

$$x = 1 + \frac{1}{x}$$

Then, replacing the $x$ in the denominator with $x = 1 + \frac{1}{x}$, we have

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{x}}$$

Repeating this process, we get

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}$$

Since $x = \Phi$, we have shown that

$$\Phi = 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+\cdots}$$

$\square$

In the first part of Example 3.17, we found a closed form for the $n$th-order convergent of the infinite continued fraction expansion of $\Phi$. This fraction had a particularly simple representation, so now we turn our attention to the $n$th-order convergents of other infinite continued fraction expansions. It turns out that $e$ has a simply stated continued fraction representation with a striking combinatorial interpretation.

## 4 The Continued Fraction Expansion of $e$

The continued fraction expansion of $e$ dates back to Euler, who developed the theory of continued fractions in the 1730's [2]. Euler introduced continued fraction expansions of $e$ and related numbers such as $\sqrt{e}$ and $\frac{e-1}{2}$. Using the fact that $e$ has an infinite continued fraction representation, Euler established that $e$ is irrational [11].

There is both a simple and nonsimple infinite continued fraction expansion for $e$. We can find the first few terms of the simple continued fraction expansion of $e$ by applying the Euclidean Algorithm to the rational approximation $e \approx \frac{271828183}{100000000}$:

$$\begin{aligned}
271828183 &= 2 \cdot 100000000 + 71828183 \\
100000000 &= 1 \cdot 71828183 + 28171817 \\
71828183 &= 2 \cdot 28171817 + 15484549 \\
28171817 &= 1 \cdot 15484549 + 12687268 \\
15484549 &= 1 \cdot 12687268 + 2797281 \\
12687268 &= 4 \cdot 2797281 + 1498144 \\
2797281 &= 1 \cdot 1498144 + 1299137 \\
1498144 &= 1 \cdot 1299137 + 199007 \\
1299137 &= 6 \cdot 199007 + 105095 \\
199007 &= 1 \cdot 105095 + 93912
\end{aligned}$$

Noticing a pattern, we conjecture that

$$e = 2 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\cdots$$

To actually prove that $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, ...]$ requires clever use of integrals [7]. While this expansion has an elegant pattern, it turns out that the nonsimple continued fraction expansion of $e$ has a more interesting combinatorial interpretation.

In this section, we will discuss the nonsimple continued fraction expansion of $e$. First, we derive the expansion. Then, we present a combinatorial interpretation of the nonsimple continued fraction expansion of $e$, which is listed as an uncounted problem in Benjamin and Quinn (2003).

## 4.1 Derivation of the Nonsimple Continued Fraction Expansion of $e$

For the following derivation of the nonsimple continued fraction expansion of $e$, we follow Loya (2006). First, we need to introduce an identity. Observe that for nonzero real numbers $\alpha_1$ and $\alpha_2$ such that $\alpha_1, \alpha_2 \neq 1$,

$$\begin{aligned}
\frac{1}{\alpha_1} - \frac{1}{\alpha_1\alpha_2} &= \frac{\alpha_2 - 1}{\alpha_1\alpha_2} \\
&= \frac{1}{\frac{\alpha_1\alpha_2}{\alpha_2 - 1}} \\
&= \frac{1}{\frac{\alpha_1(\alpha_2-1)+\alpha_1}{\alpha_2-1}} \\
&= \frac{1}{\alpha_1 + \frac{\alpha_1}{\alpha_2 - 1}}
\end{aligned}$$

This observation leads to the following theorem, which can be proved using induction (see [10] for details).

**Theorem 4.1** (Loya, 2006). *For a real sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ such that each $\alpha_i \neq 0, 1$,*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\alpha_1 \cdots \alpha_k} = \cfrac{1}{\alpha_1 + \cfrac{\alpha_1}{\alpha_2 - 1 + \cfrac{\alpha_2}{\alpha_3 - 1 + \ddots}}}$$

Recall the Maclaurin series expansion for $e^{-1}$:

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - \frac{1}{1} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots \tag{12}$$

$$= 1 - (\frac{1}{1} - \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots) \tag{13}$$

Applying Theorem 4.1 to equation (13), so that $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \ldots$, yields

$$\frac{1}{e} = 1 - \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \ddots}}}} \tag{14}$$

We subtract 1 from both sides of equation (14) and then multiply both sides by $-1$ to get

$$1 - \frac{1}{e} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \ddots}}}} \tag{15}$$

Noticing that the left side of equation (15) is equivalent to $\frac{e-1}{e}$, we invert both sides of equation (15):

$$\frac{e}{e-1} = 1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \ddots}}} \tag{16}$$

Subtracting 1 from both sides of equation (16), we have

$$\frac{1}{e-1} = \cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\ddots}}} \tag{17}$$

Then, inverting equation (17) and adding 1 to both sides, we get

$$e = 2 + \cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\ddots}}}} \tag{18}$$

Finally, we apply a transformation similar to the transformation we used in Example 3.14. We multiply by $\frac{1}{2}$ as shown below

$$e = 2 + \cfrac{\frac{1}{2}\cdot 2}{\frac{1}{2}\cdot 2+\cfrac{\frac{1}{2}\cdot 3}{3+\cfrac{4}{4+\cfrac{5}{5+\ddots}}}}$$

which is equivalent to multiplying by 1. Multiplying the next numerator and denominator by $\frac{2}{3}$, we have

$$e = 2 + \cfrac{1}{1+\cfrac{\frac{2}{3}\cdot\frac{3}{2}}{\frac{2}{3}\cdot 3+\cfrac{\frac{2}{3}\cdot 4}{4+\cfrac{5}{5+\ddots}}}}$$

Continuing this process by multiplying the next numerator and the denominator by $\frac{3}{4}$, we have

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{\frac{3}{4} \cdot \frac{8}{3}}{\frac{3}{4} \cdot 4 + \cfrac{\frac{3}{4} \cdot 5}{5 + \ddots}}}}$$

Multiplying the next numerator and denominator by $\frac{4}{5}$ and the one after by $\frac{5}{6}$ and so on, we have a nonsimple continued fraction expansion for $e$:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \ddots}}}}} \tag{19}$$

While equation (18) is certainly a clean continued fraction expansion of $e$, we will use equation (19) because its convergents have a nicer combinatorial interpretation.

## 4.2 Combinatorial Interpretation

The continued fraction expansion $[2, (1, 1), (1, 2), (2, 3), (3, 4), (4, 5), \ldots]$ of $e$ is infinite, so we cannot directly apply our combinatorial interpretation. However, we can interpret the convergents of the continued fraction expansion combinatorially.

The first few convergents of $[2, (1, 1), (1, 2), (2, 3), \ldots]$ are

$$
\begin{aligned}
c_1 &= 2 + \frac{1}{1} = 3 \\
c_2 &= 2 + \cfrac{1}{1 + \cfrac{1}{2}} = \frac{8}{3} \approx 2.666667 \\
c_3 &= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3}}} = \frac{30}{11} \approx 2.727272 \\
c_4 &= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4}}}} = \frac{144}{53} \approx 2.716981
\end{aligned}
$$

The $n$th-order convergent is:

$$
c_n = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{\ddots + \cfrac{n-1}{n}}}}}
$$

By Theorem 3.8, the numerator of the $n$th-order convergent is equal to the number of ways to tile an $(n+1)$-board with height conditions $2, (1,1), (1,2), (2,3), \ldots (n-1, n)$ and the denominator of the $n$th-order convergent is equal to the number of ways to tile an $n$-board with height conditions $1, (1,2), (2,3), \ldots (n-1, n)$ (see Figure 11). We now look at the numerator and denominator separately, in greater detail.

**Numerator.** Directly counting the number of stackable tilings, we see that

- 1-board: We can tile a 1-board using a stack of up to 2 squares, so there are 2 stackable tilings of a 1-board.

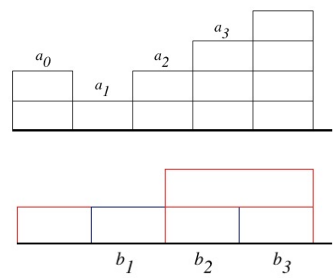


Figure 11: An $(n+1)$-board with height conditions $2, (1,1), (1,2), (2,3)\ldots$. The height conditions for the squares (top) are pictured separately from the height conditions for the dominoes (bottom).

- 2-board: We can tile a 2-board using squares ($2 \cdot 1$ tilings because we are allowed to stack up to 2 squares on the first tile) or a domino (1 tiling), so in total there are 3 stackable tilings of a 2-board.

- 3-board: We can tile a 3-board using

  - All squares. Since we can place up to 2 squares on the first tile, 1 square on the second tile, and up to 2 squares on the third tile, there are $2 \cdot 1 \cdot 2 = 4$ stackable tilings of a 3-board using all squares.

  - A domino followed by a square. Since we can place 1 domino on the first tile and up to 2 squares on the third tile, there are 2 stackable tilings of a 3-board using a domino followed by a square.

  - A square followed by a domino. Since we can place up to 2 squares on the first tile, and 1 domino on the second and third tiles, there are 2 stackable tilings of the board using a square followed by a domino.

  In total, there are 8 stackable tilings of a 3-board.

- 4-board: We can tile a 4-board using

  - All squares ($2 \cdot 1 \cdot 2 \cdot 3 = 12$ tilings)

  - A domino followed by two stacks of squares ($1 \cdot 2 \cdot 3 = 6$ tilings)

- Two stacks of squares followed by a stack of dominoes ($2 \cdot 1 \cdot 2 = 4$ tilings)
- A domino in between two stacks of squares ($2 \cdot 1 \cdot 3 = 6$ tilings)
- Two stacks of dominoes ($1 \cdot 2 = 2$ tilings)

In total, there are $12 + 6 + 4 + 6 + 2 = 30$ stackable tilings of a 4-board.

Therefore, the first few terms of the numerator of the $n$th-order convergent of $e$ are

$$2, 3, 8, 30, 144, \ldots \text{ (A001048)}$$

It turns out that the numbers in this sequence have a clean formula.

**Theorem 4.2.** *The number of stackable tilings of an* $(n+1)$*-board with height conditions* $2, (1,1), (1,2), (2,3), \ldots, (n-1, n)$ *is*

$$(n+1)! + n!.$$

*Proof.* The proof will proceed by strong induction. For $n = 0$, $1! + 0! = 2$, which is consistent with the fact that there are 2 ways to tile a 1-board with height condition 2. For $n = 1$, $2! + 1! = 3$, which is consistent with the fact that there are 3 ways to tile a 2-board with height conditions $2, (1,1)$.

Suppose we have $(n+1)! + n!$ ways to tile a $(n+1)$-board for all $1 \leq n \leq k$. Then consider a $(k+2)$-board. The $(k+2)$nd tile has height conditions $(k, k+1)$. We condition on whether the board ends in a stack of squares or dominoes. If the board ends in a stack of up to $k+1$ squares, we have $(k+1) \cdot ((k+1)! + k!)$ ways to tile the board by the induction hypothesis. If the board ends in a stack of up to $k$ dominoes, we have $k \cdot (k! + (k-1)!)$ ways to tile the board by the induction hypothesis. In total, the number of ways to tile the board is

$$\begin{aligned}
(k+1) \cdot ((k+1)! + k!) + k \cdot (k! + (k-1)!) &= (k+1) \cdot (k+1)! + (k+1)! + k \cdot k! + k! \\
&= (k+1+1) \cdot (k+1)! + (k+1) \cdot k! \\
&= (k+2)! + (k+1)!
\end{aligned}$$

This completes the proof. $\square$

**Denominator:** Directly counting the number of stackable tilings, we see that

- 1-board: We can tile a 1-board using 1 square, so there is 1 stackable tiling of a 1-board.
- 2-board: We can tile a 2-board with two squares ($2 \cdot 1$ tilings) or a single domino (1 tiling), so in total there are 3 stackable tilings of a 2-board.
- 3-board: We can tile a 3-board with all squares ($1 \cdot 2 \cdot 3 = 6$ tilings), a domino followed by a stack of squares ($1 \cdot 3 = 3$ tilings), or a square followed by a stack of dominoes (2 tilings), so in total there are 11 stackable tilings of a 3-board.
- 4-board: We can tile a 4-board using
  - All squares ($1 \cdot 2 \cdot 3 \cdot 4 = 24$ tilings)
  - A domino followed by two stacks of squares ($1 \cdot 3 \cdot 4 = 12$ tilings)
  - Two stacks of squares followed by a stack of dominoes ($1 \cdot 2 \cdot 3 = 6$ tilings)
  - A stack of dominoes in between two stacks of squares ($1 \cdot 2 \cdot 4 = 8$ tilings)
  - Two stacks of dominoes ($1 \cdot 3 = 3$ tilings)

  In total, there are $24 + 12 + 6 + 8 + 3 = 53$ stackable tilings of a 4-board.

Therefore, the first few terms of the denominator of the $n$th-order convergent of $e$ are

$$1, 3, 11, 53, \ldots \text{ (A000255)}$$

**Theorem 4.3.** *The number of stackable tilings of an $n$-board with height conditions $1, (1, 2), (2, 3), \ldots (n-1, n)$ satisfies the recursion relation*

$$a_n = n \cdot a_{n-1} + (n-1) \cdot a_{n-2}.$$

*Proof.* Let $a_n$ be the number of tilings of an $n$-board with height conditions $1, (1,2), (2,3), \dots (n-1, n)$. We count the number of possible tilings of this $n$-board in two different ways. On one hand, the number of possible tilings of the $n$-board is $a_n$. Alternatively, we can count the number of possible tilings of the $n$-board by conditioning on whether tile $n$ with height conditions $(n-1, n)$ is a stack of squares or stack of dominoes. If the last tile is a stack of up to $n$ squares, we have $n \cdot a_{n-1}$ ways to tile the rest of the board. If the last tile is a stack of up to $n-1$ dominoes, we have $(n-1) \cdot a_{n-2}$ ways to tile the rest of the board. Total, there are $n \cdot a_{n-1} + (n-1) \cdot a_{n-2}$ possible tilings of the $n$-board. We conclude that

$$a_n = n \cdot a_{n-1} + (n-1) \cdot a_{n-2}.$$

□

To summarize, we were able to write the numerator of the $n$th convergent of $e$ in a closed form: $(n+1)! + n!$. However, we have thus far only proven a recursion relation for the denominator of the $n$th convergent of $e$. Recall that our goal is to interpret $e$ combinatorially, and the closed form of the numerator suggests that to do this we may want to interpret the denominator as a type of permutation.

It turns out that the denominator is equal to a subset of the set of permutations of $\{1, 2, \cdots, n+1\}$. This result is the subject of the next section.

## 4.3 Interpretation of the Denominator of the $n$th convergent of $e$

In this section, we will prove that the denominator of the $n$th convergent of $e$ is equal to the number of permutations of $\{1, 2, \dots, n+1\}$ with no substring $(k, k+1)$. A substring $(k, k+1)$ is also called an *adjacency*.

Here we are working in two-line permutation notation. The permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix},$$

which we abbreviate 4 3 2 1, has no substring $(k, k+1)$. We say this permutation is *adjacency-free*. Conversely, the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix},$$

abbreviated 2 3 1 4, has the adjacency 2 3.

We will first show that this interpretation of the denominator holds for the first few terms, and then show that it satisfies the recursion relation we proved for the tiling interpretation of the denominator:

$$a_n = n \cdot a_{n-1} + (n-1) \cdot a_{n-2}.$$

Recall that the first few terms of the denominator of the $n$th convergent of the continued fraction expansion of $e$ are

1, 1, 3, 11, 53, ... (A000255)

We see that

- There is 1 adjacency-free permutation of $\{1\}$:

  1

- There is 1 adjacency-free permutation of $\{1, 2\}$:

  2 1

- There are 3 adjacency-free permutations of $\{1, 2, 3\}$:

  1 3 2
  2 1 3
  3 2 1

- There are 11 adjacency-free permutations of $\{1, 2, 3, 4\}$:

| | | |
|---|---|---|
| 1 3 2 4 | 1 4 3 2 | |
| 2 1 4 3 | 2 4 1 3 | 2 4 3 1 |
| 3 1 4 2 | 3 2 4 1 | 3 2 1 4 |
| 4 1 3 2 | 4 2 1 3 | 4 3 2 1 |

We now must show that number of permutations with no adjacencies follow the same recursion relation as the stackable tilings.

**Theorem 4.4.** *The number of permutations of* $\{1, 2, \ldots, n+1\}$ *with no adjacencies satisfies the recursion relation*

$$a_n = n \cdot a_{n-1} + (n-1) \cdot a_{n-2}$$

*Proof.* Let $a_n$ be the number of permutations on $n+1$ elements with no adjacencies.

First, we observe that we can build an adjacency-free permutation of $\{1, 2, \ldots, n+1\}$ from an adjacency-free permutation of $\{1, 2, \ldots, n\}$ by adding the element $n+1$, which we can place first, or to the right of any of the $n$ elements, except for $n$. There are $n$ ways to place the $(n+1)$st element, and $a_{n-1}$ adjacency-free permutations of $\{1, 2, \ldots, n\}$. So, in total, there are $n \cdot a_{n-1}$ ways to build an adjacency-free permutation of $\{1, 2, \ldots, n+1\}$ from an adjacency-free permutation of $\{1, 2, \ldots, n\}$.

Notice that this process does not get us all possible adjacency-free permutations of $\{1, 2, \ldots, n+1\}$. We are missing all of the permutations with substrings $(k, n+1, k+1)$. To create these permutations, we start with a permutation on $n$ elements with exactly one substring $(k, k+1)$ and insert $n+1$ between $k$ and $k+1$ to get an adjacency-free permutation. There are $(n-1) \cdot a_{n-2}$ permutations on $n$ elements with exactly one adjacency. To see this, observe that to create a permutation on $n$ elements with exactly one adjacency we start by placing $n-1$ of the $n$ elements so that there are no adjacencies ($a_{n-2}$ ways) and then put the remaining element $k+1$ to the right of $k$. Since one of the first $n-1$ elements we place must be 1, there are $\binom{n-1}{n-2} = n-1$ ways to choose the first $n-1$ elements. Therefore, there are $(n-1) \cdot a_{n-2}$ ways to build an adjacency-free permutation of $\{1, 2, \ldots, n+1\}$ from a permutation on $n$ elements with exactly one adjacency.

We conclude that

$$a_n = n \cdot a_{n-1} + (n-1) \cdot a_{n-2}$$

$\square$

In the following example, we create adjacency-free permutations of $n+1$ elements using the process described in the proof of Theorem 4.4 for the relatively simple case of $n = 3$.

**Example 4.5.** We will build the adjacency-free permutations of $\{1, 2, 3, 4\}$.

First, we build legal permutations of $\{1, 2, 3, 4\}$ by adding 4 to the three adjacency-free permutations of $\{1, 2, 3\}$: 1 3 2, 2 1 3, and 3 2 1. Adding 4 to each of the 3 possible places in each permutation, we get 9 permutations:

$$\begin{array}{ccc}
1\ 3\ 2 & \rightarrow & 1\ 3\ 2\ 4 \\
 & \hookrightarrow & 1\ 4\ 3\ 2 \\
 & \hookrightarrow & 4\ 1\ 3\ 2 \\
2\ 1\ 3 & \rightarrow & 2\ 1\ 4\ 3 \\
 & \hookrightarrow & 2\ 4\ 1\ 3 \\
 & \hookrightarrow & 4\ 2\ 1\ 3 \\
3\ 2\ 1 & \rightarrow & 3\ 2\ 1\ 4 \\
 & \hookrightarrow & 3\ 2\ 4\ 1 \\
 & \hookrightarrow & 4\ 3\ 2\ 1
\end{array}$$

Clearly the above permutations have no adjacencies. We observe that of the 11 legal permutations of $\{1, 2, 3, 4\}$, we are missing the permutations 2 4 3 1 and 3 1 4 2.

To construct 2 4 3 1 and 3 1 4 2, we see that there are two permutations of $\{1, 2, 3\}$ with exactly one adjacency: 2 3 1 and 3 1 2.

We add 4 to "break up" the adjacency:

$$\begin{array}{ccc}
2\ 3\ 1 & \rightarrow & 2\ 4\ 3\ 1 \\
3\ 1\ 2 & \rightarrow & 3\ 1\ 4\ 2
\end{array}$$

Now we have all 11 adjacency-free permutations of $\{1, 2, 3, 4\}$. $\square$

The fact that permutations of $\{1, 2, \ldots n + 1\}$ with no adjacencies have the same initial conditions and follow the same recursion relation as the stackable tilings of an $n$-board with height conditions $1, (1, 2), \ldots, (n - 1, n)$ suggests that there is a one-to-one correspondence between adjacency-free permutations and stackable tilings.

In the next section, we prove this correspondence by presenting an algorithm that indicates how to create an adjacency-free permutation of the set $\{1, 2, \ldots, n + 1\}$, given any stackable tiling of an $n$-board. We will also show that we can reverse this algorithm so that, given an adjacency-free permutation of the set $\{1, 2, \ldots, n + 1\}$, we can work backwards to create a stackable tiling of an $n$-board.

## 4.4 The Correspondence Between Permutations and Tilings

Recall that the proofs of the recursion relation $a_n = n \cdot a_{n-1} + (n - 1) \cdot a_{n-2}$ for adjacency-free permutations of the set $\{1, 2, \ldots, n+1\}$ and tilings of an $n$-board were each broken into two parts:

1. (a) Starting with an adjacency-free permutation of $\{1, 2, \ldots, n\}$, add $n + 1$ to one of $n$ possible places.

   (b) Starting with a tiled $(n - 1)$-board, add a stack of up to $n$ squares.

2. (a) Starting with a permutation of $\{1, 2, \ldots, n\}$ with exactly one substring $(k, k + 1)$, place $n + 1$ in between $k$ and $k + 1$.

   (b) Starting with a tiled $(n - 2)$-board, add a stack of up to $n - 1$ dominoes.

So it seems that placing the $(n+1)$st element is like placing a square, and breaking up the adjacency $(k, k + 1)$ is like placing a domino. This observation is key to developing the algorithm. One of the reasons it is helpful is that it shows we can consider two cases separately:

1. The case where the $n$-board is all squares, which corresponds to the case where the permutation of the set $\{1, 2, \ldots, n+1\}$ has no substrings $(k, i, k+1)$, where $i > k$.

2. The case where the $n$-board has at least one stack of dominoes, which corresponds to the case where the permutation of the set $\{1, 2, \ldots, n+1\}$ has at least one substring $(k, i, k+1)$, where $i > k$.

**Case 1.** Given a tiling of an $n$-board of all squares, we construct the corresponding adjacency-free permutation of $\{1, 2, \ldots, n+1\}$ by adding one element to our permutation for each tile (starting with the first tile), where the number of squares stacked on the tile indicates where to place the element.

The first tile must be a single square, which corresponds to the permutation 2 1, because 2 1 is the only adjacency-free permutation of $\{1, 2\}$.

The second tile is a stack of up to two squares. The number of squares stacked on the second tile tells us where to place 3 in the permutation 2 1. Notice that there are two possible places for 3. We can place 3 to the right of 1, resulting in the permutation 2 1 3. We say that this is placing 3 in the rightmost position. Or, we can place 3 to the left of 2, resulting in the permutation 3 2 1. We say that this is placing 3 in the second rightmost position. (Notice that if we place 3 to the right of 2, we get the permutation 2 3 1, which has the adjacency 2 3.) If we have one square stacked on the second tile, we place 3 in the rightmost position and if we have two squares stacked on the second tile, we place 3 in the second rightmost position.

Therefore, the tiling of a 2-board that consists of a square followed by a single square corresponds to the permutation 2 1 3. The tiling of a 2-board that consists of a square followed by a stack of two squares corresponds to the permutation 3 2 1.

In general, the number of squares $k$ in the $i$th stack indicates that the number $i+1$ is in the $k$th rightmost position. The $n$th stack of squares is a stack of up to $n$ squares, just as there are $n$ possible places for the number $n+1$ in a permutation of $\{1, 2, \ldots, n\}$. Next, we provide several examples of how to apply this algorithm to stackable tilings of a 5-board.

**Example 4.6.** Consider the following tiling of a 5-board:

$$SS^2S^3S^3S^3$$

where the exponent $k$ in $S^k$ signifies a stack of $k$ squares.

As we described above, we work through the board, one tile at a time, to build the corresponding permutation. As always, $S$ corresponds to the permutation 2 1.

Since there are two squares stacked on the second tile, we put 3 in the second rightmost position of the permutation 2 1. The rightmost position for 3 is to the right of the 1, and the second rightmost position for 3 is to the left of the 2. Thus we get the permutation 3 2 1.

There are 3 squares stacked on the third tile, so we put 4 in the third rightmost position of the permutation 3 2 1. This results in the permutation 4 3 2 1.

Since there are 3 squares stacked on the fourth tile, we put 5 in the third rightmost position, resulting in the permutation 4 3 5 2 1.

Finally, there are 3 squares stacked on the fifth tile, so we put 6 in the third rightmost position, resulting in the permutation 4 3 6 5 2 1.

To summarize,

$$\begin{array}{ccccccccc} S & \rightarrow & SS^2 & \rightarrow & SS^2S^3 & \rightarrow & SS^2S^3S^3 & \rightarrow & SS^2S^3S^3 \\ 2\ 1 & \rightarrow & 3\ 2\ 1 & \rightarrow & 4\ 3\ 2\ 1 & \rightarrow & 4\ 3\ 5\ 2\ 1 & \rightarrow & 4\ 3\ 6\ 5\ 2\ 1 \end{array}$$

$\square$

**Example 4.7.** Consider the following tiling of a 5-board:

$$SSS^3S^2S^4$$

Again, $S$ corresponds to the permutation 2 1. Since there is 1 square stacked on the second tile, we put 3 in the rightmost position, obtaining the permutation 2 1 3.

Since there are 3 squares stacked on the third tile, we place 4 in the third rightmost position, obtaining the permutation 4 2 1 3. Next, we see that there are two squares stacked on the fourth tile, so we place 5 in the second rightmost position, resulting in the permutation 4 2 1 5 3.

Since there are four squares stacked on the fifth tile, we place 6 in the fourth rightmost position, obtaining the permutation 4 6 2 1 5 3.

Therefore,

$$\begin{array}{ccccccccc} S & \rightarrow & SS & \rightarrow & SSS^3 & \rightarrow & SSS^3S^2 & \rightarrow & SSS^3S^2S^4 \\ 2\ 1 & \rightarrow & 2\ 1\ 3 & \rightarrow & 4\ 2\ 1\ 3 & \rightarrow & 4\ 2\ 1\ 5\ 3 & \rightarrow & 4\ 6\ 2\ 1\ 5\ 3 \end{array}$$

□

Because we want to show that this algorithm gives a one-to-one correspondence between stackable tilings and permutations, we next give an example of how to reverse this algorithm to turn a permutation into a stackable tiling.

**Example 4.8.** Consider the permutation 4 6 2 1 5 3. We see that 6 is in the fourth rightmost position, so the last tile of our 5-board must be a stack of four squares: $S^4$. Removing 6 from our permutation, we have the permutation 4 2 1 5 3.

Next, we see that 5 is in the second rightmost position, so the second to last tile of our 5-board is a stack of two squares. Thus our tiling ends in $S^2S^4$. Removing 5 from our permutation, we have the permutation 4 2 1 3.

We observe that 4 is in the third rightmost position. This means that the corresponding tile of our 5-board is a stack of three squares, so our tiling ends in $S^3S^2S^4$. Removing 4, we have the permutation 2 1 3.

Next, 3 is in the rightmost position, telling us that the corresponding tile of our 5-board is a single square, so our tiling ends in $SS^3S^2S^4$. Now we have the permutation 2 1, which corresponds to one square on the first tile. Thus we have the tiling $SSS^3S^2S^4$. We observe that this is the tiling we started with in Example 4.6.

In sum,

$$\begin{array}{ccccccccc} 4\,6\,2\,1\,5\,3 & \to & 4\,2\,1\,5\,3 & \to & 4\,2\,1\,3 & \to & 2\,1\,3 & \to & 2\,1 \\ S^4 & \to & S^2S^4 & \to & S^3S^2S^4 & \to & SS^3S^2S^4 & \to & SSS^3S^2S^4 \end{array}$$

□

**Case 2.** Now we will consider tilings that include dominoes. As we alluded to earlier, adding a stack of dominoes corresponds to breaking up an adjacency. When we add a stack of dominoes we are lengthening our board by two tiles, so we must add two elements to our permutation. With the first element, we create an adjacency, and with the second element, we break the adjacency.

The number of dominoes in the stack indicates which adjacency to create and then break. If we have a single domino, we create and then break up the adjacency 1 2. If we have a stack of two dominoes, we create and then break up the adjacency 2 3, and so on.

The placement of the stack of dominoes tells us which element we use to break up the adjacency. If our domino stack is covering tiles 1 and 2, we break up the adjacency with the element 3. If our domino stack is covering tiles 2 and 3, we break up the adjacency with the element 4, and so on.

For example, we can have up to 3 dominoes stacked on tiles 3 and 4. If we have one domino stacked on tiles 3 and 4, we create the adjacency 1 2 and break it with 5, so one domino stacked on tiles 3 and 4 corresponds to the substring 1 5 2. Similarly, two dominoes stacked on tiles 3 and 4 corresponds to the substring 2 5 3. Three dominoes stacked on tiles 3 and 4 corresponds to the substring 3 5 4.

This process is made clearer by the following examples.

**Example 4.9.** Suppose we wish to find the permutation corresponding to the tiling $SSD$. As previously mentioned, the tiling $SS$ corresponds to the permutation 2 1 3. It remains to add the domino. A single domino covering the third and fourth tile corresponds to creating the adjacency 1 2 and breaking it with 5. First, we add the adjacency 1 2 to the permutation 2 1 3. If we simply place a 2 to the right of 1 in the permutation 2 1 3, we have the nonsensical permutation 2 1 2 3, so we need to shift the 2 and 3 in the original permutation to 3 and 4, respectively. After making these shifts, we place 2 to the right of 1 in the permutation 3 1 4, resulting in the permutation 3 1 2 4. Finally, we break up the adjacency 1 2 with a 5, resulting in the adjacency-free permutation 3 1 5 2 4.

In summary,

$$\begin{array}{ccccc} S & \to & SS & \to & SSD \\ 2\ 1 & \to & 2\ 1\ 3 & \to & \underline{3}\ 1\ 2\ \underline{4} \to 3\ 1\ 5\ 2\ 4 \end{array}$$

where the underlined elements were shifted in the process of creating the adjacency.

□

Observe that in Example 4.9 the addition of a domino increased the length of the corresponding permutation by two elements.

**Example 4.10.** Suppose we have the tiling $DD^2$. We compute the corresponding permutation as follows

$$\begin{array}{ccc} D & \to & DD^2 \\ 1\ 2 \to 1\ 3\ 2 & \to & 1\ \underline{4}\ 2\ 3 \to 1\ 4\ 2\ 5\ 3 \end{array}$$

A single domino covering the first and second tiles corresponds to the permutation 1 3 2: we create the adjacency 1 2, and then break it up with 3. It remains to add the stack of two dominoes. The stack of two dominoes covering the third and fourth tile indicates that we create the adjacency 2 3, and then break it with 5. So we place 3 to the right of 2 in the permutation 1 3 2, first shifting the 3 in the original permutation to 4. This results in the permutation 1 4 2 3. Breaking up the adjacency 2 3 with a 5, we have the adjacency-free permutation 1 4 2 5 3. □

The permutation 1 4 2 5 3 may appear to have two adjacencies that were created and then broken - this permutation has substrings 2 5 3 and 4 2 5. However, recall that we only break adjacencies with elements greater than the elements that make up the adjacencies. The fact that 2 is less than 5 indicates that at no intermediate step of constructing this permutation were 4 and 5 adjacent.

We are now ready to move on to more difficult examples.

**Example 4.11.** Consider the tiling $SSDS^4D^2$. We compute the corresponding permutation as follows

$$\begin{array}{ccccccccc} S & \to & SS & \to & SSD & \to & SSDS^4 & \to & SSDS^4D^2 \\ 2\ 1 & \to & 2\ 1\ 3 & \to & \underline{3}\ 1\ 5\ 2\ \underline{4} & \to & 3\ 6\ 1\ 5\ 2\ 4 & \to & \underline{4}\ \underline{7}\ 1\ \underline{6}\ 2\ 8\ 3\ \underline{5} \end{array}$$

As usual, the first square corresponds to the permutation 2 1. Then, since there is only 1 square on the second tile, we put 3 in the rightmost position, obtaining the permutation 2 1 3.

Next, we have a single domino covering the third and fourth tile. This indicates that we must create the adjacency 1 2 by placing 2 next to 1 in the permutation 2 1 3, shifting 2 and 3 accordingly. Then we get the permutation 3 1 2 4, and we add 5 to break up the adjacency 1 2. This results in the permutation 3 1 5 2 4.

Next, we have a stack of four squares, which means that we must place 6 in the fourth rightmost position. This gives us the permutation 3 6 1 5 2 4. The last tile is a stack of two dominoes, which indicates that we must create the adjacency 2 3 and break it up with 8. So we place 3 to the right of 2, and shift the other numbers in the permutation 3 6 1 5 2 4 accordingly, to obtain 4 7 1 6 2 3 5. We then break up the adjacency 2 3 with 8, resulting in the permutation 4 7 1 6 2 8 3 5. □

Now, we give an example of how to reverse this algorithm.

**Example 4.12.** We start with the permutation 4 7 1 6 2 8 3 5 and work our algorithm backwards. We start by looking at the location of 8. We see that 8 is breaking up an adjacency, so there must be a stack of dominoes covering the last two tiles.

Specifically, since 8 is breaking up the adjacency 2 3, there is a stack of two dominoes on the last two tiles.

Now we remove the substring 8 3 from the permutation 4 7 1 6 2 8 3 5, and shift the numbers greater than 3 down accordingly. This results in the permutation 3 6 1 5 2 4. We see that 6 is not breaking up an adjacency, and that it is in the fourth rightmost position. The corresponding stack of squares is $S^4$, so the ending of our tiling is $S^4D^2$.

Removing 6 from the permutation, we have 3 1 5 2 4. We see that 5 is breaking up the adjacency 1 2, which corresponds to a single domino, so the ending of our tiling is $DS^4D^2$. Removing 5 2 from our permutation and shifting the other elements as necessary, we have the permutation 2 1 3.

Next, we see that 3 is in the rightmost position, which corresponds to a single square. Finally, we know that 2 1 corresponds to a single square. Thus the tiling corresponding to the permutation 4 7 1 6 2 8 3 5 is $SSDS^4D^2$.

In summary,

$$\begin{array}{ccccccccc}
4\,7\,1\,6\,2\,8\,3\,5 & \rightarrow & \underline{3}\,\underline{6}\,1\,\underline{5}\,2\,\underline{4} & \rightarrow & 3\,1\,5\,2\,4 & \rightarrow & \underline{2}\,1\,\underline{3} & \rightarrow & 2\,1 \\
D^2 & \rightarrow & S^4D^2 & \rightarrow & DS^4D^2 & \rightarrow & SDS^4D^2 & \rightarrow & SSDS^4D^2
\end{array}$$

$\square$

In this section, we explained our algorithm using a variety of examples. To clarify the algorithm further, we present all 11 tilings of a 3-board and the corresponding permutations of $\{1, 2, 3, 4\}$ in Table 1.

Given a tiling of an $n$-board with height conditions $1, (1, 2), (2, 3), \ldots, (n-1, n)$, we can find the corresponding adjacency-free permutation of $\{1, 2, \ldots, n+1\}$. We have shown that this process is easily reversible. Hence, we have proven a one-to-one correspondence between tilings of an $n$-board with height conditions $1, (1, 2), (2, 3), \ldots, (n-1, n)$ and adjacency-free permutations of $\{1, 2, \ldots, n+1\}$.

Table 1: Tilings of a 3-board corresponding to permutations of $\{1, 2, 3, 4\}$ with no substring $(k, k+1)$.

| | |
|---|---|
| $SS \longleftrightarrow 2\ 1\ 3$ | $SSS \longleftrightarrow 2\ 1\ 4\ 3$<br>$SSS^2 \longleftrightarrow 2\ 4\ 1\ 3$<br>$SSS^3 \longleftrightarrow 4\ 2\ 1\ 3$ |
| $SS^2 \longleftrightarrow 3\ 2\ 1$ | $SS^2S \longleftrightarrow 3\ 2\ 1\ 4$<br>$SS^2S^2 \longleftrightarrow 3\ 2\ 4\ 1$<br>$SS^2S^3 \longleftrightarrow 4\ 3\ 2\ 1$ |
| $D \longleftrightarrow 1\ 3\ 2$ | $DS \longleftrightarrow 1\ 3\ 2\ 4$<br>$DS^2 \longleftrightarrow 1\ 4\ 3\ 2$<br>$DS^3 \longleftrightarrow 4\ 1\ 3\ 2$ |
| $S \longleftrightarrow 2\ 1$ | $SD \longleftrightarrow 3\ 1\ 4\ 2$<br>$SD^2 \longleftrightarrow 2\ 4\ 3\ 1$ |

## 4.5 Scramblings and Derangements

The numerator and denominator of the $n$th-order convergent of $e$ can now both be interpreted as permutations: the numerator is equal to $(n + 1)! + n!$, and the denominator is equal to the number of permutations of $\{1, 2, \ldots, n + 1\}$ with no substring $(k, k + 1)$.

It turns out that permutations with no substring $(k, k+1)$ have been studied and are called *tertiary scramblings* [1]. As the adjective *tertiary* suggests, there are two other closely related types of scramblings, which we define in this section. But first, we need one more definition.

When an endpoint is in its natural position, we say it is *fixed* [1]. For example, in the permutation 1 5 4 3 2, the endpoint 1 is in its natural position, so we say that the left endpoint is fixed.

The three types of scramblings - perfect scramblings, secondary scramblings, and tertiary scramblings - all have no adjacencies, and only differ in how many endpoints are not fixed. For the following definitions, we follow Balof, Farmer, and Kawabata (1997).

**Definition 4.13.** If a permutation of $n$ elements has no adjacencies or fixed endpoints, it is called a *perfect scrambling*. The number of perfect scramblings is $s_n$.

**Definition 4.14.** A *secondary scrambling* is a permutation of $n$ elements in which there are no adjacencies and the left endpoint is not fixed. The number of secondary scramblings is $s'_n$.

**Definition 4.15.** A *tertiary scrambling* is a permutation of $n$ elements in which there are no adjacencies, though both endpoints may or may not be fixed. The number of tertiary scramblings is $s''_n$.

Balof et al. (1997) proved the following relationship between perfect, secondary, and tertiary scramblings.

**Theorem 4.16.** *For $n \geq 1$,*

$$s'_n = s_n + s_{n-1},$$
$$s''_n = s'_n + s'_{n-1}$$

We will not prove Theorem 4.16, but the interested reader should consult [1].

It turns out that scramblings are closely related to another subset of the permutations of $n$ elements called *derangements*.

**Definition 4.17.** A *derangement* of $\{1, 2, \ldots, n\}$ is a permutation $i_1 i_2 \ldots i_n$ of $\{1, 2, \ldots, n\}$ in which no integer is in its natural position. That is,

$$i_1 \neq 1, i_2 \neq 2, \ldots, i_n \neq n.$$

For example, the derangements of $\{1, 2, 3\}$ are 2 3 1 and 3 1 2. The derangements of $\{1, 2, 3, 4\}$ are

| | | |
|---|---|---|
| 2 1 4 3 | 2 3 4 1 | 2 4 1 3 |
| 3 1 4 2 | 3 4 1 2 | 3 4 2 1 |
| 4 1 2 3 | 4 3 2 1 | 4 3 1 2 |

We will let $d_n$ denote the number of derangements of the set $\{1, 2, \ldots, n\}$. The following are relatively well known facts about derangements so they will be presented without proof. For proofs of these results, see Brualdi (2004).

**Proposition 4.18.** *For $n \geq 3$,*

$$d_n = (n-1)(d_{n-2} + d_{n-1})$$

**Theorem 4.19.** *For $n \geq 1$,*

$$d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

Using Theorem 4.19, we can connect derangements to $e$ [5]. Recalling the Maclaurin series expansion of $\frac{1}{e}$

$$\frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$$

we realize that

$$\frac{1}{e} = \frac{d_n}{n!} + \frac{(-1)^{n+1}}{(n+1)!} + \frac{(-1)^{n+2}}{(n+2)!} + \ldots$$

so

$$\lim_{n \to \infty} \frac{d_n}{n!} = \frac{1}{e}$$

Finally, we present the relationship between perfect scramblings and derangements.

**Theorem 4.20.** *For $n > 2$,*

$$s_n + s_{n-1} = d_n$$

For a proof of this result, see [1]. Next, we combine the results from this section to take the limit of the $n$th order convergent of $e$.

## 4.6 Re-establishing $e$

We have shown that the $n$th order convergent of $e$ is

$$c_n = \frac{(n+1)! + n!}{s''_{n+1}}$$

Using Theorems 4.16 and 4.20, we have

$$\begin{aligned} d_{n+1} &= s_{n+1} + s_n \\ &= s'_{n+1} \\ &= s''_{n+1} - s'_n \end{aligned}$$

Then, by the recursion relation for $d_n$ (Proposition 4.18), we have

$$\begin{aligned} s''_{n+1} &= d_{n+1} + s'_n \\ &= d_{n+1} + d_n \\ &= \frac{d_{n+2}}{n+1} \end{aligned}$$

Now, taking the limit of the $n$th convergent of $e$ as $n$ goes to infinity,

$$\begin{aligned}
\lim_{n\to\infty} \frac{(n+1)! + n!}{s''_{n+1}} &= \lim_{n\to\infty} \frac{(n+1)! + n!}{\frac{d_{n+2}}{n+1}} \\
&= \lim_{n\to\infty} \frac{(n+1)\cdot(n+1)! + (n+1)!}{d_{n+2}} \\
&= \lim_{n\to\infty} \frac{(n+2)!}{d_{n+2}} \\
&= e.
\end{aligned}$$

Alternatively, we can show this by writing

$$\frac{(n+1)! + n!}{s''_{n+1}} = \frac{(n+1)! + n!}{d_{n+1} + d_n}$$

And observing that

$$\frac{n!}{d_n} \leq \frac{(n+1)! + n!}{d_{n+1} + d_n} \leq \frac{(n+1)!}{d_{n+1}}$$

Since $\lim_{n\to\infty} \frac{n!}{d_n} = e$ and $\lim_{n\to\infty} \frac{(n+1)!}{d_{n+1}} = e$,

$$\lim_{n\to\infty} \frac{(n+1)! + n!}{d_{n+1} + d_n} = e.$$

Thus, using tilings, permutations, scramblings, and derangements, we have presented a combinatorial interpretation of the continued fraction expansion of $e$.

# 5 Harmonic and Stirling Numbers

To a calculus student, the phrase harmonic likely brings to mind the harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$, the canonical example of a divergent series with the property that the limit of the $n$th term as $n$ tends to infinity is 0.

The Harmonic numbers defined by

$$
\begin{aligned}
H_1 &= 1 \\
H_2 &= 1 + \frac{1}{2} = \frac{3}{2} \\
H_3 &= 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} \\
&\vdots \\
H_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}
\end{aligned}
$$

are also important in combinatorics; they frequently arise in solutions to combinatorial problems [4].

It may not be immediately obvious that we can interpret the harmonic numbers combinatorially. Similar to finite continued fractions, we consider the numerator and denominator of $H_n$ separately.

It turns out that $H_n = \frac{\left[{n+1 \atop 2}\right]}{n!}$ for all $n \geq 0$, where the numerator $\left[{n+1 \atop 2}\right]$ is a Stirling number of the first kind. Before proving this equality, we need to introduce Stirling numbers of the first kind.

## 5.1 Stirling Numbers of the First Kind

**Definition 5.1.** (Benjamin & Quinn, 2003). The Stirling number of the first kind, $\left[{n \atop k}\right]$, counts the number of permutations of $n$ elements with $k$ cycles.

To illustrate Definition 5.1, we present the first few Stirling numbers of the first kind when $k = 2$. Since there are no permutations of 1 element with 2 cycles, $\left[{1 \atop 2}\right] = 0$. The only permutation of 2 elements with 2 cycles is (1)(2), so $\left[{2 \atop 2}\right] = 1$.

The permutations of 3 elements with 2 cycles are $(12)(3), (13)(2)$, and $(1)(23)$, so $\left[{3 \atop 2}\right] = 3$. There are 11 permutations of 4 elements with 2 cycles:

$$
\begin{array}{lll}
(12)(34) & (13)(24) & (14)(23) \\
(123)(4) & (132)(4) & \\
(124)(3) & (142)(3) & \\
(134)(2) & (143)(2) & \\
(1)(234) & (1)(243) &
\end{array}
$$

so $\left[{4 \atop 2}\right] = 11$.

Stirling numbers of the first kind are also defined as coefficients in the expansion of the rising factorial function [3]:

**Definition 5.2.** The Stirling number of the first kind is given by

$$x(x+1)(x+2)\cdots(x+n-1) = \sum_{m=1}^{n} \begin{bmatrix} n \\ m \end{bmatrix} x^m$$

It is not immediately obvious why Definitions 5.1 and 5.2 are equivalent. To show the equivalence of these two definitions, we generalize the following argument from Benjamin and Quinn (2003), who prove the equivalence for the specific case of $n = 10$ and $k = 3$.

To show that Definitions 5.1 and 5.2 are equivalent, we must show that the coefficient of $x^k$ in the polynomial $x(x+1)\cdots(x+n-1)$ is equal to the number of ways to put the numbers $0, 1, 2, \ldots, n-1$ in $k$ cycles.

Observe that each term of $x^k$ in the polynomial $x(x+1)\cdots(x+n-1)$ is the product of $n-k$ numbers between 0 and $n-1$. It follows that the coefficient of $x^k$ in the expansion $x(x+1)(x+2)\cdots(x+n-1)$ is the sum of all possible products of $n-k$ numbers chosen from $0, 1, 2, \ldots, n-1$.

We will show that a product of $n-k$ numbers between 0 and $n-1$ counts the number of ways to put $n$ numbers into $k$ cycles, once we have decided which $k$ numbers begin each cycle.

To place the numbers $0, 1, 2, \ldots, n-1$ in $k$ cycles, we first pick $k$ of the $n$ numbers.

Each of those $k$ numbers will serve as the first number in one of the $k$ cycles. Since the first number of a cycle must be the smallest number in that cycle, one of these $k$ numbers must be 0, which will guarantee that our product is nonzero. After deciding which $k$ numbers start the cycles, we have $n-k$ numbers remaining.

Since 0 starts a cycle, the first number we place is 1. If 1 is not one of the $k$ numbers at the beginning of a cycle, there is only one place to put it: to the right of 0. If 1 is one of the $k$ numbers at the beginning of a cycle, we consider the number 2. If 2 is not one of the $k$ numbers starting a cycle, we can put it to the right of 0 or to the right of 1, so there are two ways to place it. This holds when 0 and 1 are in different cycles and when 0 and 1 are next to each other in the same cycle. Next, if 3 is not one of the $k$ numbers starting a cycle, we can put it to the right of 0, the right of 1, or the right of 2, so there are three ways to place it.

We continue this process, so that there are $i$ ways to place the number $i$, provided that $i$ is not the first number of a cycle. It follows that the number of ways to place $n$ numbers in $k$ cycles is the product of some $n-k$ numbers chosen from $0, 1, 2, \ldots, n-1$.

Summing over all possible choices of the $k$ numbers, the number of ways to place $0, 1, 2, \ldots, n-1$ in $k$ cycles is the sum of all products of $n-k$ numbers chosen from $0, 1, 2, \ldots, n-1$. As discussed earlier, the sum of all products of $n-k$ numbers chosen from $0, 1, 2, \ldots, n-1$ is also the coefficient of $x^k$ in the expansion $x(x+1)(x+2)\cdots(x+n-1)$. We conclude that Definitions 5.1 and 5.2 are equivalent. ∎

We can prove many Stirling number identities relying just on Definition 5.1.

**Example 5.3.** (Benjamin & Quinn, 2003).

For $m, n \geq 0$,

$$\sum_{k=m}^{n} \begin{bmatrix} k \\ m \end{bmatrix} \frac{n!}{k!} = \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}$$

*Proof.* We count the number of ways to permute $n+1$ elements into exactly $m+1$

cycles in two different ways.

**Method 1.** By definition, the number of ways to permute $n+1$ elements into exactly $m+1$ cycles is $\left[{n+1 \atop m+1}\right]$.

**Method 2.** Condition on the number of elements in the first $m$ cycles. To permute $n+1$ elements into exactly $m+1$ cycles, we first choose $k$ elements from the first $n$ elements and permute them into exactly $m$ cycles where, since there is at least one element in each cycle, $k \geq m$. There are $\binom{n}{k}$ ways to choose the $k$ elements and $\left[{k \atop m}\right]$ ways to permute them into $m$ cycles.

Next, we permute the remaining $n-k+1$ elements into 1 cycle. The lowest element is placed first by convention, so there are $(n-k)!$ ways to arrange the remaining $n-k$ elements. We have permuted $n+1$ elements into $m+1$ cycles. Summing over all possible values of $k$, we have that the number of ways to permute $n+1$ elements into exactly $m+1$ cycles is

$$\sum_{k=m}^{n} \begin{bmatrix} k \\ m \end{bmatrix} \binom{n}{k} (n-k)! = \sum_{k=m}^{n} \begin{bmatrix} k \\ m \end{bmatrix} \frac{n!}{k!}$$

We conclude that

$$\sum_{k=m}^{n} \begin{bmatrix} k \\ m \end{bmatrix} \frac{n!}{k!} = \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}$$

□

The values of the first few Stirling numbers suggest that the Stirling numbers may follow a triangular pattern similar to the pattern in Pascal's triangle. To investigate this, we list the Stirling numbers for $n = 1$ through $n = 5$ in Table 2.

Looking at Table 2, we notice that

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

Table 2: The first few Stirling numbers.

| $n$ | $\begin{bmatrix} n \\ k \end{bmatrix}$ |
|---|---|
| 1 | 1 |
| 2 | 1 1 |
| 3 | 2 3 1 |
| 4 | 6 11 6 1 |
| 5 | 24 50 35 10 1 |

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + 4 \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

This relationship between Stirling numbers is reminiscent of the relationship between binomial coefficients:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{20}$$

It turns out that we can generalize the observations from Table 2 to get an identity similar to equation (20). The proof of this identity is from Benjamin and Quinn (2003).

**Theorem 5.4** (Benjamin & Quinn, 2003)**.** *For $n \geq k \geq 1$,*

$$\left[{n \atop k}\right] = \left[{n-1 \atop k-1}\right] + (n-1)\left[{n-1 \atop k}\right]$$

*Proof.* (Benjamin & Quinn, 2003). There are, by definition, $\left[{n \atop k}\right]$ ways to permute $n$ elements into $k$ cycles. We can also count the number of ways to permute $n$ elements into $k$ cycles by conditioning based on whether element $n$ is alone in a cycle. If element $n$ is alone, there are $\left[{n-1 \atop k-1}\right]$ ways to permute the remaining $n-1$ elements into the remaining $k-1$ cycles. If element $n$ is not alone in a cycle, we first permute the other $n-1$ elements into the $k$ cycles. Then we place $n$ to the right of any of the first $n-1$ elements. By the multiplication rule, there are $(n-1)\cdot\left[{n-1 \atop k-1}\right]$ ways to permute $n$ elements into $k$ cycles if $n$ is not alone in a cycle. Thus the total number of ways to permute $n$ elements into $k$ cycles is $\left[{n-1 \atop k-1}\right]+(n-1)\left[{n-1 \atop k}\right]$, so we conclude that

$$\left[{n \atop k}\right]=\left[{n-1 \atop k-1}\right]+(n-1)\left[{n-1 \atop k}\right].$$

□

Theorem 5.4 is essential to our proof of the combinatorial interpretation of the harmonic numbers.

**Theorem 5.5** (Benjamin & Quinn, 2003)**.** *For $n \geq 0$,*

$$H_n=\frac{\left[{n+1 \atop 2}\right]}{n!}.$$

*That is, the numerator of the $n$th harmonic number counts the number of permutations of $n+1$ elements into two cycles, and the denominator of the $n$th harmonic number counts the number of permutations of $n$ elements.*

*Proof.* Finding a common denominator, we have

$$H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=\frac{a_n}{n!}$$

We will show that $a_n=\left[{n+1 \atop 2}\right]$ by showing that $a_n$ and $\left[{n+1 \atop 2}\right]$ satisfy the same initial conditions and recursion relation.

We see that since $H_0=0, a_0=0$; $H_1=1$, so $a_1=1$; $H_2=\frac{3}{2}$, so $a_2=3$. Earlier, we noted that $\left[{1 \atop 2}\right]=0$, $\left[{2 \atop 2}\right]=1$, $\left[{3 \atop 2}\right]=3$ and thus $a_n$ and $\left[{n+1 \atop 2}\right]$ have the same initial conditions.

It remains to show that $a_n$ and $\left[{n+1 \atop 2}\right]$ follow the same recursion relation. By Theorem 5.4, for $n \geq 0$,

$$\left[{n+1 \atop 2}\right]=\left[{n \atop 1}\right]+n\cdot\left[{n \atop 2}\right].$$

Note that since typical cycle notation requires placing the lowest element first, the number of permutations of $n$ elements into 1 cycle is equal to the number of permutations of $n-1$ elements. Thus $\left[ {n \atop 1} \right] = (n-1)!$, and we have for all $n \geq 0$,

$$\left[ {n+1 \atop 2} \right] = (n-1)! + n \cdot \left[ {n \atop 2} \right].$$

To find a recursion relation for $a_n$, we observe that

$$H_n = H_{n-1} + \frac{1}{n}$$

so

$$\frac{a_n}{n!} = H_{n-1} + \frac{1}{n}.$$

Then we have that

$$\begin{aligned} a_n &= n! \cdot H_{n-1} + \frac{n!}{n} \\ &= n! \cdot \frac{a_{n-1}}{(n-1)!} + \frac{n!}{n} \\ &= (n-1)! + n \cdot a_{n-1} \end{aligned}$$

And thus $a_n$ and $\left[ {n+1 \atop 2} \right]$ satisfy the same recursion relation. We conclude that $a_n = \left[ {n+1 \atop 2} \right]$, so, for $n \geq 0$,

$$H_n = \frac{\left[ {n+1 \atop 2} \right]}{n!}$$

□

The fact that the numerator of the $n$th harmonic number is equal to $\left[ {n+1 \atop 2} \right]$ is not surprising given that Stirling numbers are defined as coefficients in the expansion of the rising factorial function (Definition 5.2). In the process of finding a common denominator for $H_n$, we add $n$ terms, each of which is $n-1$ elements from the set $\{1, 2, 3, \ldots n\}$. For example, one of these $n$ terms is $1 \cdot 2 \cdot 3 \cdot 5 \cdot 6 \cdots n$. Since each term of the numerator of $H_n$ and each term of the coefficient of $x^2$ in the function $x(x+1)(x+2)\cdots(x+n)$ is a product of $n-1$ elements from $\{1, 2, 3, \ldots, n\}$, the numerator of $H_n$ is equal to $\left[{n+1 \atop 2}\right]$.

## 5.2 Hyperharmonic Numbers

There are several ways to generalize the harmonic numbers, but the generalization that we will focus on is the *hyperharmonic* numbers.

**Definition 5.6.** (Benjamin, Gaebler, & Gaebler, 2003). For $r, n \geq 1$, the hyperharmonic number of order $r$ is

$$H_n^r = \sum_{i=1}^{n} {H_i}^{r-1}$$

Note that for $r < 0$ or $n \leq 0$, we define $H_n^r = 0$ and for $r = 0$, $n \geq 1$, $H_n^0 = \frac{1}{n}$.

Table 3: The first few hyperharmonic numbers. Bolded cells indicate the cells that we sum to compute $H_3^4$.

| | n =1 | n =2 | n=3 | n=4 |
|---|---|---|---|---|
| $H_n^1$ | 1 | $1+\frac{1}{2}$ | $1+\frac{1}{2}+\frac{1}{3}$ | $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$ |
| $H_n^2$ | 1 | $2\cdot 1+\frac{1}{2}$ | $3\cdot 1+2\cdot\frac{1}{2}+\frac{1}{3}$ | $4\cdot 1+3\cdot\frac{1}{2}+2\cdot\frac{1}{3}+\frac{1}{4}$ |
| $H_n^3$ | **1** | $\mathbf{3\cdot 1+\frac{1}{2}}$ | $\mathbf{6\cdot 1+3\cdot\frac{1}{2}+\frac{1}{3}}$ | $10\cdot 1+6\cdot\frac{1}{2}+3\cdot\frac{1}{3}+\frac{1}{4}$ |
| $H_n^4$ | 1 | $4\cdot 1+\frac{1}{2}$ | $10\cdot 1+4\cdot\frac{1}{2}+\frac{1}{3}$ | $20\cdot 1+10\cdot\frac{1}{2}+4\cdot\frac{1}{3}+\frac{1}{4}$ |

In Table 3, we give the first few terms of the hyperharmonic numbers $H_n^1$, $H_n^2$, $H_n^3$, and $H_n^4$. This table is helpful in understanding how to compute an arbitrary hyperharmonic number. For example,

$$H_3^4 = \sum_{i=1}^{3} {H_i}^3 = H_1^3 + H_2^3 + H_3^3 = 1 + 3\cdot 1 + \frac{1}{2} + 6\cdot 1 + 3\cdot\frac{1}{2} + \frac{1}{3} = 10\cdot 1 + 4\cdot\frac{1}{2} + \frac{1}{3}$$

It turns out that the hyperharmonic numbers have a combinatorial interpretation similar to that of the harmonic numbers:

$$H_n^r = \frac{\left[{n+r \atop r+1}\right]_r}{n!}$$

where the numerator, as the notation suggests, is closely related to the Stirling numbers of the first kind.

**Definition 5.7.** (Benjamin et al., 2003). The $r$-Stirling number $\left[{n \atop k}\right]_r$ is the number of ways to permute $n$ elements into exactly $k$ cycles, where elements 1 through $r$ are in different cycles.

To clarify Definition 5.7, we look at a few simple cases for $r = 2$ and $k = 3$. We find that $\left[{3 \atop 3}\right]_2 = 1$ since there is only one way to permute 3 elements into 3 cycles. Next, we see that $\left[{4 \atop 3}\right]_2 = 5$ since there are 5 ways to permute 4 elements into 3 cycles when elements 1 and 2 must be in different cycles:

$$\begin{gathered}(14)(2)(3)\\(1)(24)(3)\\(1)(2)(34)\\(13)(2)(4)\\(1)(23)(4)\end{gathered}$$

We see that $\left[{5 \atop 3}\right]_2 = 26$, since there are 26 ways to permute 5 elements into 3 cycles when elements 1 and 2 must be in different cycles:

$$\begin{array}{ccccc}
(145)(2)(3) & (154)(2)(3) & (1)(245)(3) & (1)(254)(3) & (1)(2)(345)\\
(135)(2)(4) & (153)(2)(4) & (1)(235)(4) & (1)(2)(354) & (1)(253)(4)\\
(134)(2)(5) & (143)(2)(5) & (1)(234)(5) & (1)(243)(5) & (14)(25)(3)\\
(15)(24)(3) & (13)(25)(4) & (15)(23)(4) & (13)(24)(5) & (14)(23)(5)\\
(14)(2)(35) & (15)(2)(34) & (13)(2)(45) & (1)(24)(35) & (1)(25)(34)\\
(1)(23)(45) & & & &
\end{array}$$

Since $\left[{4 \atop 3}\right]_2 = 5$, $\left[{3 \atop 3}\right]_2 = 1$, and $\left[{3 \atop 2}\right]_2 = 2$, we observe that

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_2 + 3 \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix}_2.$$

Similarly, since $\left[{5 \atop 3}\right]_2 = 26$, $\left[{4 \atop 2}\right]_2 = 6$, and $\left[{4 \atop 3}\right]_2 = 5$ we see that

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_2 + 4 \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix}_2.$$

We can generalize these observations to hold for all $r$ and $n$.

**Theorem 5.8** (Benjamin et al., 2003)**.** *For $n \geq r \geq 1$,*

$$\begin{bmatrix} n+r \\ r+1 \end{bmatrix}_r = \begin{bmatrix} n+r-1 \\ r \end{bmatrix}_r + (n+r-1) \begin{bmatrix} n+r-1 \\ r+1 \end{bmatrix}_r$$

*Proof.* By definition, the number of permutations of $n+r$ elements into $r+1$ cycles where elements 1 through $r$ are in different cycles is $\left[{n+r \atop r+1}\right]_r$.

Another way to count the number of permutations of $n+r$ elements into $r+1$ cycles where elements 1 through $r$ are in different cycles is to condition on whether element $n+r$ is in its own cycle. If element $n+r$ is in its own cycle, there are $\left[{n+r-1 \atop r}\right]_r$ ways to arrange the remaining elements. If element $n+r$ is not in its own cycle, we arrange the first $n+r-1$ elements ($\left[{n+r-1 \atop r+1}\right]_r$ ways), and then place element $n+r$ to the right of any of the $n+r-1$ elements ($n+r-1$ ways). By the multiplication rule, there are $(n+r-1)\left[{n+r-1 \atop r+1}\right]_r$ ways to arrange the elements. Therefore, the number of permutations of $n+r$ elements into $r+1$ cycles where elements 1 through $r$ are in different cycles is $\left[{n+r-1 \atop r}\right]_r + (n+r-1)\left[{n+r-1 \atop r+1}\right]_r$. □

Comparing Theorems 5.4 and 5.8, we see that the Stirling and $r$-Stirling numbers follow the same recursion relation.

Now that we have an understanding of the $r$-Stirling numbers, we return to discussing the combinatorial interpretation of the hyperharmonic numbers. We will prove the combinatorial interpretation in general, but first we prove it for the simpler case of $r = 2$.

**Theorem 5.9** (Benjamin et al., 2003). *For $n \geq 2$,*

$$H_n^2 = \frac{\left[{n+2 \atop 3}\right]_2}{n!}$$

*Proof.* We will show that

$$\begin{bmatrix} n+2 \\ 3 \end{bmatrix}_2 = n! H_n^2$$

where recall that

$$H_n^2 = H_1 + H_2 + \cdots + H_n.$$

To count the number of permutations of $n+2$ elements into 3 cycles where elements 1 and 2 are in different cycles, we condition on the number of elements in the third cycle. Without loss of generality, we may assume that element 1 is in the first cycle and element 2 is in the second cycle. Suppose we have $k$ elements in the third cycle, for $1 \leq k \leq n$. There are $\binom{n}{k}$ ways to pick these $k$ elements from the set $\{3, 4, \ldots, n+2\}$. Then, there are $(k-1)!$ ways to arrange the $k$ elements.

Next, we must arrange the remaining $n-k$ elements. There are two ways to place the first of the $n-k$ elements: to the right of 1 or to the right of 2. Then there are three ways to place the second element, four ways to place the third element, ... , and $n-k+1$ ways to place the $(n-k)$th element. In total, there are $(n-k+1)!$ ways to arrange the $n-k$ elements.

Summing over all possible values of $k$, there are $\sum_{k=1}^{n} \binom{n}{k}(k-1)!(n-k+1)!$ ways to permute $n+2$ elements into 3 cycles where elements 1 and 2 are in different cycles.

We see that

$$\begin{aligned}
\sum_{k=1}^{n} \binom{n}{k}(k-1)!(n-k+1)! &= \sum_{k=1}^{n} \frac{n!(k-1)!(n-k+1)!}{k!(n-k)!} \\
&= \sum_{k=1}^{n} \frac{n!(n-k+1)}{k} \\
&= n! \sum_{k=1}^{n} \frac{(n-k+1)}{k}
\end{aligned}$$

Observe that

$$\sum_{k=1}^{n} \frac{(n-k+1)}{k} = n + \frac{n-1}{2} + \frac{n-2}{3} + \cdots + \frac{1}{n}$$

and thus $\sum_{k=1}^{n} \frac{(n-k+1)}{k}$ consists of $n$ 1's, $n-1$ $\frac{1}{2}$ 's, $n-2$ $\frac{1}{3}$ 's, ... , 2 $\frac{1}{n-1}$ 's, and 1 $\frac{1}{n}$. It follows that

$$\begin{aligned}
\sum_{k=1}^{n} \frac{(n-k+1)}{k} &= H_1 + H_2 + H_3 + \cdots + H_n \\
&= H_n^2
\end{aligned}$$

We have shown that

$$\sum_{k=1}^{n} \binom{n}{k}(k-1)!(n-k+1)! = n!H_n^2$$

so we conclude that

$$\begin{bmatrix} n+2 \\ 3 \end{bmatrix}_2 = n!H_n^2.$$

□

Next, we provide a combinatorial interpretation of the hyperharmonic numbers for any $r$. First, we need two lemmas.

**Lemma 5.10.** (Benjamin & Quinn, 2003).

$$\sum_{m=1}^{n} \binom{m-k+r-1}{r-1} = \binom{n-k+r}{r}$$

*Proof.* (Benjamin & Quinn, 2003). We will count the number of $r$-subsets in the set $\{1, 2, \ldots, n-k+r\}$ in two different ways.

**Method 1.** By definition, the number of $r$-subsets in the set $\{1, 2, \ldots, n-k+r\}$ is $\binom{n-k+r}{r}$.

**Method 2.** Condition on the largest number in the subset. A size $r$ subset with maximum element $m-k+r$ can be created $\binom{m-k+r-1}{r-1}$ ways since once we have the maximum number, we just need to choose $r-1$ more numbers from the remaining $m-k+r-1$ numbers. Summing over all possible values of $m$, we have that the number of $r$ subsets in the set $\{1, 2, \ldots, n-k+r\}$ is $\sum_{m=1}^{n} \binom{m-k+r-1}{r-1}$. $\square$

**Lemma 5.11.**

$$H_n^r = \sum_{k=1}^{n} \frac{1}{k} \cdot \binom{n-k+r-1}{r-1} \tag{21}$$

*Proof.* To prove equation (21), we will show that in $H_n^r$, the number $\frac{1}{k}$ appears $\binom{n-k+r-1}{r-1}$ times for $1 \leq k \leq n$.

We will use induction on $r$. When $r = 1$, $H_n^1 = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, so each $\frac{1}{k}$, $1 \leq k \leq n$, appears once. Since $\binom{n-k}{0} = 1$, equation (21) holds for $r = 1$.

Suppose that $\frac{1}{k}$ appears $\binom{n-k+r-1}{r-1}$ times in $H_n^r$ for some $r \geq 1$. Then consider

$$H_n^{r+1} = \sum_{i=1}^{n} H_i{}^r = H_1^r + H_2^r + H_3^r + \ldots H_n^r. \tag{22}$$

By the induction hypothesis, $\frac{1}{k}$ appears

$$\binom{1-k+r-1}{r-1} + \binom{2-k+r-1}{r-1} + \ldots + \binom{n-k+r-1}{r-1} = \sum_{m=1}^{n} \binom{m-k+r-1}{r-1}$$

times in equation (22). By Lemma 5.10, $\frac{1}{k}$ appears $\binom{n-k+r}{r}$ times. Thus by the Principle of Mathematical Induction, equation (21) holds for all positive integers

$r$. $\square$

**Theorem 5.12** (Benjamin et al., 2003)**.** *For $n \geq r \geq 0$,*

$$H_n^r = \frac{\left[{n+r \atop r+1}\right]_r}{n!}$$

*Proof.* We will show that

$$n!H_n^r = \left[{n+r \atop r+1}\right]_r$$

by counting the number of permutations of $n+r$ elements into exactly $r+1$ cycles, where elements 1 through $r$ are in different cycles, in two different ways.

**Method 1.** By definition, the number of permutations $n+r$ elements into exactly $r+1$ cycles where elements 1 through $r$ are in different cycles is $\left[{n+r \atop r+1}\right]_r$.

**Method 2.** We condition on the number of elements in cycle $r+1$. Suppose we have $k$ elements in cycle $r+1$, for some $1 \leq k \leq n$. There are $\binom{n}{k}$ ways to pick these $k$ elements from the set $\{r+1, r+2, \ldots, n+r\}$ and then there are $(k-1)!$ ways to arrange the elements in the $(r+1)$st cycle.

We next need to arrange the remaining $n-k$ elements into the first $r$ cycles. Without loss of generality, assume that element 1 is in cycle 1, element 2 is in cycle 2, …, element $r$ is in cycle $r$. Since we can place the first of the $n-k$ remaining elements to the right of any of these $r$ elements, there are $r$ ways to place the first of the $n-k$ elements. After we place this element, the same reasoning reveals that there are $r+1$ ways to place the second of the $n-k$ elements, and so on, so that there are $(n-k-1+r)$ ways to place the $(n-k)$th element. Then we have that the number of permutations of $n+r$ elements into exactly $r+1$ cycles is

$$\sum_{k=1}^{n} \binom{n}{k} (k-1)!(n-k+r-1) \cdot (n-k+r-2) \cdots (r+1) \cdot r \qquad (23)$$

Simplifying equation (23), we have

$$\begin{aligned}
\sum_{k=1}^{n}\binom{n}{k}(k-1)!(n-k+r-1)\cdots r &= \sum_{k=1}^{n}\frac{n!(k-1)!}{k!(n-k)!}(n-k+r-1)\cdots r \\
&= n!\sum_{k=1}^{n}\frac{1}{k}\cdot\frac{(n-k+r-1)\cdots r}{(n-k)!} \\
&= n!\sum_{k=1}^{n}\frac{1}{k}\cdot\frac{(n-k+r-1)!}{(n-k)!(r-1)!} \\
&= n!\sum_{k=1}^{n}\frac{1}{k}\cdot\binom{n-k+r-1}{r-1}
\end{aligned}$$

By Lemma 5.11, $n!\sum_{k=1}^{n}\frac{1}{k}\cdot\binom{n-k+r-1}{r-1}=n!H_n^r$. That is, the number of permutations of $n+r$ elements into exactly $r+1$ cycles where elements 1 through $r$ are in different cycles is $n!H_n^r$.

We have shown that $n!H_n^r = \left[{n+r \atop r+1}\right]_r$. □

We conclude this section with an example of how we can use Theorem 5.12 to prove an identity involving the hyperharmonic numbers.

**Example 5.13.** Benjamin et al. (2003) present the identity

$$nH_n^r = \binom{n+r-1}{r} + rH_{n-1}^{r+1} \tag{24}$$

To prove this identity using Theorem 5.12, we first write equation (24) so that the left side of the equation is an $r$-Stirling number. We see equation (24) is equivalent to

$$\frac{n\cdot\left[{n+r \atop r+1}\right]_r}{n!} = \binom{n+r-1}{r} + \frac{r\cdot\left[{n+r \atop r+2}\right]_{r+1}}{(n-1)!} \tag{25}$$

Multiplying both sides of equation (25) by $(n-1)!$, we have

$$\left[{n+r \atop r+1}\right]_r = (n-1)!\binom{n+r-1}{r} + r\cdot\left[{n+r \atop r+2}\right]_{r+1}.$$

Finally, observing that

$$(n-1)!\binom{n+r-1}{r} = \frac{(n-1)!(n+r-1)!}{r!(n-1)!} = \frac{(n+r-1)!}{r!}$$

we have

$$\begin{bmatrix} n+r \\ r+1 \end{bmatrix}_r = \frac{(n+r-1)!}{r!} + r\begin{bmatrix} n+r \\ r+2 \end{bmatrix}_{r+1}$$

*Proof.* We will count the number of permutations of $n+r$ elements into exactly $r+1$ cycles, where elements 1 through $r$ are in different cycles.

**Method 1.** By definition, the number of permutations of $n+r$ elements into exactly $r+1$ cycles, where elements 1 through $r$ are in different cycles, is $\left[\begin{smallmatrix} n+r \\ r+1 \end{smallmatrix}\right]_r$.

**Method 2.** Without loss of generality, assume element 1 is in cycle 1, element 2 is in cycle 2, …, element $r$ is in cycle $r$. Condition on whether $r+1$ is in the $(r+1)$st cycle.

If $r+1$ is in the $(r+1)$st cycle, then place the remaining $n-1$ elements in the $r+1$ cycles. There are $r+1$ ways to place the first of the $n-1$ elements since it may go to the right of any of the first $r+1$ elements. After we have placed this element, there are $r+2$ ways to place the second of the $n-1$ elements, $r+3$ ways to place the third of the $n-1$ elements, and so on, so that there are $n+r-1$ ways to place the last element. In total, there are

$$(n+r-1)\cdot(n+r-2)\cdots(r+2)\cdot(r+1) = \frac{(n+r-1)!}{r!}$$

ways to permute $n+r$ elements into $r+1$ cycles where elements 1 through $r$ are in distinct cycles, if $r+1$ is in the $(r+1)$st cycle.

If $r+1$ is not in the $(r+1)$st cycle, we arrange the $n+r$ elements into $r+2$ cycles, where elements 1 through $r+1$ must be in different cycles. Then, we choose one of the first $r$ cycles, and add the cycle that starts with $r+1$ to the end of this cycle.

Now we have a permutation of $n+r$ elements into $r+1$ cycles, where the elements 1 through $r$ are in different cycles, but element $r+1$ is not in the $(r+1)$st cycle.

We conclude that there are $\frac{(n+r-1)!}{r!} + r\left[{n+r \atop r+2}\right]_{r+1}$ ways to permute $n+r$ elements into $r+1$ cycles where elements 1 through $r$ are in distinct cycles, so

$$\left[{n+r \atop r+1}\right]_r = \frac{(n+r-1)!}{r!} + r\left[{n+r \atop r+2}\right]_{r+1}$$

$\square$

It is quite remarkable that the proof of a seemingly complicated identity involving hyperharmonic numbers reduces to counting permutations. A reader interested in seeing more examples of proofs of hyperharmonic number identites using Theorem 5.12 should consult [4].

# 6 Conclusion

In this paper, we presented combinatorial interpretations for the Fibonacci numbers, the Lucas numbers, continued fractions, harmonic numbers, and hyperharmonic numbers. We dedicated the most time to continued fractions, going beyond the combinatorial intepretation of finite continued fractions. We explored how we can combinatorially interpret infinite continued fractions and ultimately presented a combinatorial interpretation of the continued fraction expansion of $e$. It would be interesting to see if we can interpret other infinite continued fraction expansions, such as the expansion we derived for $\log 2$, combinatorially.

In Benjamin and Quinn's *Proofs that Really Count*, Benjamin and Quinn also present proofs by direct counting of identities involving linear recurrences, binomial coefficients, and number theory. Given more time, it would have been interesting to look at examples of proofs by direct counting of famous results from number theory, such as Fermat's Little Theorem and Wilson's Theorem. Additionally, Benjamin and

Quinn present quite a few open problems involving the Fibonacci numbers. With more time, we would have liked to look at these problems and further explore Zeckendorf's Theorem (Section 2.5). The curious reader is encouraged to consult *Proofs that Really Count*: there are many identities just waiting to be counted!

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## References

[1] B. Balof, E. Farmer, J. Kawabata, The Link Between Scrambling Numbers and Derangements (1997).

[2] C. Baltus, Euler: Continued Fractions and Divergent Series (and Nicholas Bernoulli) (2008).

[3] A. T. Benjamin and J. J. Quinn, *Proofs That Really Count*, The Mathematical Association of America Inc., 2003.

[4] A.T. Benjamin, D. Gaebler, and R. Gaebler, A Combinatorial Approach to Hyperharmonic Numbers, *Integers: The Electronic Journal of Combinatorial Theory* **3** (2003) 1-9.

[5] R. A. Brualdi, *Introductory Combinatorics*, Pearson Prentice Hall, 2004.

[6] J. H. Conway and R. K. Guy, *The Book of Numbers*, Springer-Verlag Inc., New York, 1996.

[7] H. Cohn, A Short Proof of the Simple Continued Fraction Expansion of $e$, *Mathematical Association of America* **113** (2006) 57-62.

[8] L. Debnath, A short history of the Fibonacci and golden numbers with their applications, *International Journal of Mathematical Education in Science and Technology* **42.3** (2011) 337-367.

[9] A. Ya. Khinchin, *Continued Fractions*, University of Chicago Press, 1964.

[10] P. Loya, *Amazing and Aesthetic Aspects of Analysis: On the incredible infinite*, 2006.

[11] E. Sandifer, How Euler Did It, *Mathematical Association of America Online* (2006).