PÓLYA'S COUNTING THEORY

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1 Introduction

In combinatorics, there are very few formulas that apply comprehensively to all cases of a given problem. Pólya's Counting Theory is a spectacular tool that allows us to count the number of distinct items given a certain number of colors or other characteristics. Basic questions we might ask are, "How many distinct squares can be made with blue or yellow vertices?" or "How many necklaces with n beads can we create with clear and solid beads?" We will count two objects as 'the same' if they can be rotated or flipped to produce the same configuration. While these questions may seem uncomplicated, there is a lot of mathematical machinery behind them. Thus, in addition to counting all possible positions for each weight, we must be sure to not recount the configuration again if it is actually the same as another.

We can use Burnside's Lemma to enumerate the number of distinct objects. However, sometimes we will also want to know more information about the characteristics of these distinct objects. Pólya's Counting Theory is uniquely useful because it will act as a picture function - actually producing a polynomial that demonstrates what the different configurations are, and how many of each exist. As such, it has numerous applications. Some that will be explored here include chemical isomer enumeration, graph theory and music theory.

This paper will first work through proving and understanding Pólya's theory, and then move towards surveying applications. Throughout the paper we will work heavily with examples to illuminate the simplicity of the theorem beyond its notation.

2 Burnside's Lemma

2.1 Group Theory

We will first clarify some basic notation. Let S be a finite set. Then $|S|$ denotes the number of its elements. If G is a group, then $|G|$ represents the number of elements in G and is called the *order* of the group. Finally, if we have a group of permutations of a set S, then $|G|$ is the *degree* of the permutation group.

Gallian [3] provides the following definitions, necessary for Theorem 1 and Theorem 2.

Definition 1. Coset of H in G *Let G be a group and H a subset of G. For any $a \in G$, the set $\{ah|h \in H\}$ is denoted by aH. Analogously, $Ha = \{ha|h \in H\}$ and $aHa^{-1} = \{aha^{-1}|h \in H\}$. When H is a subgroup of G, the set aH is called the left coset of H in G containing a, whereas Ha is called the right coset of H in G containing a.*

Definition 2. Stabilizer of a Point *Let G be a group of permutations of a set S. For each $i \in S$, let $stab_G(i) = \{\phi \in G|\phi(i) = i\}$. We call $stab_G(i)$ the stabilizer of i in G.*

The set $stab_G(i)$ is a subset of G and is also a subgroup of G. We know it is nonempty because the identity element will certainly fix $i \in S$. If ϕ_a, $\phi_b \in stab_G(i)$, then $\phi_a(i) = i$ and $\phi_b(i) = i$. Thus $\phi_a\phi_b^{-1}(i) = \phi_a(\phi_b^{-1}(i)) = \phi_a(i) = i$, which confirms that $\phi_a\phi_b^{-1} \in stab_G(i)$.

Definition 3. Orbit of a Point *Let G be a group of permutations of a set S. For each $s \in S$, let $orb_G(s) = \{\phi(s)|\phi \in G\}$. The set $orb_G(s)$ is a subset of S called the orbit of s under G.*

All the objects in a given orbit will be considered the same, or isomorphic. For counting purposes, we will be counting the number of orbits.

Tucker [10] offers the following definition for the *elements fixed by ϕ.*

Definition 4. Elements Fixed by ϕ *For any group G of permutations on a set S and any ϕ in G, we let $fix(\phi) = \{i \in S|\phi(i) = i\}$. This set is called the elements fixed by ϕ.*

2.2 The Orbit-Stabilizer Theorem

Gallian [3] also proves the following two theorems.

Theorem 1. Lagrange's Theorem *If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$. Moreover, the number of distinct left cosets of H in G is $\frac{|G|}{|H|}$.*

Proof. Let a_1H,a_2H,..., a_rH denote the distinct left cosets of H in G. For each a in G, we have $aH = a_iH$ for some i and $a \in aH$. Thus, each member of G belongs to one of the cosets a_iH. In symbols,

$$G = a_iH \cup \cdots \cup a_rH.$$

Since these cosets are disjoint,

$$|G| = |a_1H| + |a_2H| + \cdots + |a_rH|.$$

Because $|a_iH| = |H|$ for each i, we have $|G| = r|H|$.

□

Here are some basic examples of this theorem.

Example 1. Consider the group $\mathbb{Z}_{15}$ and its subgroup $H = \{0, 3, 6, 9, 12\}$. Then $|H| = 5$, which divides 15, and there are $\frac{15}{5} = 3$ distinct left cosets. They are H, $1 + H = \{1, 4, 7, 10, 13\}$, and $2 + H = \{2, 5, 8, 10, 14\}$.

Example 2. Consider the dihedral group, D_4. The dihedral group D_4 consists of the elements R_0, R_{90}, R_{180}, R_{270}, H, V, D, and D', where the R_i's are rotations by i degrees. H is a flip with respect to the horizontal line, V is a flip with respect to the vertical line, D is a flip with respect to the diagonal through the lower left corner and the upper right corner, and D' is a flip with respect to the other diagonal. Then consider the subgroup $S = \{R_0, H\}$. As expected, $|S| = 2$ divides $|D_4| = 8$ and the subgroup has four distinct left cosets, R_0H, $R_{90}H$,$R_{180}H$, and $R_{270}H$.

Theorem 2. Orbit-Stabilizer Theorem *Let G be a finite group of permutations of a set S. Then, for any i from S,*

$$|G| = |orb_G(i)||stab_G(i)|.$$

Proof. We know that $\frac{|G|}{|stab_G(i)|}$ is the number of distinct left cosets of $stab_G(i)$ in G by Lagrange's Theorem. Define a correspondence $T : G \times S \to S$ by $T((\phi, i)) = \phi(i)$. If there is one-to-one correspondence between the left cosets of $stab_G(i)$ and the elements in the orbit of i, we are done. To show that T is well-defined, we must show that $\alpha stab_G(i) = \beta stab_G(i)$ implies $\alpha(i) = \beta(i)$. But $\alpha stab_G(i) = \beta stab_G(i)$ implies $\alpha^{-1}\beta \in stab_G(i)$, so that $(\alpha^{-1}\beta)(i) = i$ and, therefore, $\beta(i) = \alpha(i)$. Because these arguments can be reversed from the last step to the first step, T is one-to-one. To show T is onto $orb_G(i)$, let $j \in orb_G(i)$. Then $\alpha(i) = j$ for some $\alpha \in G$ and $T(\alpha stab_G(i)) = \alpha(i) = j$. Thus T is onto.

□

2.3 Burnside's Lemma

Tucker [10] provides the following theorem.

Theorem 3. Burnside's Theorem *If G is a finite group of permutations on a set S, then the number of orbits of G on S is*

$$\frac{1}{|G|}\sum_{\phi \in G} |fix(\phi)|.$$

Burnside's Lemma can be described as finding the number of distinct orbits by taking the average size of the fixed sets. Gallian [3] provides the proof.

Proof. Let n denote the number of pairs (ϕ, i), with $\phi \in G$, $i \in S$, and $\phi(i) = i$. We begin by counting these pairs in two ways. First, for each particular ϕ in G, the number of such pairs is exactly $|fix(\phi)|$, as i runs over S. So,

$$n = \sum_{\phi \in G} |fix(\phi)|. \tag{1}$$

Second, for each particular i in S, observe that $|stab_G(i)|$ is exactly the number of pairs of the form (ϕ, i), as ϕ runs over G. So,

$$n = \sum_{i \in S} |stab_G(i)|. \tag{2}$$

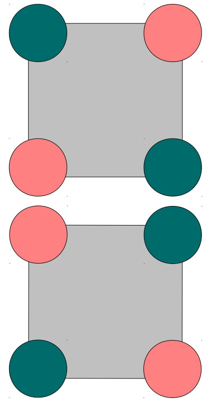

Figure 1: Fixed by R_{180}

We know that if s and t are in the same orbit of G, then $orb_G(s) = orb_G(t)$ and $|stab_G(s)| = |stab_G(t)|$. So if we choose $t \in S$, sum over $orb_G(s)$, and appeal to Theorem 2, we have

$$\sum_{t \in orb_G(s)} |stab_G(t)| = |orb_G(s)||stab_G(s)| = |G|. \tag{3}$$

Finally, by summing over all the elements of G, one orbit at a time, it follows from Equations 1, 2, and 3 that

$$\sum_{\phi \in G} |fix(\phi)| = \sum_{i \in S} |stab_G(i)| = |G| * (number\ of\ orbits)$$

and then we are done.

□

2.4 Enumerating Squares

Now that we have established sufficient theory, we will show how this can be applied to a problem involving distinct squares. Two squares will be considered distinct if they cannot be made identical by rotating and/or flipping one of the squares.

Example 3. How many distinct squares are there with turquoise and salmon beads at the corners? Let G be the dihedral group, D_4, consisting of the elements defined in Example 2, and let S be the set of not necessarily distinct squares that can be created with turquoise and salmon beads. Since there are 4 corners and 2 colors, there are 2^4 elements in S. Now we must find how many square configurations each element of D_4 fixes.

The identity, R_0, fixes all 16 configurations. Both R_{90} and R_{270} only fix two elements, namely the square with all turquoise dots and the square with all salmon dots. Note that every element in D_4 fixes these two monochrome squares. In addition to these two, R_{180} also fixes the squares shown in Figure 1. H also fixes the squares in Figure 2 and V fixes those in Figure 3. The six additional squares fixed by D and D' are shown in Figure 4 and Figure 5.

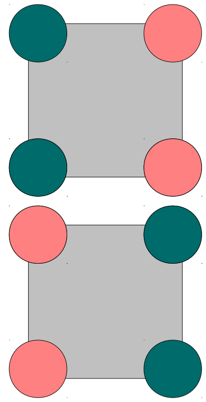

Figure 2: Fixed by H

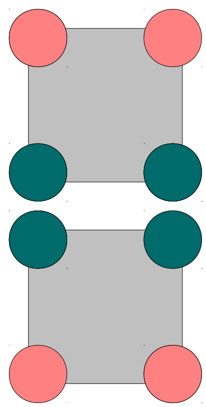

Figure 3: Fixed by V

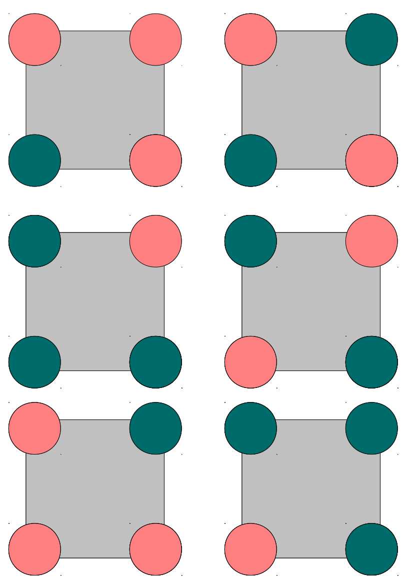

Figure 4: Fixed by D

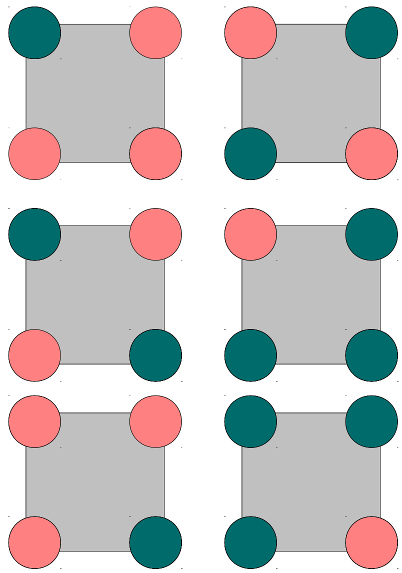

Figure 5: Fixed by D'

$g \in D_4$	$\|fix(g)\|$
R_0	16
R_{90}	2
R_{180}	4
R_{270}	2
H	4
V	4
D	8
D'	8

Figure 6: Table of $|fix(g)|$ values

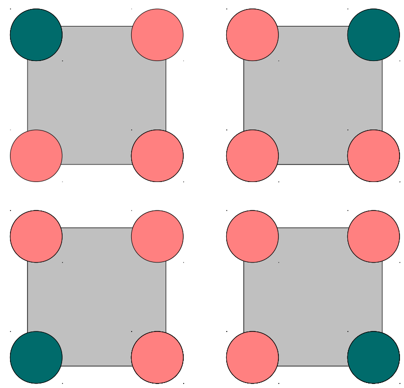

Figure 7: Orbit of s

Figure 6 shows the order of $|fix(g)|$ for all elements of D_4.

Now we can finally use Theorem 3 to count the number of orbits of D_4 on S.

$$\frac{1}{|G|}\sum_{g\in G}|fix(g)| = \frac{1}{8}(16+2+4+2+4+4+8+8) = 6.$$

We conclude that it is possible to create 6 distinct squares with turquoise and salmon beads at the corners. These 6 orbits can be seen in Figure 8, where each row shows a different orbit, except the first row which shows the two monochrome figures which are each alone in their own orbits.

Moreover, Theorem 2 is easily viewed in this context. Consider the upper left square in Figure 7 and call this configuration s. Using elements of D_4, s can be mapped to itself or the other three squares in the same figure. It is fixed by two elements, D' and R_0. Thus we have that $|orb_{D_4}(s)||stab_{D_4}(s)| = 4*2 = 8$. Unsurprisingly, $|D_4| = 8$.

2.5 Cube Vertices and Faces

Example 4. Let S be the set of vertices of a cube colored black or white, and let G be the set of permutations of S that can be produced by rotating the cube. Since there are 8 vertices, $|S| = 2^8$. There are 24 elements in $|G|$, which we can split into five categories:

1. The identity

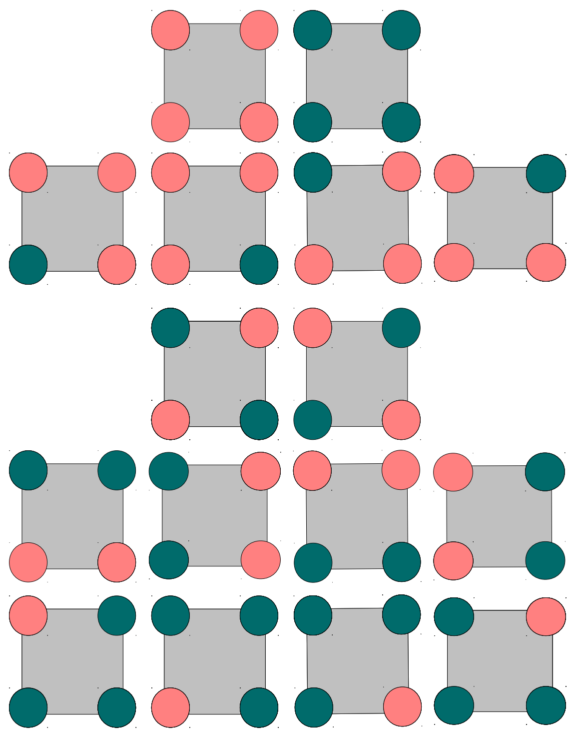

Figure 8: All possible square configurations arranged into equivalent rows

2. Three 180 degree rotations about lines through the centers of opposite faces
3. Six 180 degree rotations about lines through the centers of opposite edges
4. Three 90 degree rotations and three 270 degree rotations about the lines through the centers of opposite faces
5. Four 120 degree rotations and four 240 degree rotations about lines connecting opposite vertices.

The identity fixes all 2^8 elements of S.

The three 180 degree rotations about lines through the centers of opposite faces fix configurations of a cube where, if the cube were sliced in half, the vertices on one side of the cube must match or mirror the vertices on the other side. For example, if the cube in Figure 9 is rotated about the centers of face $abcd$ and face $efgh$, then c would be colored the same as a, d the same as b, h the same as f, and e the same as g. Then we have 2^4 choices, since the second four vertices are dependent on the first four.

The six 180 degree rotations about lines through the centers of opposite edges fix 2^4 configurations. If the cube was rotated about a line connecting eh and ab, then e and h, a and b, c and f, and d and f would have the same color. Again, there are four vertices that are independent, but the other four are dependent on them.

The 90 degree rotations fix cubes that have one side of identical vertices and another side of identical vertices. For example, using labels from Figure 9, if we rotated about the centers of face $abcd$ and face $efgh$, then a, b, c, and d would need to be colored black or white, and e,f,g, and h would all need to be colored either black or white. This gives 2^2 choices.

Finally, the 120 degree rotations and 240 rotations each give 2^4 possible choices. For example, if we rotate the cube in Figure 9 about the line through c and f, then the coloring of both c and f is arbitrary. The vertices b, d and h must be colored identically, and vertices a, g, and e must be colored identically.

Then, using Burnside's Lemma, we have

$$(2^8 + 3 * 2^4 + 6 * 2^4 + 6 * 2^2 + 8 * 2^4)/24 = 23.$$

Thus there are 23 distinct cubes that can be created using either black or white vertices.

For the cube, we can also count face colorations.

Example 5. Let S be the set of faces of a cube colored black or white, and let G be the set of permutations of S that can be produced by rotating the cube. Since there are 6 faces, $|S| = 2^6$. There are 24 elements in G, which we can split into the same five categories as in Example 4.

The identity fixes all 2^6 elements of S.

The second type of rotation fixes 2^3 elements of S. If we rotate about a line through the center of

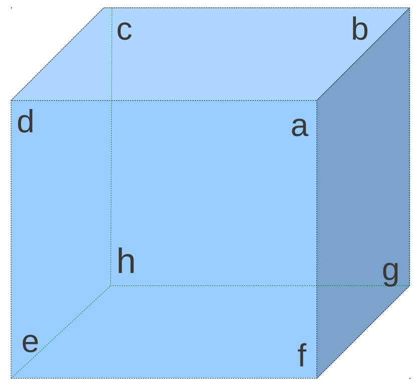

Figure 9: The cube

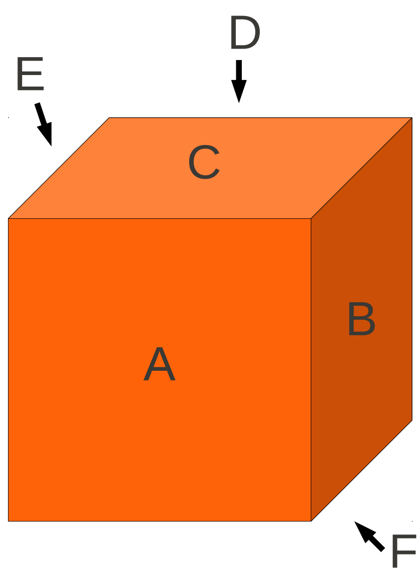

Figure 10: The cube

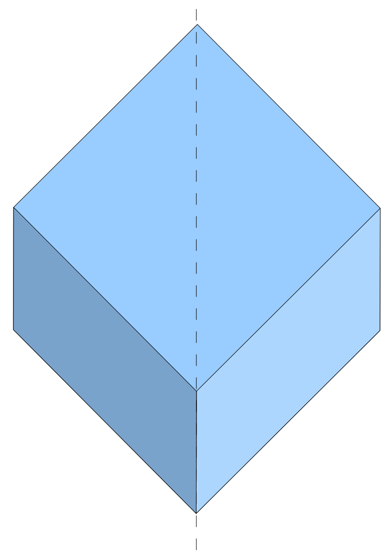

Figure 11: The cube for the fourth rotation

face A and face D of Figure 10, then A and D can be either color but B, C, E, and F must be the same.

The third type of rotation fixes 2^3 elements. If we rotate about a line through the center of the edge connecting A and B and the center of the edge connecting E and D, then the pairs C and F, A and B, and E and D must match.

The fourth type of rotation fixes 2^4 elements. If we rotate about a line through the center of face A and face D of Figure 10, then A and D can be either color but B and E must match, as must C and F.

The fifth type of rotation fixes 2^2 elements. If we rotate the cube around a line connecting two opposite vertices as in Figure 11, the "top" three faces must all be colored the same, as must the "bottom" three.

Then, using Burnside's Lemma, we have

$$(2^6 + 6 * 2^3 + 3 * 2^4 + 8 * 2^2 + 6 * 2^3)/24 = 10.$$

Thus there are 10 distinct cubes that can be created using either black or white faces.

Since our numbers are still small, we can figure out what these 10 cubes look like without the help of Pólya's Theorem. There are 2 monochrome cubes and 2 with one face colored differently (either all black with one white or all white with one black). Then there are 2 with two adjacent same-colored faces and 2 with two same-colored faces on opposite sides. Finally, there is 1 with three co-corner faces (only one since it will include both the three co-corner black faces and three co-corner white faces) and 1 with three non co-corner faces.

$g \in D_4$	$\lvert fix(g)\rvert$
R_0	3^4
R_{90}	3
R_{180}	3^2
R_{270}	3
H	3^2
V	3^2
D	3^3
D'	3^3

Figure 12: Table of $|fix(g)|$ values

2.6 More Colors

We will now extend our square and cube examples to three colors. This is exceedingly easy since our previous examples already have set up the entire problem. We can extend them to the case with n colors by simply replacing the three with n.

Example 6. How many distinct squares are there with turquoise, salmon, and brown beads at the corners?
Let G be the dihedral group, $D_4 = \{R_0, R_{90}, R_{180}, R_{270}, H, V, D, D'\}$, and let S be the set of not necessarily distinct squares that can be created with turquoise, salmon, and brown beads. Since there are 4 corners and 3 colors, there are 3^4 elements in S. Now we must find how many square configurations each element of D_4 fixes. We can simply use the basic logic followed in Example 3, but since there are 3 color choices there will be 3^i instead of 2^i fixed elements.

Figure 12 shows the order of $|fix(g)|$ for all elements g in D_4.

Now we can use Burnside's Lemma to count the number of orbits of D_4 on S.

$$\frac{1}{|G|}\sum_{g\in G}|fix(g)| = \frac{1}{8}(81+3+9+3+9+9+27+27) = 21.$$

We conclude that it is possible to create 21 distinct squares with turquoise and salmon beads at the corners.

We can use similar logic to extend Example 13 to the case with three colors.

Example 7. Let S be the set of faces of a cube colored black, white, or brown, and let G be the set of permutations of S that can be produced by rotating the cube. Since there are 6 faces, $|S| = 3^6$. There are 24 elements in G, which we can again split into five categories.

We will use the exact same logic as in Example 13 to use Burnside's Lemma, but we will now raise 3 to the various powers, since we have 3 instead of 2 colors. It follows that

$$(3^6 + 6*3^3 + 3*3^4 + 8*3^2 + 6*3^3)/24 = 57.$$

Thus there are 57 distinct cubes that can be created using either black, white, or brown faces.

Example 8. Finally, we will extend the example to find the number of distinct cubes that can be created using either black, white, or brown vertices. Without revisiting the details of the rotations again, we will apply Burnside's Lemma to find

$$(3^8 + 3 * 3^4 + 6 * 3^4 + 6 * 3^2 + 8 * 3^4)/24 = 333.$$

Thus there are 333 distinct cubes that can be created with three different colors at the vertices.

3 Pólya's Counting Theory

3.1 The cycle index

De Buijn [2] introduces this notation. Note that if S is a finite set, a *permutation* of S is a one-to-one mapping of S onto itself. If a permutation π is given, then we can split S into *cycles*, which are subsets of S that are cyclically permuted by π. If l is the length of a cycle, and s is any element of that cycle, then the cycle consists of

$$s, \pi s, \pi^2 s, \ldots, \pi^{l-1} s,$$

where $\pi^2 = \pi\pi$, $\pi^3 = \pi\pi\pi$, etc.

Definition 5. *Let S be a finite set. If S splits into b_1 cycles of length 1, b_2 cycles of length 2, etc., then π is of type $\{b_1, b_2, b_3, \ldots\}$.*

Definition 6. *Let G be a group whose elements are the permutations of S, where $|S| = m$. We define the polynomial in m variables $x_1, x_2, \ldots, x_m$, with nonnegative coefficients, where for each $\phi \in G$ we form the product $x_1^{b_1} x_2^{b_2} \cdots x_m^{b_m}$, if $\{b_1, b_2, b_3, \ldots\}$ is the type of ϕ. Then the polynomial*

$$P_G(x_1, x_2, \ldots, x_m) = \frac{1}{|G|} \sum_{\phi \in G} x_1^{b_1} x_2^{b_2} \cdots x_m^{b_m}$$

is called the cycle index of G.

This formula closely resembles Burnside's Lemma. The key difference is that now we differentiate between cycles of different lengths, and specify how many of each cycle there are. Later, this will allow us to not only count the number of different objects we seek, but also have an idea of what the appearance of each different object is like.

Example 9. Consider the simple example when G consists of only the identity permutation. Then the identity permutation is of type $\{m, 0, 0, \ldots\}$ and thus $P_G = x_1^m$.

Example 10. Suppose we have a necklace that is made from three beads and four colors. For this example, we can rotate the necklace but we cannot flip it. Then $G = \{R_0, R_{120}, R_{270}\}$ where the three elements of G represent the three possible rotations of the necklace. These three permutations are of type $\{3, 0, 0, \ldots\}$, $\{0, 0, 1, 0, \ldots\}$, and $\{0, 0, 1, 0, \ldots\}$, respectively. The cycle index is then

$$P_G = \frac{1}{3}(x_1^3 + 2x_3).$$

It follows that the number of 4-colored strings of three beads is $P_G(4,4,4) = \frac{1}{3}(4^3 + 2 * 4) = 24$. To generalize, the number of n-colored strings of three beads is $P_G(n,n,n) = \frac{1}{3}(n^3 + 2 * n)$.

Example 11. Returning to the example of distinct cube face colorations, the cycle index will allow us to differentiate between permutations of different lengths. Recall that we can split the 24 elements in G into the following five categories:

1. The identity
2. Three 90 degree rotations and three 270 degree rotations about the lines through the centers of opposite faces
3. Three 180 degree rotations about lines through the centers of opposite faces
4. Eight 120 degree rotations about lines connecting opposite vertices
5. Six 180 degree rotations about lines through the centers of opposite edges

The five categories produce the types $\{6,0,0,\dots\}$, $\{2,0,0,1,0,\dots\}$,$\{2,2,0,\dots\}$,$\{0,0,2,0,\dots\}$, and $\{0,3,0,0,\dots\}$ respectively. This corresponds to the cycle index

$$P_G = \frac{1}{24}(x_1^6 + 6x_1^2x_4 + 3x_1^2x_2^2 + 8x_3^2 + 6x_2^3).$$

If we find $P_G(2,2,2,2)$, the function will yield the same solution as found in Example 13. And if we find $P_G(n,n,n,n)$ to represent n different colors, we can find the number of distinct cubes that can be created using n colors.

Known Cycle Indices

There are actually formulas for some of the better known cycle indices. The ones that will be of use in this paper are the indices for symmetric groups, cyclic groups, and dihedral groups.

But first, Gallian [3] defines what a symmetric group actually is.

Definition 7. *Let $A = \{1,2,\dots,n\}$. The set of all permutations of A is called the symmetric group of degree n and is denoted by S_n.*

The order of the group is $n!$ since there are n choices for mapping the first element, $n-1$ choices for the mapping of the second, and so on. This group will be particularly useful for our discussion of graph applications.

Symmetric groups quickly get large for large values of n, but here are the smallest four:

$$S_0 = S_1 = \{e\}$$

$$S_2 = \{e, (12)\}$$

$$S_3 = \{e, (12), (13), (23), (123), (132)\}$$

.

Theorem 4. The cycle index of the symmetric group S_n is given by

$$Z(S_n) = \sum_{(j)} \frac{1}{\prod_k k^{j_k} j_k!} \prod_k s_k^{j_k},$$

where the summation is taken over all partitions (j) of n.

Proof. Consider some partition (j) of n where $(j) = (j_1, j_2, \ldots, j_n)$. Assume that the cycles of some permutation having cycles given by (j) are ordered in $n!$ different ways. However, for each k, the j_k cycles can be ordered in $j_k!$ different ways, and can begin in k different elements. Thus, any permutation is represented $\prod_k k^{j_k} j_k!$ times, so that there are a total of $\frac{n!}{\prod_k k^{j_k} j_k!}$ permutations with cycle structure given by (j). This allows us to reindex over (j) rather than the individual permutations, obtaining the cycle structure as stated in the theorem. □

Before we venture to the formula for cyclic groups, C_n and dihedral groups, D_n, we need the following definition given by Weisstein [12]:

Definition 8. The totient function $\phi(n)$, also called Euler's totient function or the Euler-phi function, is defined as the number of positive integers $\leq n$ that are relatively prime to n, where 1 is counted as being relatively prime to all numbers.

For example, there are eight numbers relatively prime to 24 that are less than 24 $(1, 5, 7, 11, 13, 17, 19,$ and $23)$, so $\phi(24) = 8$.

Harary and Palmer [4] give the following formulas:

$$Z(C_n) = \frac{1}{n} \sum_{k|n} \phi(k) s_k^{n/k},$$

and

$$Z(D_n) = \frac{1}{2} Z(C_n) + \begin{cases} \frac{1}{2} s_1 s_2^{(n-1)/2} & \text{if } n \text{ odd}, \\ \frac{1}{4}(s_2^{n/2} + s_1^2 s_2^{(n-2)/2}) & \text{if } n \text{ even}. \end{cases} \tag{4}$$

We can easily understand why $Z(D_n)$ is split into two cases by imagining a necklace of four beads versus a necklace of five beads. With four beads, there are two types of flips: those over the line through the centers of opposite edges, and those over the line connecting opposite vertices. However, with five beads, there is only one type of flip: those over lines connecting a vertex with the center of an opposite edge. This can be extended to any even or odd number of beads, respectively.

3.2 Pólya's Counting Theory (simple version)

Tucker [10] provides the following theorem, which is central to this entire paper. It will allow us to not only count the number of distinct orbits, but to also list out and view what all the objects look like.

Theorem 5. ***Pólya's Enumeration Formula*** *Let S be a set of elements and G be a group of permutations of S that acts to induce an equivalence relation on the colorings of S. The inventory of nonequivalent colorings of S using colors $c_1, c_2, \ldots, c_m$ is given by the generating function*

$$P_G\left(\sum_{j=1}^{m} c_j, \sum_{j=1}^{m} c_j^2, \ldots, \sum_{j=1}^{m} c_j^k\right), \tag{5}$$

where k corresponds to the largest cycle length.

So the inventory of colorings of S using three colors, A, B, and C, is given by

$$P_G\Big((A+B+C), (A^2+B^2+C^2), \ldots, (A^k+B^k+C^k)\Big).$$

3.3 Pólya's Counting Theory

De Buijn [2] provides the following slightly more formalized version of the theorem. Instead of just colors, De Buijn will use *weights*, which can represent, for example, a number, variable, or an element of a commutative ring containing the rational numbers. Before we define weights, we must clarify a few other terms.

Let D and R be finite sets, so we consider mappings of D into R. The set of all such functions is denoted by R^D. Suppose that we are given a group G of permutations of D. This group then introduces an equivalence relation in R^D: Two functions f_1,$f_2 \in R^D$ are called *equivalent* (denoted $f_1 \sim f_2$) if there exists an element $g \in G$ such that

$$f_1(g(d)) = f_2(d)$$

for all $d \in D$. We will verify that this is an equivalence relation.

1. We know $f \sim f$ since G contains the identity permutation.
2. If $f_1 \sim f_2$, then $f_2 \sim f_1$ because if $g \in G$, then $g^{-1} \in G$.
3. Finally, if $f_1 \sim f_2$ and $f_2 \sim f_3$, then $f_1 \sim f_3$. This is because if $g_1 \in G$, $g_2 \in G$, then the composite mapping $g_1 g_2$ is in G.

So $\sim$ is an equivalence relation, and the set R^D splits into equivalence classes that will be called *patterns*. In Example 13 with the sepia or teal face colorations of the cube, the ten distinct colorations were the ten patterns.

Now we have the notation to define weights. De Buijn [2] gives the following definition:

Definition 9. *Let D and R be finite sets, and G be a permutation group of D. We will assign a weight to each element $r \in R$, call it $w(r)$. The weight $W(f)$ of a function $f \in R^D$ is the product*

$$W(f) = \prod_{d \in D} w[f(d)]. \tag{6}$$

Note that if $f_1 \sim f_2$, that is, if they belong to the same pattern, then they have the same weight. Therefore, we may define the weight of the pattern as a common value. Thus if F denotes a pattern, we will denote the weight of F by $W(F)$.

Theorem 6. *Pólya's Fundamental Theorem*
Let D and R be finite sets and G be a permutation group of D. The elements of R have weights $w(r)$. The functions $f \in R^D$ and the patterns F have weights $W(f)$ and $W(F)$, respectively. Then the pattern inventory is

$$\sum_F W(F) = P_G\Big\{\sum_{r \in R} w(r), \sum_{r \in R}[w(r)]^2, \sum_{r \in R}[w(r)]^3, \dots, \Big\}, \tag{7}$$

where P_G is the cycle index. In particular, if all weights are chosen to be equal to unity, then we obtain

$$\text{the number of patterns} = P_G(|R|, |R|, |R|, \dots). \tag{8}$$

Corollary 7. *The coefficient of x^r in $Z(A, 1+x)$ is the number of A-equivalence classes of r-sets of X.*

Proof. In the figure counting series $1 + x$, the term $1 = x^0$ can indicate the absence of an object in X while $x = x^1$ stands for its presence. The corollary now follows immediately from Pólya's Enumeration Theorem 5. □

3.4 Examples

Example 12. Suppose we have a necklace that is made from three beads and n colors. Then, by Example 10 the cycle index is

$$P_G = \frac{1}{3}(x_1^3 + 2x_3).$$

Suppose we want to examine the necklaces made by amber, A, and beige, B, beads. Then

$$\begin{aligned} P_G\left((A+B), (A^2+B^2), (A^3+B^3)\right) &= \frac{1}{3}((A+B)^3 + 2 * (A^3+B^3)) \\ &= \frac{1}{3}(A^3 + 3A^2B + 3AB^2 + B^3 + 2A^3 + 2B^3) \qquad (9) \\ &= \frac{1}{3}(3A^3 + 3A^2B + 3AB^2 + 3B^3) \\ &= A^3 + A^2B + AB^2 + B^3. \qquad (10) \end{aligned}$$

Equation 9 shows us that the configurations fixed by R_0 will be the necklace with three amber beads, three necklaces with two amber beads and one beige bead, three necklaces with one amber bead and two beige beads, and the necklace with three beige beads. The configurations fixed by R_{120} and R_{240} will be the necklace of three amber beads and the necklace with three beige beads. Equation 10 displays all the possible necklace colorations that are completely distinct.

The next example, which should be very familiar by now, will also be given with two colors. For now, we are avoiding an example with three colors simply because the expansion of the generating function would be tedious. If the number of colors is large or if the sums are raised to a large power, computer algebra software can be used to do the work.

Example 13. In Example 11 we found the cycle index of distinct cube face colorations to be

$$P_G = \frac{1}{24}(x_1^6 + 6x_1^2x_4 + 3x_1^2x_2^2 + 8x_3^2 + 6x_2^3).$$

We also found that the number of distinct cubes that could be created from two colors was 10. Suppose a face can be colored with sepia or teal. Substituting $x_i = (s^i + t^i)$ for each $i = \{1, 2, 3, 4\}$, we get

$$\begin{aligned}
P_G &= \frac{1}{24}\left((s+t)^6 + 6(s+t)^2(s^4+t^4) + 3(s+t)^2(s^2+t^2)^2 + 8(s^3+t^3)^2 + 6(s^2+t^2)^3\right) \\
&= \frac{1}{24}\big((s^6 + 6s^5t + 15s^4t^2 + 20s^3t^3 + 15s^2t^4 + 6st^5 + t^6) + 6(s^2+2st+t^2)(s^4+t^4) \\
&\quad + 3(s^2+2st+t^2)(s^4+2s^2t^2+t^4) + 8(s^6+2s^3t^3+t^6) + 6(s^4+2s^2t^2+t^4)(s^2+t^2)\big) \\
&= \frac{1}{24}(24s^6 + 24s^5t + 48s^4t^2 + 48s^3t^3 + 48s^2t^4 + 24st^5 + 24t^6) \\
&= s^6 + s^5t + 2s^4t^2 + 2s^3t^3 + 2s^2t^4 + st^5 + t^6.
\end{aligned}$$

Now, not only can we count that there are ten distinct colorations by adding the coefficients, we also have a list of what they are. For instance, the term $2s^4t^2$ indicates that there are two different configurations where there are four sepia sides and two teal sides. The two teal sides could either be touching on an edge (one configuration) or they could be on opposite faces (the second configuration).

Example 14. To find the pattern inventory for the edge colorings of a tetrahedron, we first need to consider the symmetries of the tetrahedron as permutations of the edges. Let S be the set of edges of a tetrahedron colored silver or tangerine, and G be the set of permutations of S that can be produced by rotating the tetrahedron.

Figure 13 shows the 120 and 240 rotations of the tetrahedron, and Figure 14 shows the 180 degree rotations of the tetrahedron. Both figures assume a bird's-eye perspective looking down on a translucent tetrahedron. In Figure 14, the tetrahedron is rotated about the opposite edges A and E. Thus C and D switch, and B and F switch.

The 12 elements in G and their cycle structures are as follows:

1. The identity leaves all 6 edges fixed and has structure representation x_1^6.
2. Four 120 degree rotations about a corner and the middle of the opposite face give two cycles of length three. They cyclically permute the edges incident to that corner and also the edges

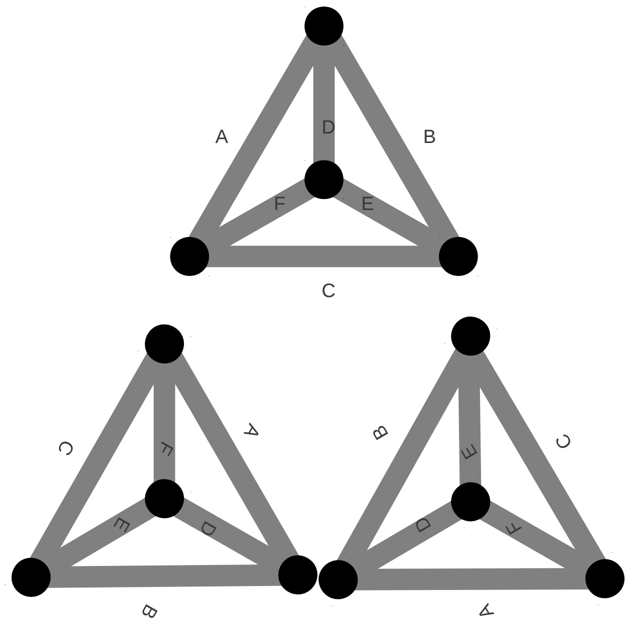

Figure 13: Rotation of tetrahedron by 120 degrees and 240 degrees

bounding the opposite face, so the cycle structure representation is x_3^2. The four colorations this fixes are shown in Figure 15.

3. Four 240 degree rotations essentially do the same thing as the 120 rotations, also giving a structure of x_3^2. The four colorations this fixes are the same as shown in Figure 15.

4. Three 180 degree rotations about opposite edges leave the two edges fixed. The other four edges are left in cycles of length 2. Thus we have the structure $x_1^2 x_2^2$. The sixteen fixed colorations are shown in Figures 16 and 17, and are rotated about sides A and E, as labeled in Figure 14.

We then have the function

$$P_G = \frac{1}{12}(x_1^6 + 8x_3^2 + 3x_1^2 x_2^2).$$

If we use n colors, the number of distinct tetrahedron edge colorations will be given by

$$P_G = \frac{1}{12}(n^6 + 8n^2 + 3n^4).$$

More specifically, if we have the two colors silver and teal, then we will substitute $x_i = (s^i + t^i)$ for

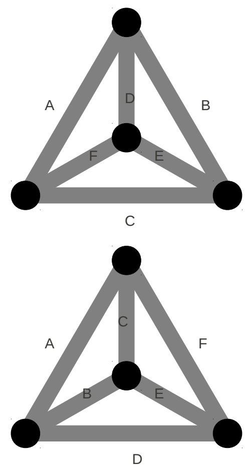

Figure 14: Rotation of tetrahedron by 180 degrees about edges A and E

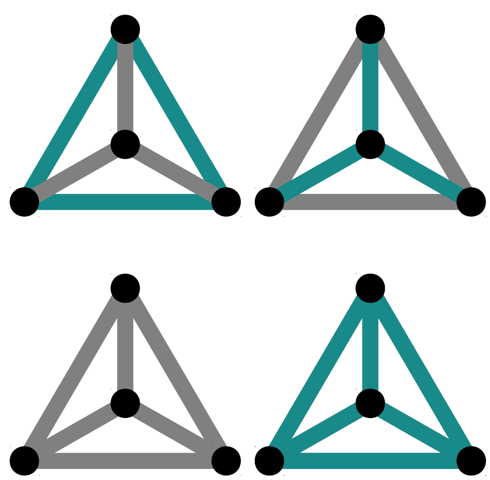

Figure 15: Colorations of the tetrahedron fixed by rotations of 120 or 240 degrees

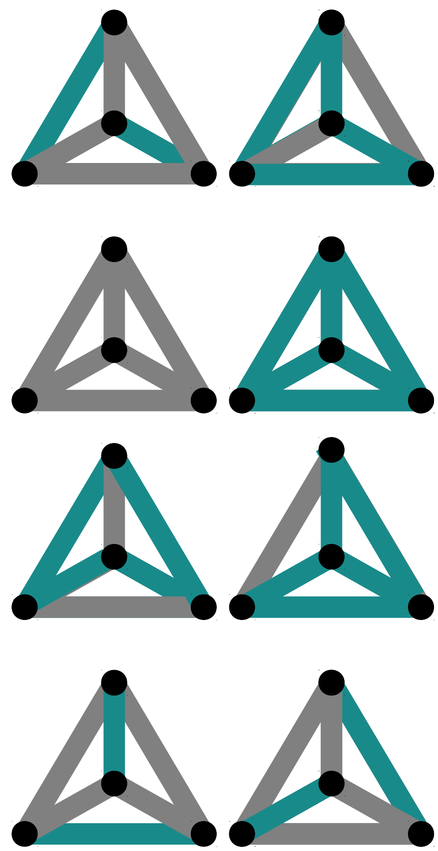

Figure 16: First eight colorations of the tetrahedron fixed by rotations of 180 degrees

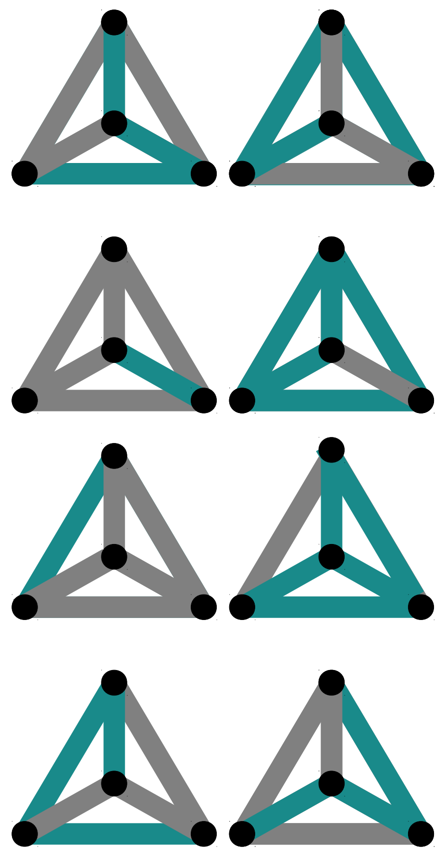

Figure 17: Second eight colorations of the tetrahedron fixed by rotations of 180 degrees

each $i = \{1, 2, 3, 4\}$ to get

$$\begin{aligned}
P_G &= \frac{1}{12}\left((s+t)^6 + 8(s^3+t^3)^2 + 3(s+t)^2(s^2+t^2)^2\right) \\
&= \frac{1}{12}\big((s^6 + 6s^5t + 15s^4t^2 + 20s^3t^3 + 15s^2t^4 + 6st^5 + t^6) + (8s^6 + 16s^3t^3 \\
&\quad + 8t^6) + (3s^6 + 6s^5t + 9s^4t^2 + 12s^3t^3 + 9s^2t^4 + 6st^5 + 3t^6)\big) \\
&= \frac{1}{12}(12s^6 + 12s^5t + 24s^4t^2 + 48s^3t^3 + 24s^2t^4 + 12st^5 + 12t^6) \\
&= s^6 + s^5t + 2s^4t^2 + 4s^3t^3 + 2s^2t^4 + st^5 + t^6. \qquad (11)
\end{aligned}$$

Thus there are twelve distinct colorations of the edges of a tetrahedron using the colors silver and teal. The list of these colorations is given by the terms of Equation 11. For instance, the term $2s^2t^4$ indicates that there are two configurations with two edges that are silver and four edges that are teal.

4 Application 1: Chemical Enumeration

As stated earlier, one of the main applications of Pólya theory is chemical isomer enumeration. Isomers are compounds which have the same number and types of atoms but different structural formulas because the atoms are bonded to each other differently. We will now examine the derivatives of benzene and cyclopropane. Reade [7] walks through both examples.

Example 15. The benzene molecule can be seen in Figure 18. It consists of six carbon atoms in a ring, to each of which a hydrogen atom is attached. Since its structure is "flat," we can do the same analysis that we would to count distinct six bead necklaces. Derivatives of benzene are formed by replacing the hydrogen atoms by other atoms or groups of atoms. We will first consider dichlorobenzenes, in which chlorines can be substituted for hydrogens. Since the molecule does not change when rotated or flipped, the group G that we will consider is the dihedral group D_6. It contains elements $R_0, R_{60}, R_{120}, R_{180}, R_{240}, R_{300}, F_0, F_{30}, F_{60}, F_{90}, F_{120}.$ and F_{150}. The rotations should be clear, and the flips are made about the lines in Figure 19 that are i degrees clockwise away from the vertical axis.

We then have twelve permutations with the subsequent structure properties:

1. The identity leaves all six atoms fixed, giving six cycles of length one and thus the structure x_1^6.

2. The two rotations R_{120} and R_{240} each give two cycles of length three. In particular, they each fix the configurations that are all chlorine or all hydrogen, or the alternating H, Cl configurations, shown in Figure 20. Thus we have index $2x_3^2$.

3. The three flips F_{30}, F_{90}, and F_{150} and the rotation R_{180} all have cycle length two. For the the flips, we can imagine the molecule being symmetric across the axes that they are flipped

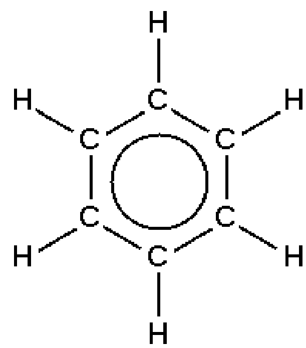

Figure 18: The benzene molecule

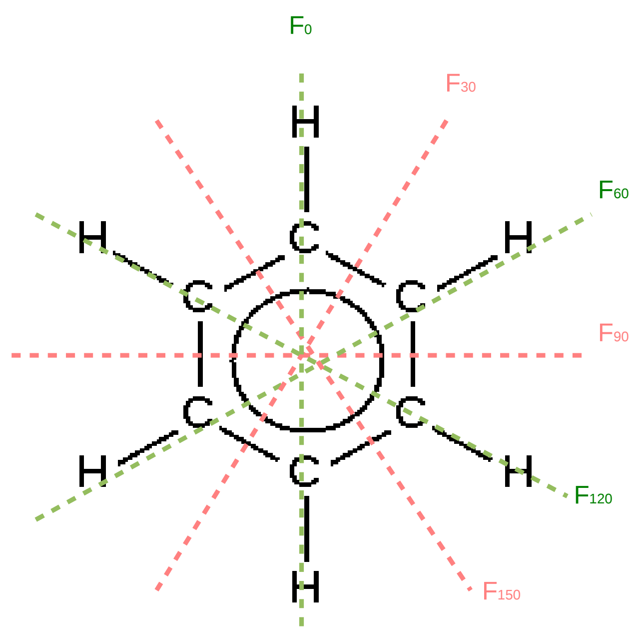

Figure 19: The axes for the F_i flips

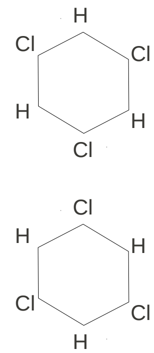

Figure 20: Dichlorobenzenes fixed by R_{120} and R_{240}

across, and thus leaving half the atoms dependent on the first three. For the rotation, we can let three atoms be hydrogen or chlorine, but then the three that are rotated into their places must match. This gives index $3x_2^3 + x_2^3 = 4x_2^3$.

4. Now consider the remaining flips F_0, F_{60}, and F_{120}. The two atoms on the axes will be fixed by the flip and therefore have cycle length 1. The other four other atoms are left in two cycles of length two. Thus we have the structure $3x_1^2x_2^2$.

5. Finally, the two rotations R_{60} and R_{300} each have just one cycle length six, and thus produce the structure $2x_6$.

We then have the function

$$P_{D_6} = \frac{1}{12}(x_1^6 + 2x_3^2 + 4x_2^3 + 3x_1^2x_2^2 + 2x_6).$$

If we use n different atoms or groups of atoms, the number of distinct isomers will be given by

$$P_{D_6} = \frac{1}{12}(n^6 + 2n^2 + 4n^3 + 3n^4 + 2n)$$

by Burnside's Lemma. But if we want to find how many isomers there are of a certain type, it is useful to substitute the precise weights we are concerned about. More specifically, for dichlorobenzenes, we will substitute $x_i = (H^i + Cl^i)$ for each $i = \{1, 2, \ldots, 6\}$ to get

$$\begin{aligned} P_{D_6} &= \frac{1}{12}\Big((H+Cl)^6 + 2(H^3+Cl^3)^2 + 4(H^2+Cl^2)^3 + 3(H+Cl)^2(H^2+Cl^2)^2 + 2(H^6+Cl^6)\Big) \\ &= H^6 + H^5Cl + 3H^4Cl^2 + 3H^3Cl^3 + 3H^2Cl^4 + HCl^5 + Cl^6. \end{aligned} \tag{12}$$

We used WolframAlpha to simplify the above polynomial.
Therefore, not counting the original benzene molecule, we can create twelve distinct dichlorobenzene isomers. We can also know the number of distinct isomers with two hydrogen and four chlorine atoms (3) or of any other certain combination, given by Equation 12. The formula can be generalized to other isomers if we include the other atoms or groups of atoms in the x_i substitution.

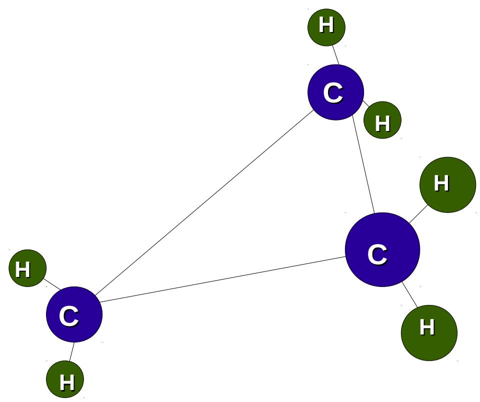

Figure 21: Three dimensional image of cyclopropane structure

Now we will count distinct derivatives of cyclopropane. Sometimes an extension of Pólya's Theorem is necessary when one variable is not sufficient, as is the case in this next example.

Example 16. Cyclopropane differs from benzene in that we must consider its structure in three dimensions, as seen in Figure 21. It also differs from benzene in that we must draw from two families of variables. To obtain derivatives of cyclopropane, each hydrogen atom H may be left unchanged or can be replaced by a fluorine atom F or an iodine atom I, and each carbon atom c can be left unchanged or replaced by an atom of radioactive carbon r. We will use variables x_i for the first set and y_i for the second.

Since the molecule can be rotated or flipped, the group we will consider is the dihedral group D_3, consisting of the identity, two rotations and three flips. We then have six types of permutations with the subsequent structure representations:

1. The identity leaves all six hydrogen locations fixed and all three carbon locations fixed, giving the structure $x_1^6 y_1^3$.
2. The two rotations of 120 degrees or 240 degrees give one cycle of length three for the carbon locations, and 2 cycles of length three for the hydrogen locations. Their structure is given by $2x_3^2 y_3$.
3. There are three flips which occur along each median of the central triangle. There are three cycles of length two for the hydrogen locations as well as one cycle of length one for the fixed carbon location, and one cycle of length two for the two other interchanged carbon locations. Thus we have the structure $3x_2^3 y_1 y_2$.

We then have what is called the *compound cycle index*

$$P_{D_3} = \frac{1}{6}(x_1^6 y_1^3 + 2x_3^2 y_3 + 3x_2^3 y_1 y_2).$$

We will substitute $x_i = (H^i + F^i + I^i)$ and $y_i = (c^i + r^i)$ for each $i = \{1, 2, 3\}$ to get

$$P_{D_3} = \frac{1}{6}\bigg((H+F+I)^6(c+r)^3 + 2(H^3+F^3+I^3)^2(c^3+r^3) + 3(H^2+F^2+I^2)^3(c+r)(c^2+r^2)\bigg).$$

If simplified, this equation would give a complete description of all distinct derivatives of cyclopropane.
Finding the number of distinct derivatives is a much easier calculation. Let $H = F = I = c = r = 1$ and we have

$$P_{D_3} = \frac{1}{6}\bigg((3)^6(2)^3 + 2(3)^2(2) + 3(3)^3(2)(2)\bigg) = 1032.$$

5 Application 2: Graphs

Pólya's Theorem can be used to calculate the number of graphs up to isomorphism with a fixed number of vertices.

Definition 10. *A* ***graph*** *is a collection of points and lines connecting some (possibly empty) subset of them. The points of a graph are most commonly known as* ***vertices****, and the lines connecting the vertices are known as* ***edges****.*

To apply Pólya's Theorem and count the number of different graphs, we need a weight structure. If the edge exists, it will have weight 1, and it if does not, it will have weight 0, giving $f(x) = 1 + x$ as the generating function for the set of colors, as seen in Corollary 3.3. Remember that for any positive integer n, the symmetric group on the set $\{1, 2, 3, \ldots, n\}$ is called the *symmetric group on n elements*, and is denoted by S_n.

Example 17. We will find the number of graphs on three vertices. We will consider the symmetric group S_3 (although we could also use D_3 again for this example). For the identity permutation (1)(2)(3), there are three cycles of length 1, giving the structure s_1^3. The three permutations (1)(23),(2)(13), and (3)(12) each have one cycle of length 1 and one of length 2, so the term is given by $3s_1s_2$. Finally, the two permutations (123) and (132) both are of length three, giving $2s_3$.
Then

$$P_{S_3} = \frac{1}{3!}(s_1^3 + 3s_1s_2 + 2s_3).$$

Thus we have

$$\begin{aligned} P_{S_3}(x+1, x^2+1, x^3+1) &= \frac{1}{3!}((x+1)^3 + 3(x+1)(x^2+1) + 2(x^3+1)) \\ &= x^3 + x^2 + x + 1 \end{aligned} \tag{13}$$

which shows there is one graph with each number of edges. These can be seen in Figure 22.
As a demonstration of Corollary 3.3, the coefficient of x^3 shows the number of equivalence classes of graphs on three vertices with three edges. (There is only one.)

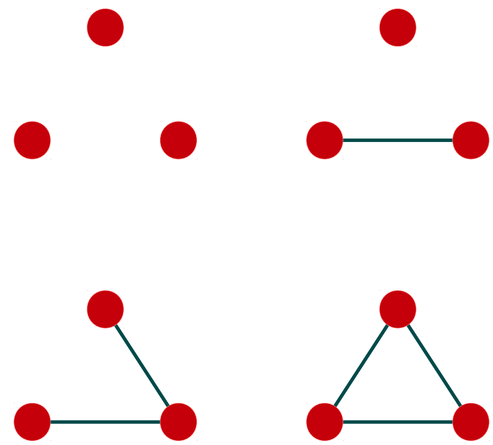

Figure 22: All graphs on three vertices, up to isomorphism

The logical next consideration is a generalization of this to p vertices. Harary and Palmer [4] give us the following and unsightly formula.

Theorem 8. *The polynomial $g_p(x)$ which enumerates graphs of order p by number of lines is given by*

$$g_p(x) = Z(S_p^{(2)}, 1 + x), \tag{14}$$

where

$$Z(S_p^{(2)}) = \frac{1}{p!} \sum_{(j)} \frac{p!}{\prod k^{j_k} j_k!} \prod_k s_{2k+1}^{kj_{2k+1}} \prod_k (s_k s_{2k}^{k-1})^{j_{2k}} s_k^{k\binom{j_k}{2}} \prod_{r<t} s_{[r,t]}^{(r,t)j_r j_t}. \tag{15}$$

In the above equation, j_k represents the number of k-cycles in a given permutation α, (r,t) represents the greatest common divisor of r and t, and $[r,t]$ represents the least common multiple of r and t.

Though this notation is intense, we will look at the simple demonstration of the transition from S_p to S_p^2.

Example 18. Consider S_4. We will review the permutations in S_4 and their cycle structures.

1. The permutation (1)(2)(3)(4) has four cycles of length one, giving the term s_1^4.
2. The permutation (1)(2)(34) has structure $s_1^2 s_2$. There are six of these permutations, for each of the six possible pairs of vertices: (12), (13), (14), (23), (24), and (34).
3. The permutation (1)(234) has structure $s_1 s_3$. There are eight possible differing cycles of length three, and thus eight permutations with this structure. The other seven cycles of length three are (123), (132), (134), (143), (124), (142), and (243).
4. The permutations (12)(34),(13)(24), and (14)(23) have structure s_2^2.

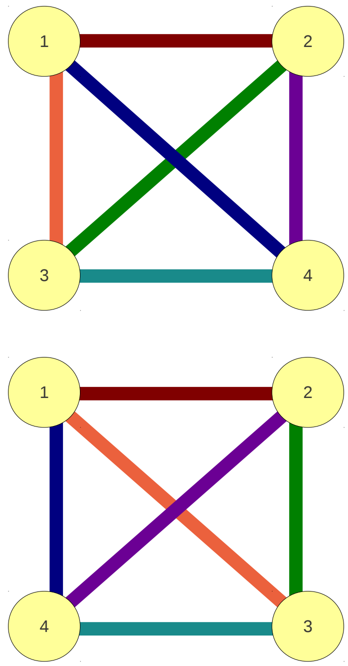

Figure 23: The permutation (1)(2)(34) in S_4, or $(\overline{12})$ $(\overline{34})$ $(\overline{14}\ \overline{13})$ $(\overline{23}\ \overline{24})$ in $S_4^{(2)}$

5. The permutation (1234) has structure s_4. The other five with this structure include (1243), (1423), (1342),(1324), (1432).

Thus

$$Z(S_4) = \frac{1}{24}(s_1^4 + 6s_1^2 s_2 + 8s_1 s_3 + 3s_2^2 + 6s_4). \tag{16}$$

To evaluate permutations in $S_4^{(2)}$ we will switch from permuting vertices to permuting *pairs* of vertices, since we are trying to eventually count edges. For notation, we will let $\overline{ij}$ be the pair of vertices i and j, and $\overline{ij} = \overline{ji}$.

1. If our term in S_4 is s_1^4, the corresponding term in $S_4^{(2)}$ is s_1^6. The new permutation is $(\overline{12})(\overline{13})(\overline{14})(\overline{23})(\overline{24})(\overline{34})$. Remember that $(\overline{12})$ is not permuting vertices 1 and 2, rather it is a cycle of length one that fixes the pair of vertices 1 and 2 together, or fixes the edge connecting 1 and 2. This makes intuitive sense since there are six possible edges, and the identity fixes them in cycles of length 1.

2. If our term in S_4 is $s_1^2 s_2$, the corresponding term in $S_4^{(2)}$ is $s_1^2 s_2^2$. The permutation (1)(2)(34) would become $(\overline{12})$ $(\overline{34})$ $(\overline{14}\ \overline{13})$ $(\overline{23}\ \overline{24})$. This is reflected in Figure 23 where the brown and teal edges stay put, but the orange and blue switch, and the purple and green switch, creating two cycles of length two.

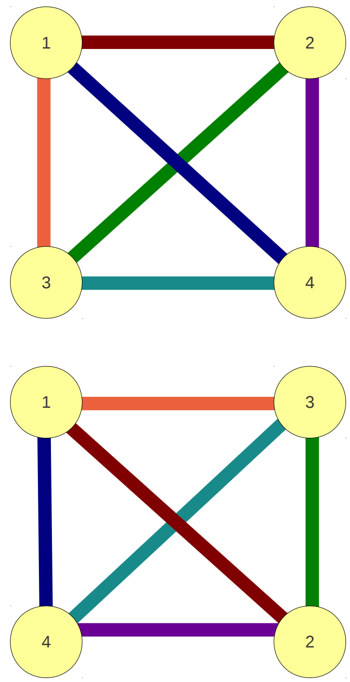

Figure 24: The permutation $(1)(243)$ in S_4, or $(\overline{12}\ \overline{14}\ \overline{13})(\overline{23}\ \overline{24}\ \overline{34})$ in $S_4^{(2)}$

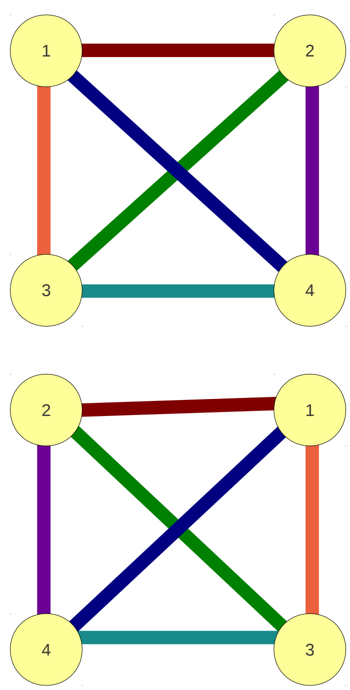

Figure 25: The permutation $(12)(34)$ in S_4, or $(\overline{12})(\overline{34})(\overline{24}\ \overline{13})(\overline{23}\ \overline{14})$ in $S_4^{(2)}$

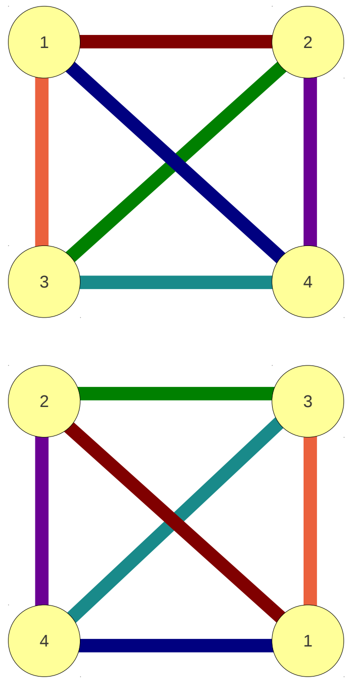

Figure 26: The permutation (1432) in S_4, or $(\overline{24}\ \overline{13})(\overline{12}\ \overline{23}\ \overline{34}\ \overline{41})$ in $S_4^{(2)}$

3. If our term in S_4 is s_1s_3, the corresponding term in $S_4^{(2)}$ is s_3^2. The permutation (1)(243) would become $(\overline{12}\ \overline{14}\ \overline{13})(\overline{23}\ \overline{24}\ \overline{34})$. This is reflected in Figure 24 where the brown, blue and orange edges are a permuted, and the teal, green, and purple edges are permuted, creating two cycles of length three.

4. If our term in S_4 is s_2^2, the corresponding term in $S_4^{(2)}$ is $s_1^2s_2^2$. The permutation (12)(34) would become $(\overline{12})(\overline{34})(\overline{24}\ \overline{13})(\overline{23}\ \overline{14})$. This is reflected in Figure 25 where the brown and teal edges are fixed, the orange and purple are permuted, and the green and blue are permuted.

5. If our term in S_4 is s_4, the corresponding term in $S_4^{(2)}$ is s_2s_4. The permutation (1432) would become $(\overline{24}\ \overline{13})(\overline{12}\ \overline{23}\ \overline{34}\ \overline{41})$. This is reflected in Figure 26 where the the orange and purple are permuted in a cycle of length two, and the remaining four colors are permuted with each other in a cycle of length four.

Thus

$$Z(S_4^{(2)}) = \frac{1}{24}(s_1^6 + 6s_1^2s_2^2 + 8s_3^2 + 3s_1^2s_2^2 + 6s_2s_4) \tag{17}$$

$$= \frac{1}{24}(s_1^6 + 9s_1^2s_2^2 + 8s_3^2 + 6s_2s_4). \tag{18}$$

Finally, we can substitute $1 + x$ in this cycle index and use the powers of WolframAlpha to give

$$\begin{aligned}
g_4(x) &= Z(S_4^{(2)}, 1+x) && (19)\\
&= \frac{1}{24}((x+1)^6 + 9(x+1)^2(x^2+1)^2 + 8(x^3+1)^2 + 6(x^2+1)(x^4+1)) && (20)\\
&= x^6 + x^5 + 2x^4 + 3x^3 + 2x^2 + x + 1. && (21)
\end{aligned}$$

From Equation 21, we gather that there are eleven distinct graphs on four vertices including one graph each with six edges, five edges, one edge, or no edges, two graphs each with four edges or two edges, and three graphs with three edges.

Example 19. Now we will find the number of graphs on five vertices. First consider the cycle index for S_5. We will review the permutations in S_5 and their cycle structures. Since $|S_5| = 5! = 120$, we will not list out most of the permutations. Rather, we will take one permutation of each type, and explain how we found how many of each exist.

1. The permutation (1)(2)(3)(4)(5) has five cycles of length one, giving the term s_1^5.
2. The permutation (12)(3)(4)(5) has structure $s_1^3 s_2$. There are ten of these permutations, which can be found by all the combinations of five objects into pairs, i.e. $\binom{5}{2}$.
3. The permutation (123)(4)(5) has structure $s_1^2 s_3$. There are twenty, which can be counted by first recognizing the ten combinations for the cycle of length two, and then for each remaining cycle of length three there are two possibilities for their arrangement.
4. The permutation (1234)(5) has structure $s_1 s_4$. There are six different ways that four vertices can be permuted, and there are five choices for our cycle of length one. Thus there are thirty of these permutations.
5. The permutation (12)(34)(5) has structure $s_1 s_2^2$. From the previous example with four vertices, we saw that there were three possible permutations with the same structure as (12)(34). Now we have five possibilities for which number we choose to be the cycle of length one, so we have a total of fifteen permutations with this structure.
6. The permutation (123)(45) has structure $s_2 s_3$. There are twenty, which can be counted in the same way as permutation 3.
7. Finally, the permutation (12345) has structure s_5 and there are $4! = 24$ of them.

Thus

$$Z(S_5) = \frac{1}{120}(s_1^5 + 10s_1^3 s_2 + 20s_1^2 s_3 + 30s_1 s_4 + 15s_1 s_2^2 + 20s_2 s_3 + 24s_5). \tag{22}$$

Now we will examine each structure in $S_5^{(2)}$.

1. The permutation (1)(2)(3)(4)(5) becomes $(\overline{12})(\overline{13})(\overline{14})(\overline{15})(\overline{23})(\overline{24})(\overline{25})(\overline{34})(\overline{35})(\overline{45})$, giving the term s_1^{10}. This again makes intuitive sense, because there are a total of ten edges on a graph of five vertices.

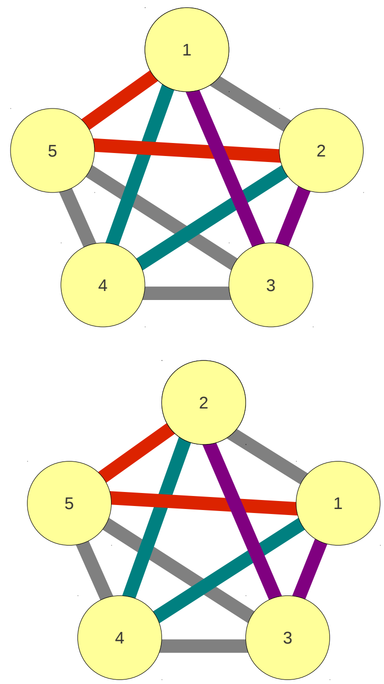

Figure 27: The permutation $(12)(3)(4)(5)$ in S_4, or $(\overline{12})(\overline{34})(\overline{35})(\overline{45})(\overline{15}\ \overline{25})(\overline{14}\ \overline{24})(\overline{13}\ \overline{23})$ in $S_4^{(2)}$

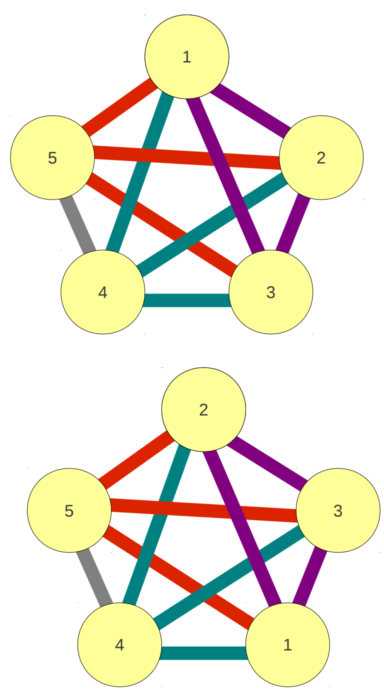

Figure 28: The permutation $(123)(4)(5)$ in S_4, or $(\overline{45})(\overline{12}\ \overline{23}\ \overline{31})(\overline{14}\ \overline{24}\ \overline{34})(\overline{15}\ \overline{25}\ \overline{35})$ in $S_4^{(2)}$

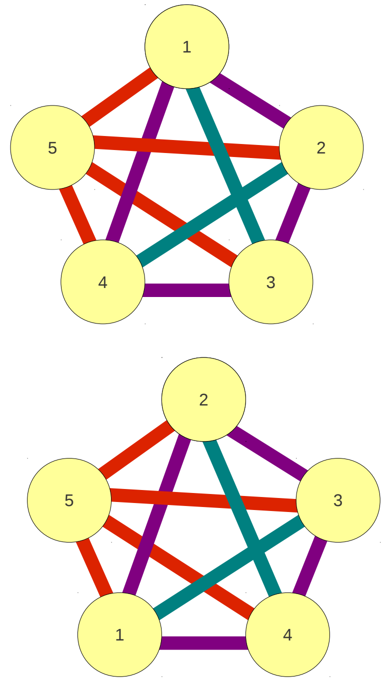

Figure 29: The permutation (1234)(5) in S_4, or $(\overline{24}\ \overline{13})(\overline{12}\ \overline{23}\ \overline{34}\ \overline{14})(\overline{15}\ \overline{25}\ \overline{35}\ \overline{45})$ in $S_4^{(2)}$

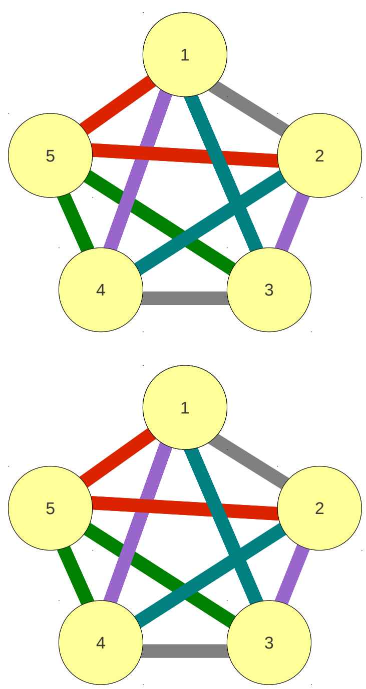

Figure 30: The permutation (12)(34)(5) in S_4, or $(\overline{12})(\overline{34})(\overline{14}\ \overline{23})(\overline{13}\ \overline{24})(\overline{15}\ \overline{25})(\overline{35}\ \overline{45})$ in $S_4^{(2)}$

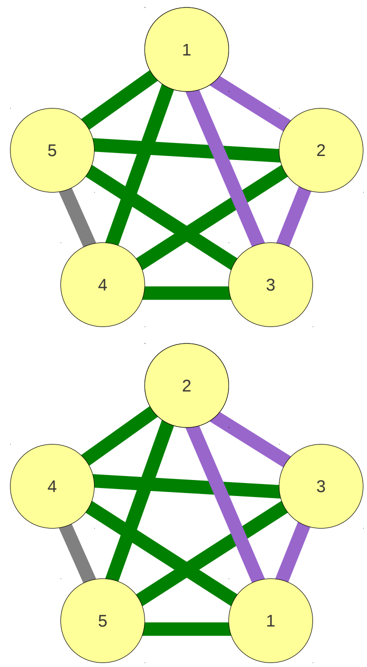

Figure 31: The permutation (123)(45) in S_4, or $(\overline{45})(\overline{13}\ \overline{12}\ \overline{23})(\overline{14}\ \overline{35}\ \overline{24}\ \overline{15}\ \overline{34}\ \overline{25})$ in $S_4^{(2)}$

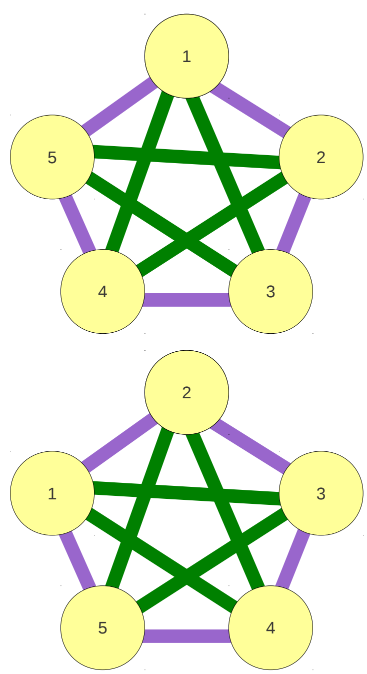

Figure 32: The permutation (12345) in S_4, or $(\overline{12}\ \overline{23}\ \overline{34}\ \overline{45}\ \overline{51})(\overline{13}\ \overline{24}\ \overline{35}\ \overline{41}\ \overline{52})$ in $S_4^{(2)}$

2. The permutation (12)(3)(4)(5) becomes $(\overline{12})(\overline{34})(\overline{35})(\overline{45})(\overline{15}\ \overline{25})(\overline{14}\ \overline{24})(\overline{13}\ \overline{23})$, giving the term $s_1^4 s_2^3$. The cycles of length two are depicted by the red, blue, and purple edges in Figure 27.

3. The permutation (123)(4)(5) becomes $(\overline{45})(\overline{12}\ \overline{23}\ \overline{31})(\overline{14}\ \overline{24}\ \overline{34})(\overline{15}\ \overline{25}\ \overline{35})$, giving the term $s_1 s_3^3$, as in Figure 28.

4. The permutation (1234)(5) becomes $(\overline{24}\ \overline{13})(\overline{12}\ \overline{23}\ \overline{34}\ \overline{14})(\overline{15}\ \overline{25}\ \overline{35}\ \overline{45})$, giving the term $s_2 s_4^2$, as in Figure 29.

5. The permutation (12)(34)(5) becomes $(\overline{12})(\overline{34})(\overline{14}\ \overline{23})(\overline{13}\ \overline{24})(\overline{15}\ \overline{25})(\overline{35}\ \overline{45})$, giving structure $s_1^2 s_2^4$. These cycles are shown in Figure 30, where the gray edges are fixed.

6. The permutation (123)(45) becomes $(\overline{45})(\overline{13})(\overline{12}\ \overline{23})(\overline{14}\ \overline{35}\ \overline{24}\ \overline{15}\ \overline{3425})$. The cycle structure is then $s_1 s_3 s_6$, as reflected in Figure 31.

7. Finally, the permutation (12345) would become $(\overline{12}\ \overline{23}\ \overline{34}\ \overline{45}\ \overline{51})(\overline{13}\ \overline{24}\ \overline{35}\ \overline{41}\ \overline{52})$, giving the term s_5^2. The cycles are shown in Figure 32.

Thus

$$Z(S_5^{(2)}) = \frac{1}{120}(s_1^{10} + 10s_1^4 s_2^3 + 20s_1 s_3^3 + 30s_2 s_4^2 + 15s_1^2 s_2^4 + 20s_1 s_3 s_6 + 24s_5^2). \tag{23}$$

Finally, we can substitute $x+1$ in this cycle index and use WolframAlpha to give

$$\begin{aligned} g_5(x) &= Z(S_5^{(2)}, x+1) \\ &= \frac{1}{120}((x+1)^{10} + 10(x+1)^4(x^2+1)^3 + 20(x+1)(x^3+1)^3 + 30(x^2+1)(x^4+1)^2 \\ &\quad + 15(x+1)^2(x^2+1)^4 + 20(x+1)(x^3+1)(x^6+1) + 24(x^5+1)^2) \\ &= x^{10} + x^9 + 2x^8 + 4x^7 + 6x^6 + 6x^5 + 6x^4 43x^3 + 2x^2 + x + 1, \end{aligned} \tag{24}$$

which is, unsurprisingly, the same result we acheived using Theorem 8. There is 1 graph with each of zero, one, nine, or ten edges, 2 graphs with each of two or eight edges, 4 graphs with each of three or seven edges, and 6 graphs with each of five, six, or seven edges. This gives a total of 34 graphs on five vertices.

Butler [1] put together a table of enumerations of unlabeled graphs, as seen in Figure 33. As can be observed on the table, the number of graphs quickly becomes very large, so that a computer is needed to apply Pólya's Theory.

6 Application 3: Music Theory

Reiner [8] gives us the below definitions.

Number of Edges	<---------- Number of Vertices ----------->											
	2	3	4	5	6	7	8	9	10	11	12	13
0	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	2	2	2	2	2	2	2	2	2	2
3	0	1	3	4	5	5	5	5	5	5	5	5
4	0	0	2	6	9	10	11	11	11	11	11	11
5	0	0	1	6	15	21	24	25	26	26	26	26
6	0	0	1	6	21	41	56	63	66	67	68	68
7	0	0	0	4	24	65	115	148	165	172	175	176
8	0	0	0	2	24	97	221	345	428	467	485	492
9	0	0	0	1	21	131	402	771	1,103	1,305	1,405	1,446
10	0	0	0	1	15	148	663	1,637	2,769	3,664	4,191	4,435
11	0	0	0	0	9	148	980	3,252	6,759	10,250	12,763	14,140
12	0	0	0	0	5	131	1,312	5,995	15,772	28,259	39,243	46,415
13	0	0	0	0	2	97	1,557	10,120	34,663	75,415	119,890	154,658
14	0	0	0	0	1	65	1,646	15,615	71,318	192,788	359,307	517,121
15	0	0	0	0	1	41	1,557	21,933	136,433	467,807	1,043,774	1,711,908
16	0	0	0	0	0	21	1,312	27,987	241,577	1,069,890	2,911,086	5,546,619
17	0	0	0	0	0	10	980	32,403	395,166	2,295,898	7,739,601	17,422,984
18	0	0	0	0	0	5	663	34,040	596,191	4,609,179	19,515,361	52,664,857
19	0	0	0	0	0	2	402	32,403	828,728	8,640,134	46,505,609	152,339,952
20	0	0	0	0	0	1	221	27,987	1,061,159	15,108,047	104,504,341	420,048,805
21	0	0	0	0	0	1	115	21,933	1,251,389	24,630,887	221,147,351	1,101,083,128
22	0	0	0	0	0	0	56	15,615	1,358,852	37,433,760	440,393,606	2,739,261,020
23	0	0	0	0	0	0	24	10,120	1,358,852	53,037,356	825,075,506	6,461,056,816
24	0	0	0	0	0	0	11	5,995	1,251,389	70,065,437	1,454,265,734	14,441,470,390
25	0	0	0	0	0	0	5	3,252	1,061,159	86,318,670	2,411,961,516	30,583,652,956
26	0	0	0	0	0	0	2	1,637	828,728	99,187,806	3,765,262,970	61,372,294,334
27	0	0	0	0	0	0	1	771	596,191	106,321,628	5,534,255,092	116,724,411,757
28	0	0	0	0	0	0	1	345	395,166	106,321,628	7,661,345,277	210,474,287,115
29	0	0	0	0	0	0	0	148	241,577	99,187,806	9,992,340,187	359,954,668,522
30	0	0	0	0	0	0	0	63	136,433	86,318,670	12,281,841,209	584,089,835,857
31	0	0	0	0	0	0	0	25	71,318	70,065,437	14,229,503,560	899,632,282,299
32	0	0	0	0	0	0	0	11	34,663	53,037,356	15,542,350,436	1,315,729,343,451
33	0	0	0	0	0	0	0	5	15,772	37,433,760	16,006,173,014	1,827,823,498,798
34	0	0	0	0	0	0	0	2	6,759	24,630,887	15,542,350,436	2,412,694,353,115
35	0	0	0	0	0	0	0	1	2,769	15,108,047	14,229,503,560	3,026,821,673,656
36	0	0	0	0	0	0	0	1	1,103	8,640,134	12,281,841,209	3,609,810,088,490
37	0	0	0	0	0	0	0	0	428	4,609,179	9,992,340,187	4,093,273,437,761
38	0	0	0	0	0	0	0	0	165	2,295,898	7,661,345,277	4,413,678,080,790
39	0	0	0	0	0	0	0	0	66	1,069,890	5,534,255,092	4,525,920,859,198
40	0	0	0	0	0	0	0	0	26	467,807	3,765,262,970	4,413,678,080,790
41	0	0	0	0	0	0	0	0	11	192,788	2,411,961,516	4,093,273,437,761
42	0	0	0	0	0	0	0	0	5	75,415	1,454,265,734	3,609,810,088,490
43	0	0	0	0	0	0	0	0	2	28,259	825,075,506	3,026,821,673,656
44	0	0	0	0	0	0	0	0	1	10,250	440,393,606	2,412,694,353,115
45	0	0	0	0	0	0	0	0	1	3,664	221,147,351	1,827,823,498,798
46	0	0	0	0	0	0	0	0	0	1,305	104,504,341	1,315,729,343,451
47	0	0	0	0	0	0	0	0	0	467	46,505,609	899,632,282,299
48	0	0	0	0	0	0	0	0	0	172	19,515,361	584,089,835,857
49	0	0	0	0	0	0	0	0	0	67	7,739,601	359,954,668,522
50	0	0	0	0	0	0	0	0	0	26	2,911,086	210,474,287,115
51	0	0	0	0	0	0	0	0	0	11	1,043,774	116,724,411,757
52	0	0	0	0	0	0	0	0	0	5	359,307	61,372,294,334
53	0	0	0	0	0	0	0	0	0	2	119,890	30,583,652,956
54	0	0	0	0	0	0	0	0	0	1	39,243	14,441,470,390
55	0	0	0	0	0	0	0	0	0	1	12,763	6,461,056,816
56	0	0	0	0	0	0	0	0	0	0	4,191	2,739,261,020
57	0	0	0	0	0	0	0	0	0	0	1,405	1,101,083,128
58	0	0	0	0	0	0	0	0	0	0	485	420,048,805
59	0	0	0	0	0	0	0	0	0	0	175	152,339,952
60	0	0	0	0	0	0	0	0	0	0	68	52,664,857
61	0	0	0	0	0	0	0	0	0	0	26	17,422,984
62	0	0	0	0	0	0	0	0	0	0	11	5,546,619
63	0	0	0	0	0	0	0	0	0	0	5	1,711,908
64	0	0	0	0	0	0	0	0	0	0	2	517,121
65	0	0	0	0	0	0	0	0	0	0	1	154,658
66	0	0	0	0	0	0	0	0	0	0	1	46,415
67	0	0	0	0	0	0	0	0	0	0	0	14,140
68	0	0	0	0	0	0	0	0	0	0	0	4,435
69	0	0	0	0	0	0	0	0	0	0	0	1,446
70	0	0	0	0	0	0	0	0	0	0	0	492
71	0	0	0	0	0	0	0	0	0	0	0	176
72	0	0	0	0	0	0	0	0	0	0	0	68
73	0	0	0	0	0	0	0	0	0	0	0	26
74	0	0	0	0	0	0	0	0	0	0	0	11
75	0	0	0	0	0	0	0	0	0	0	0	5
76	0	0	0	0	0	0	0	0	0	0	0	2
77	0	0	0	0	0	0	0	0	0	0	0	1
78	0	0	0	0	0	0	0	0	0	0	0	1
Totals	2	4	11	34	156	1,044	12,346	274,668	12,005,168	1,018,997,864	165,091,172,592	50,502,031,367,952

Figure 33: Counting graphs

Definition 11. *The n-scale is taken to be equal (well) tempered and consists of the integers from 0 to $n-1$. We equate octave notes, so our scale is mathematically $\mathbb{Z}_n$ with addition.*

For example, Western music has $n = 12$, Debussy used a whole tone scale with $n = 6$, and Ives used a quarter tone scale with $n = 24$.

6.1 Chords

Definition 12. *A k-chord in the n-scale is an equivalence class of subsets of k elements each of $\mathbb{Z}_n$.*

We can use the dihedral group D_n to induce the equivalence relation between chords. A group element would have the form $T^i I : a \to i - a (\text{mod } n)$. For example, the C major chord is equivalent to the C minor chord: $\{C, E, G\} = \{0, 4, 7\} \sim \{0, 3, 7\} = \{C, E\flat, G\}$, under the element $T^7 I$.

This is actually no different from the two-color necklace problems we have worked with previously. We now have a circular "necklace" of notes to choose from, and we choose the notes in the chord by coloring them one color, and coloring the rest another color.

Chords in twelve-tone music

To examine twelve-tone music with dihedral symmetry, we will need to obtain the cycle index for D_{12}. As a sidenote, we could also go through a similar analysis using the cyclic group C_{12}, but we will use D_{12} since we are more familiar with it thus far. There are 24 elements in D_{12} and their cycle structures are as follows:

1. There are two types of flips. Because we now have an even number of vertices, we can flip over lines through pairs of vertices or lines through pairs of edges.
   (a) We'll first deal with the flips over opposite pairs of vertices. As seen in Figure 34, this will fix the two green vertices and leave the rest of the blue "beads" on cycles of length two. Note that the matching shades of blue correspond with the beads that are in the same cycle. There are six such flips, and their structure is $x_1^2 x_2^5$.
   (b) The second type flips over the centers of opposite edges . As seen in Figure 34, there will be six cycles of length two. There are also six of these, and their structure is x_2^6.
2. There are twelve rotations which break into six different types. We will call them $R_0, R_{30}, R_{60}, \ldots, R_{330}$.
   (a) The identity, R_0 leaves everything fixed and thus has the cycle index x_1^{12}.
   (b) The rotations R_{30}, R_{150}, R_{210}, and R_{330} all have structures x_{12}^1. With these rotations, each bead would visit every spot before returning to its orginial place after the twelve rotations.

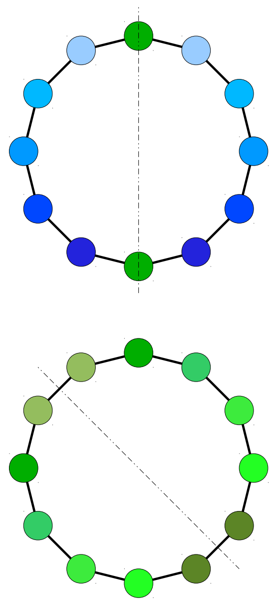

Figure 34: Examples of the two types of flip permutations in D_{12}

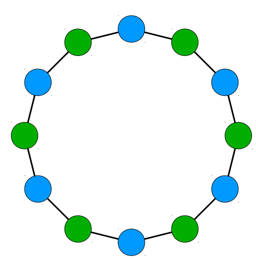

Figure 35: Rotations R_{60} and R_{300} both have structures x_6^2

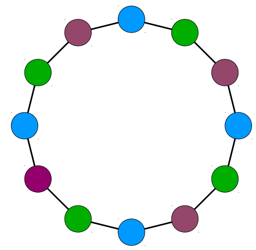

Figure 36: Rotations R_{90} and R_{270} have structures x_4^3

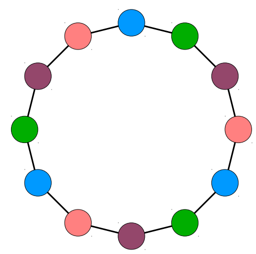

Figure 37: Rotations R_{120} and R_{240} have structures x_3^4

(c) The rotations R_{60} and R_{300} both have structures x_6^2. As seen in Figure 35, these permutations will rotate the blue beads in a cycle of length six, and rotate the green beads in a cycle of length six.

(d) The rotations R_{90} and R_{270} have structures x_4^3, as these rotations create three cycles of length four. These can be demonstrated by the three colors in Figure 36.

(e) The rotations R_{120} and R_{240} have structures x_3^4, creating four cycles of length three. This is shown in Figure 37.

(f) Finally, the rotation R_{180} has six cycles of length 2, not unlike the second type of flip, giving the structure x_2^6. These cycles can be seen in Figure 38 with the matching pairs of shades of blue.

Thus we have

$$Z_{D_{12}} = \frac{1}{24}(6x_1^2x_2^5 + 6x_2^6 + x_1^{12} + 4x_{12}^1 + 2x_6^2 + 2x_4^3 + 2x_3^4 + x_2^6). \tag{25}$$

We will use the substitution $x_i = (x^i + 1)$, as the term $1 = x^0$ indicates the absence of an object while $x = x^1$ represents its presence. So x^i indicates that i distinct objects are present.

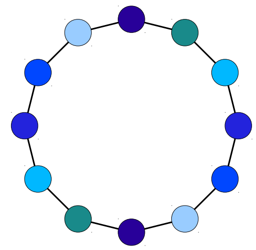

Figure 38: The rotation R_{180} has structure x_2^6

Substituting $x_i = (x^i + 1)$ for each $i = \{1, 2, \ldots, 12\}$ and using WolframAlpha to simplify the polynomial we get

$$Z_{D_{12}} = \frac{1}{24}(6(x+1)^2(x^2+1)^5 + 6(x^2+1)^6 + (x+1)^{12} + 4(x^{12}+1) + 2(x^6+1)^2 \tag{26}$$
$$+ (x^4+1)^3 + 2(x^3+1)^4 + (x^2+1)^6)$$
$$= x^{12} + x^{11} + 6x^{10} + 12x^9 + 29x^8 + 38x^7 + 50x^6 + 38x^5 + 29x^4 + 12x^3 + 6x^2 + x + 1. \tag{27}$$

The number of k-chords is the coefficient of x^k in Equation 27. Thus, in twelve-tone music, there are 12 types of tetrachords, 29 types of pentachords, 50 types of hexachords, and so on. These can be summed for a total of 224 distinct chords.

We could go through the above analysis for each n-scale to find the cycle index of D_n or C_n. However, it would be more convenient to use the general formula for D_n or C_n if we want to find indices for large values of n or a large number of n's. These formulas can be found in the discussion of cycle indices.

6.2 Tones

The Virginia Tech Multimedia Music Dictionary defines a *Tone Row* as a specific arrangement of the twelve tones of the twelve-tone scale as a basis for a twelve-tone composition. We already know there are 12! such arrangements, but 'symmetry' will make some of these equivalent. We will shortly define what these symmetries are.

Rumery [9] brings us the following example with figures.

Example 20. Consider a row extracted from Schoenberg's first twelve-tone composition, as in Figure 6.2. The pitches are numbered according to their position in the chromatic scale beginning on E.

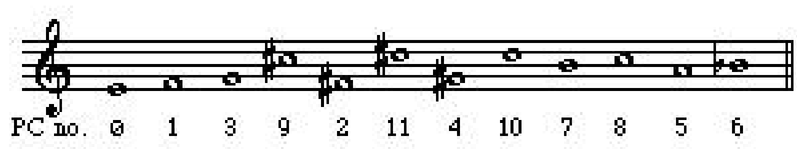

Figure 39: Row from Suite, Op. 25, Schoenberg

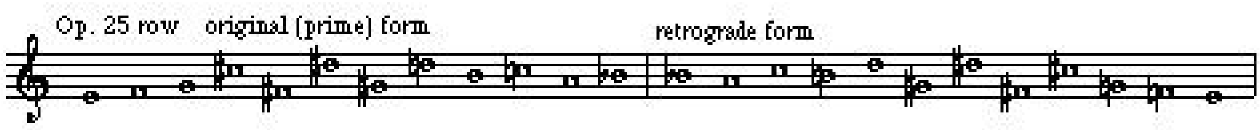

Figure 40: Row in original and retrograde form

E is numbered 0 because it is the first note of the row. The pitch class numbers of the remaining notes indicate their distance above E in half-steps.

We can extend this to n rows, which Reiner [8] does in the following definition:

Definition 13. *An n-tone row is an equivalence class of permutations in* S_n.

Our equivalence classes are induced by the group generated by transposition,

$$T : S_n \to S_n : (a_1, \ldots, a_n) \to (a_1 + 1, \ldots, a_n + 1)(\text{mod } n);$$

inversion,

$$I : (a_1, \ldots, a_n) \to (a_1, 2a_1 - a_2, \ldots, 2a_1 - a_n)(\text{mod } n);$$

and retrogradation,

$$R : (a_1, \ldots, a_n) \to (a_n, \ldots, a_1).$$

According to Reiner [8], music theorists regard transposition as such a basic transformation that, instead of working with all the permutations, they work with equivalence classes of permutations under transposition, those beginning with 0 be regarded as class representatives. So our set is the set of $(n-1)!$ permutations of $\{1, \ldots, n-1\}$, and our group is generated by R and I. Since $RI = IR$, we are working with what is known as the *Klein four-group*, which is also known as $\mathbb{Z}_2 \times \mathbb{Z}_2$.

To better grasp the group elements, *retrograde* form is created by writing the notes in the original version in reverse order, as in Figure 40. The *inversion* form has all the intervals written upside down, with the interval directions changed, as in the left side of Figure 41. Lastly, the *retrograde inversion* from is just the inversion form, but with the notes in reverse order, as in the right side of Figure 41.

For further clarification, Pickett [6] provides the matrix in Figure 6.2. Tone rows in their original forms are listed from left to right. The retrograde version can be read from right to left, and the inverted version can be read by starting at the same note but then reading down the column. Finally, the row's retrograde inverted version can be found by reading the column from bottom to top.

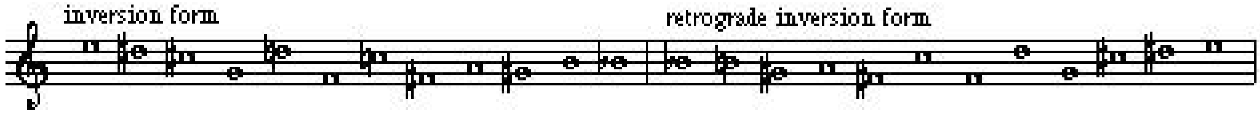

Figure 41: Row in inversion and retrograde inversion form

P R

I

	I6	I5	I2	I4	I3	I0	I9	I1	I8	I11	I10	I7	
P6	F#	F	D	E	Eb	C	A	C#	G#	B	Bb	G	R6
P7	G	F#	Eb	F	E	C#	Bb	D	A	C	B	G#	R7
P10	Bb	A	F#	G#	G	E	C#	F	C	D#	D	B	R10
P8	Ab	G	E	F#	F	D	B	D#	Bb	C#	C	A	R8
P9	A	Ab	F	G	F#	D#	C	E	B	D	C#	Bb	R9
P0	C	B	G#	Bb	A	F#	D#	G	D	F	E	C#	R0
P3	Eb	D	B	C#	C	A	F#	Bb	F	G#	G	E	R3
P11	B	A#	G	A	G#	F	D	F#	C#	E	D#	C	R11
P4	E	D#	C	D	C#	Bb	G	B	F#	A	G#	F	R4
P1	C#	C	A	B	Bb	G	E	Ab	D#	F#	F	D	R1
P2	D	C#	Bb	C	B	G#	F	A	E	G	F#	D#	R2
P5	F	E	C#	D#	D	B	G#	C	G	Bb	A	F#	R5
	RI6	RI5	RI2	RI4	RI3	RI0	RI9	RI1	RI8	RI11	RI10	RI7	

RI

To count the number of distinct tone rows, we will count the number of equivalence classes. By Burnside's Lemma, the number of equivalence classes for a group G acting on D is

$$\frac{1}{|G|}\sum_{g\in G}(\#\ of\ elements\ of\ D\ fixed\ by\ g).$$

Closely following the analysis done by Reiner [8], we find the number of elements fixed by each permutation e, I, R, and IR. To avoid the trivial cases, assume $n \geq 3$.

1. Since our set has $(n-1)!$ elements, the identity e fixes $(n-1)!$ elements.
2. Consider I. $I(0, a_2, \ldots, a_n) = (0, -a_2, \ldots, -a_n) \sim (0, a_2, \ldots, a_n)$ implies $a_i \equiv -a_i (\text{mod } n)$ for $i = 2, \ldots, n$. The first element 0 is fixed. There is at most one nonzero solution to $x \equiv -x$ (mod n), which is not sufficient to fill out the permutation. So there are are 0 elements fixed by I.
3. Consider R. $R(a_1, \ldots, a_n) = (a_n, \ldots, a_1) \sim (a_1, \ldots, a_n)$ implies that t exists such that $a_1 \equiv a_n + t, a_2 \equiv a_{n-1} + t, \ldots$ (mod n). If n is odd, the middle element is fixed and no transposition is allowed: $t = 0$. But then we have the contradiction that $a_n = a_1$. So R fixes 0 elements if n is odd. However, if n is even, the first and last congruences imply that $2t = 0$; thus $t = 0$ or $t = n/2$. We already saw that $t = 0$ fails, but the other gives fixed permutations. Since $a_1 = 0$, then $a_n = n/2$. For a_2, we can choose any of $n-2$ elements, and this determines a_{n-1}. For a_3, we have $n-4$ choices, and so on. Hence if n is even, the number of elements fixed by R is $(n-2)(n-4)\cdots(2)$.
4. Consider IR. $IR(a_1, \ldots, a_n) = (-a_n, \ldots, -a_1) \sim (a_1, \ldots, a_n)$ implies that t exists such that $a_1 + a_n \equiv t$, $a_2 + a_{n-1} \equiv t, \ldots$ (mod n). The last congruence for n odd is $2a_{(n+1)/2} \equiv t$ (mod n). It is not essential that we fix the first element as 0, as we could fix $a_{(n+1)/2}$ as 0 and find the same count. Hence we may assume $t = 0, a_{(n+1)/2} = 0$, giving $n-1$ choices for a_1, which determines a_n, $n-3$ choices for a_2, etc. Thus if n is odd, we have $(n-1)(n-3)\cdots(2)$ fixed elements. If n is even, we fix a_1 so $a_n \neq t \neq 0$. We see that t must be odd in order to complete the permutation. If $t = 2k$, there is no mate for k in the permutation. So there are $n/2$ choices for $t = a_n$. For a_2, there are $n-2$ choices, which determines a_{n-1}, and so on. Thus there are $(n/2)(n-2)(n-4)\cdots(2)$ elements fixed by IR.

Finally, we conclude that the number of n-tone rows is

$$\frac{1}{4}[(n-1)! + (n-1)(n-3)\cdots(2)] \qquad \text{if } n \text{ is odd;} \qquad (28)$$

$$\frac{1}{4}[(n-1)! + (n-2)(n-4)\cdots(2)(1+\frac{n}{2})] \qquad \text{if } n \text{ is even.} \qquad (29)$$

Thus in twelve-tone music, there are 9985920 tone rows.

7 Conclusion

We have already seen that Pólya's Counting Theory has numerous and remarkable applications. The two most extensively researched are graphical and chemical isomer enumeration, but there are plenty of other uses as well. Without exploring the details too extensively, Reade [7] mentions the other following applications:

1. Investigating crystal structure and studying nuclear magnetic resonance and NMR spectroscopy, as well as other applications in theoretical chemistry.
2. Counting *Latin squares*, which are $n \times n$ arrays filled with n different symbols, each occurring exactly once in each row and exactly once in each column.
3. Enumerating the number of Boolean functions under various conditions.
4. Counting the number of essentially different propositions of n statements, and showing that the problem is equivalent to coloring the vertices of a hypercube.
5. Counting finite automata and certain binary matrices.
6. Counting graphs in statistical mechanics.
7. Studying kinetic structures.

And there certainly are more which have not been mentioned. Though we will not explore them here, we can easily conclude that Pólya's Counting Theory is one of the most useful and satisfying tools to exist in combinatorics.

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