PATHOLOGICAL
APPLICATIONS OF LEBESGUE MEASURE TO THE CANTOR SET AND NON-INTEGRABLE DERIVATIVES

PRICE HARDMAN

WHITMAN COLLEGE

1. Introduction

Pathological is an oft used word in the mathematical community, and in that context it has quite a different meaning than in everyday usage. In mathematics, something is said to be "pathological" if it is particularly poorly behaved or counterintuitive. However, this is not to say that mathematicians use this term in an entirely derisive manner. Counterintuitive and unexpected results often challenge common conventions and invite a reevaluation of previously accepted methods and theories. In this way, pathological examples can be seen as impetuses for mathematical progress. As a result, pathological examples crop up quite often in the history of mathematics, and we can usually learn a great deal about the history of a given field by studying notable pathological examples in that field. In addition to allowing us to weave historical narrative into an otherwise noncontextualized mathmematical discussion, these examples shed light on the boundaries of the subfield of mathematics in which we find them.

It is the intention then of this piece to investigate the basic concepts, applications, and historical development of measure theory through the use of pathological examples. We will develop some basic tools of Lebesgue measure on the real line and use them to investigate some historically significant (and fascinating) examples. One such example is a function whose derivative is bounded yet not Riemann integrable, seemingly in violation of the Fundamental Theorem of Calculus. In fact, this example played a large part in the development of the Lebesgue integral in the early years of the 20^{th} century. To construct this function, however, will require a fair amount of work, and will allow us to apply the concepts of measure theory to a number of other examples along the way.

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Although ideally suited for a student of real analysis, most sections of this piece are largely accessible to an astute student of calculus.

2. Lebesgue Measure

2.1. **Introduction and History.** Measure theory is, in general, a branch of mathematical analysis concerned with determining the sizes of sets. As stated in the introduction, it is the intent of this piece to develop some basic concepts of measure theory in the context of the real numbers, $\mathbb{R}$. As such, measure is, in this case, a generalization of the notion of length, the standard measurement in one dimension. Much of elementary calculus and real analysis is concerned with arbitrary intervals of real numbers and their lengths. As one begins to move into more advanced analysis, however, more complicated and exotic sets tend to crop up. This leads to the question: How can we give meaning to the notion of length for a set that is not an interval? Measure theory was developed for precisely this purpose.

During the second half of the 19^{th} century mathematicians were largely concerned both with generalizing previous results as well as putting them on more rigorous theoretical footings. This endeavor led to many familiar contemporary concepts, such as modern calculus, axiomatic set theory, and point-set topology. Measure theory came into being as part of this larger trend within mathematics as a whole. The development of measure theory is tied up inextricably with the development of modern analysis, particularly the Riemann integral and its more sophisticated descendent, the Lebesgue integral. Although it was Bernhard Riemann (1826-1866) who developed the first widely-accepted rigorous definition of the integral, in the years following his death it became clear to many that the Riemann integral had serious theoretical shortcomings, and mathematicians realized that the fledging field of measure theory could give insight into the exact nature of these shortcoming. In the 1880s and 1890s Camille Jordan (1838-1922) and Emile Borel (1871-1956) developed what are now known as *Jordan measure* and *Borel measure*, respectively. These were the first attempts at defining the size of arbitrary sets, but it was not until the turn of the century that a truly robust definition of measure was developed.

Henri Lebesgue[1] (1875-1941), in the process of developing his revolutionary definition of integration, created the concept of **Lebesgue measure** between 1899 and 1901. Recall that measure[2] in this case is simply an extension of the concept of length to sets that are not intervals, and in light of this we should develop measure in such a way that it behaves in the same manner as length. In this way, the development of measure theory can be

[1]Pronounced "le-BEG".

[2]From now on throughout this piece, we will use the terms "Lebesgue measure" and "measure" interchangeably, since no other measures will be discussed further.

seen as a constructive endeavor, where the theory is formed around a list of requirements it is desired to possess.

2.2. **Lebesgue Outer Measure.** Since we wish to develop the concept of measure as an extension of the notion of length, we require that measure behave in fundamentally the same way as length does.

If A is a set of real numbers, then the Lebesgue measure of A is denoted as $\mu(A)$ and we wish μ to possess the following properties.

(a) If I is an interval, then $\mu(I) = \ell(I)$, the length of the set.
(b) If A is a subset of B, $\mu(A) \leq \mu(B)$.
(c) Translation invariance. That is, if x_0 is a constant and we define $A + x_0$ to be $\{x + x_0 : x \in A\}$, then $\mu(A + x_0) = \mu(A)$.
(d) If A and B are disjoint sets, then $\mu(A \cup B) = \mu(A) + \mu(B)$. Furthermore, for a sequence of disjoint sets $\{A_i\}$, $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.

With this wish list of desired properties in hand, we can begin to construct the Lebesgue measure.

The essential concept of how Lebesgue measure determines the length of a set is the notion of a **countable interval cover** of a set. Luckily, the term "cover" gives an intuitive sense of its precise defintion.

Definition 2.1. *A set A has a* **countable interval cover** *if there exists a countable collection of open intervals $\{I_n\}$ whose union contains A, or*

$$A \subseteq \bigcup_{n=1}^{\infty} I_n.$$

Since we can easily find the length of intervals, we can use interval covers to measure the length of a set. Lebesgue measure determines the length of a set indirectly by taking the sum of the lengths of the intervals in every possible countable cover and taking the infimum, or greatest lower bound. In this way, the total length of the intervals in the cover approximate the length of the set as closely as possible. The following definition formalizes this idea.

Definition 2.2. *Let $E \subseteq \mathbb{R}$. The* **Lebesgue outer measure** *of E, denoted $\mu^*(E)$, is defined as*

$$\mu^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) : \{I_k\} \text{ is a countable cover of } E \right\}.$$

Lebesgue outer measure, although it comes close, does not satisfy the desired properties of measure set forth in this section. The problem is that outer measure is not countably additive, but rather is countably *subadditive*, where the measure of the union of disjoint sets can be less than the sum of their individual measures. Due to this property, The following theorem highlights the properties that outer measure does possess. The proofs of parts (e) and (f) are rather long and tedious, so the interested reader is directed to either Gordon [2] for an elementary proof, or Kolmogorov [4] for a proof that appeals to the Heine -Borel theorem.

Theorem 2.3. *Lebesgue outer measure has the following properties:*

a) *If $E_1 \subseteq E_2$, then $\mu^*(E_1) \leq \mu^*(E_2)$.*
b) *If E is countable, then $\mu^*(E) = 0$.*
c) $\mu^*(\emptyset) = 0$.
d) *$\mu^*(E)$ is invariant under translation. That is, $\mu^*(E + x_0) = \mu^*(E)$ for any real number x_0.*
e) *μ^* is countably subadditive: Given a sequence $\{E_i\}$ of sets,*

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$$

f) *If I is an interval, then $\mu^*(I) = \ell(I)$.*

Proof. Part (a) follows directly from the definition of outer measure. For (b), suppose that the set $E = \{x_k : k \in \mathbb{Z}^+\}$ is countably infinite. Let $\epsilon > 0$ and $\{\epsilon_k\}$ be a sequence of positive numbers such that $\displaystyle\sum_{k=1}^{\infty} \epsilon_k = \frac{\epsilon}{2}$. Since $\displaystyle E \subseteq \bigcup_{k=1}^{\infty} (x_k - \epsilon_k, x_k + \epsilon_k)$, it follows from the definition of outer measure that $\mu^*(E) < \epsilon$. Therefore, we conclude that $\mu^*(E) = 0$.

Parts (a) and (b), when combined, provide a proof of (c). As for part (d), since an open cover of E also yields an open cover of $E + x_0$ with intervals of the same length, we find that $\mu^*(E + x_0) \leq \mu^*(E)$. But since $\mu^*(E) = \mu^*(E+x_0-x_0) \leq \mu^*(E+x_0)$, we conclude that $\mu^*(E) = \mu^*(E+x_0)$. □

It should be clear that every set of real numbers has a Lebesgue outer measure.[3] However, outer measure does not satisfy property (d) in our wish list at the start of this section. Therefore, we divide the real numbers into two distinct camps: the collection of sets satisfying property (d) and those sets that do not. This will be the criterion for what it means for a set to be measurable.

2.3. **Measurable vs Nonmeasurable Sets.** Since in the previous section we saw that every set of real numbers has a Lebesgue outer measure and that this outer measure is countably subadditive, we were led to divide the real numbers into the collection of sets whose outer measures are countably additive and those whose outer measures are not. For the sets with countably additive outer measures, it would appear that outer measure is a suitable candidate for the measure that we have been looking for, since the only thing holding outer measure back was that it failed to be countably additive. In

[3]This is a consequence of the Completeness Axiom. See Gordon [3].

fact, this is equivalent to the common definition of Lebesgue measurability, although on the surface this equivalency is hard to see.

Definition 2.4. *A set of real numbers E is* **Lebesgue measurable** *if for every set of real numbers A,*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*\left(A \cap E^{\mathcal{C}}\right).$$

For any Lebesgue measurable set E its Lebesgue measure, denoted $\mu(E)$, is equal to its Lebesgue outer measure $\mu^*(E)$.

In some texts, this is known as **Carathéodory's condition**, named for the Ottoman mathematican Constantin Carathéodory (1873-1950) who developed this definition in 1914. Since that time, Lebesgue's original definition of measurability, which involved an additional concept called inner measure, has largely been supplanted by Carathéodory's.

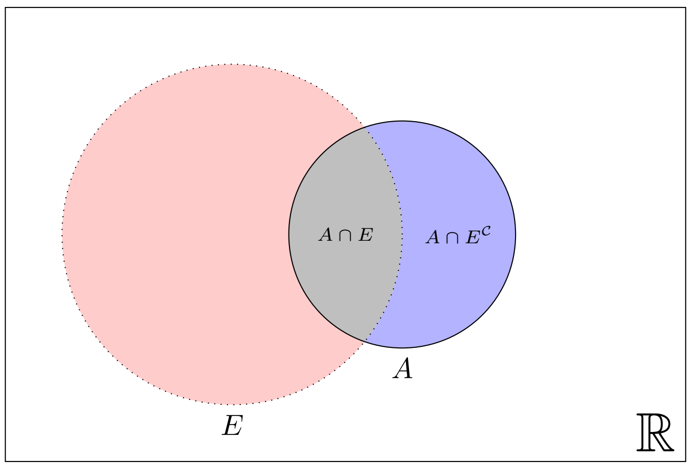

FIGURE 1. A set E is only measurable if it is sufficiently well-behaved to "cleanly" divide any other set A into disjoint parts whose summed measures are commensurate with the measure of the original set.

This definition is a bit strange and is also hard to intuitively grasp. The reason is that the concept of measurability strikes right at the heart of the foundations of modern mathematics. It is very difficult to imagine and visualize two sets E and A for which $\mu^*(A) \neq \mu^*(A \cap E) + \mu^*\left(A \cap E^{\mathcal{C}}\right)$. This would mean that E is a **nonmeasurable set**. For a long time, the existence of nonmeasurable sets was controversial. During the first half of the 20^{th} century the Axiom of Choice, one of the ten fundamental axioms upon which modern mathematics is based – axioms which are collectively known as Zermelo-Fraenkel set theory with the Axiom of Choice, or ZFC –, was very contentious because of some counterintuitive results that arise from its adoption. In addition, it was proven finally in 1964 that the Axiom of Choice is independent of the other nine Zermelo-Fraenkel axioms, meaning that those axioms are consistent regardless of whether we adopt the Axiom of Choice or its negation. Today, the vast majority of mathematicians accept the the Axiom of Choice and the fascinating (albeit frustrating) results to which it leads.

Despite being difficult to visualize, the following theorem proves the existence of nonmeasurable sets.

Theorem 2.5. *There exist sets that are not Lebesgue measurable.*

Proof. We begin by defining a relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if $x - y$ is rational. Take note that $\sim$ is an equivalence relation.[4] We can see then that this equivalence relation delineates a collection of equivalence classes of the form $\{x + r : r \in \mathbb{Q}\}$. Since each equivalence class contains a point in $[0, 1]$, we define $E \subseteq [0, 1]$ to be a set consisting of one point from each equivalence class.[5] For each positive integer i, let the set $[-1, 1] \cap \mathbb{Q} = \{r_i : i \in \mathbb{Z}^+\}$ and let $E_i = E + r_i$. We then claim that

$$[0, 1] \subseteq \bigcup_{i=1}^{\infty} E_i \subseteq [-1, 2] .$$

We can see that $\bigcup\limits_{i=1}^{\infty} E_i \subseteq [-1, 2]$ is true since each E_i is the the sum of two objects from $[0, 1]$ and $[-1, 1]$, respectively. To see that the first inclusion is true, let $x \in [0, 1]$. We can then see that there exists a $y \in E$ such that $x - y$ is rational. Seeing as $-1 \leq x - y \leq 1$, there exists some index j such that $x - y = r_j$. Thus, $x = y + r_j \in E_j \subseteq \bigcup\limits_{i=1}^{\infty} E_i$. In addition, we can see that if i and j are distinct, then $E_i \cap E_j = \emptyset$, otherwise there exists $y, z \in E$ such that $y + r_i = z + r_j$, which tells us that $y \sim z$, which would be a contradiction.

Now let us suppose that the set E is measurable. This implies that each E_i is measurable and that $\mu(E_i) = \mu(E + r_i) = \mu(E)$. Since the sets E_i are disjoint, we have

$$1 \leq \mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E) \leq 3.$$

If the measure of $\mu(E) = 0$, then $1 \leq \sum_{i=1}^{\infty} \mu(E) = 0$, a contradiction. On the other hand, if $\mu(E) > 0$, then $\infty = \sum_{i=1}^{\infty} \mu(E) \leq 3$, which is also a contradiction. Either way, we are left with the conclusion that E is not measurable. □

While nonmeasurable sets are fascinating and mysterious objects, to study them any more would take us too far afield. Seeing as we are concerned here with measurable sets, we ought to specify some of the most basic properties of measurable sets. Proofs of each property can be found in Gordon [2].

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[4] An equivalence relation is a relation that is reflective ($a \sim a$), symmetric ($a \sim b$ implies $b \sim a$), and transitive ($a \sim b$ and $b \sim c$ together imply $a \sim c$).

[5] Our ability to make this choice of exactly one element from each set in an uncountable collection is a direct result of the Axiom of Choice.

Theorem 2.6. *The collection of Lebesgue measurable sets has the following properties*

a) *Both the empty set $\emptyset$ and the real line $\mathbb{R}$ are measurable, with measures 0 and ∞, respectively.*
b) *If E is a measurable set, then $E^{\mathcal{C}}$ is also measurable.*
c) *Any set with outer measure 0 is measurable.*
d) *If two sets A and B are measurable, then $A \cap B$ and $A \cup B$ are also measurable.*
e) *Measurable sets are translation invariant. If E is measurable, then so is $E + x_0$, $x \in \mathbb{R}$.*

2.4. **Further Properties of Lebesgue Measure.** In this section we will prove some important properties of Lebesgue measure, including finite additivity, countable additivity, and limit properties. For the sake of brevity, we will assume without proof that countable unions and intersections of measurable sets are themselves measurable sets. A proof of this fact can be found in Gordon [2].

Theorem 2.7. *If $\{E_i : 1 \leq i \leq n\}$ is a finite collection of disjoint measurable sets, then*

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i)\,.$$

Proof. This will be a proof by induction. For the case $n = 1$, the result is trivial. Let us then assume that $\mu\left(\bigcup_{i=1}^{k-1} E_i\right) = \sum_{i=1}^{k-1} \mu(E_i)$ for a positive integer $k > 1$. It follows then by the induction hypothesis that

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{k} E_i\right) &= \mu\left(\left(\bigcup_{i=1}^{k} E_i\right) \cap E_k\right) + \mu\left(\left(\bigcup_{i=1}^{k} E_i\right) \cap E_k^{\mathcal{C}}\right) \\
&= \mu(E_k) + \mu\left(\bigcup_{i=1}^{k-1} E_i\right) \\
&= \mu(E_k) + \sum_{i=1}^{k-1} \mu(E_i) \\
&= \sum_{i=1}^{k} \mu(E_i)
\end{aligned}$$

Thus, the result holds for any positive integer n.

□

We can use the case of finite additivity to prove that Lebesgue measure is countably additive as well.

Theorem 2.8. *If $\{E_i\}$ is a countable collection of disjoint measurable sets, then*

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)\,.$$

Proof. By Theorem 2.7, we know that for every positive integer n

$$\sum_{i=1}^{n} \mu(E_i) = \mu\left(\bigcup_{i=1}^{n} E_i\right) \leq \mu\left(\bigcup_{i=1}^{\infty} E_i\right).$$

However, since by the countable subadditivity of Lebesgue outer measure and the measurability of countable unions of measurable sets, we also know that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

When combined with the previous inequality and the fact that $\sum_{i=1}^{n} \mu(E_i)$ is convergent, we have equality and the proof is complete.

□

Thus, we find that Lebesgue measure satisfies the four conditions set forth in Section 2.1. Although measure has many other interesting properties, the only other one that will serve our purposes in this piece is the limit property of intersections of measurable sets.

Theorem 2.9. *Let $\{E_n\}$ be a sequence of measurable sets such that $\mu(E_1)$ is finite. If $E_{n+1} \subseteq E_n$ for all n and $E = \bigcap_{n=1}^{\infty} E_n$, then $\mu(E) = \lim_{n\to\infty} \mu(E_n)$.*

Proof. Since $\mu(E_1) < \infty$, $\mu(E_{n+1}) \leq \mu(E_n)$ for all n, and $\mu(E_n) \geq 0$ for all n, we have that $\{\mu(E_n)\}$ is a bounded, nonincreasing sequence of nonnegative numbers. Therefore, $\lim_{n\to\infty} \mu(E_n)$ exists.[6] Let $A_k = E_k \backslash E_{k+1}$ for all k. Then $\{A_k\}$ is a sequence of disjoint measurable sets with $E_1 \backslash E = \bigcup_{k=1}^{\infty} A_k$. Then, since $\mu(E_1 \backslash E) = \mu(E_1) - \mu(E)$ and $\mu(A_k) = \mu(E_k) - \mu(E_{k+1})$ for all k, it follows that

$$\begin{aligned}
\mu(E_1) - \mu(E) &= \mu(E_1 \backslash E) \\
&= \sum_{k=1}^{\infty} \mu(A_k) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n-1} (\mu(E_k) - \mu(E_{k+1})) \\
&= \lim_{n\to\infty} (\mu(E_1) - \mu(E_n)) \\
&= \mu(E_1) - \lim_{n\to\infty} \mu(E_n)
\end{aligned}$$

Therefore, we find that $\mu(E) = \lim_{n\to\infty} \mu(E_n)$.

□

[6]This is a consequence of the fact that bounded monotone sequences converge, a proof of which can be found in Gordon [3].

With this result in hand, we have finished our investigation of the properties of Lebesgue measure. As mentioned in the introduction, the aim of this piece is twofold: to develop the basics of measure theory on the real line, and then to use those methods to construct and explore a problematic example that sheds light on the theoretical shortcomings of the Riemann integral. Through this process we will gain insight into what motivated Lebesgue to develop his revolutionary new integration process, the Lebesgue integral.

3. Riemann Integration

3.1. Background and Notation.

For over 200 years following the development of calculus by Newton and Leibniz, mathematicians were differentiating and integrating functions with what some would call reckless abandon. They were astounded by the fact that calculus so elegantly solved previously-intractable problems, and that for lack of a better description, it "just worked." It was not until the 19$^{\text{th}}$ century however that mathematicians began to devote themselves to the problem of putting calculus on a firm theoretical footing. Questions began to be asked such as "What exactly does it *mean* to take an integral of a function?" and "What are the conditions under which integration can be performed?" Although it was preceded by other definitions, the definition of the integral developed by Bernhard Riemann (1826-1866) has been regarded as the first real step towards a rigorous definition of an integration process. The Riemann integral works by partitioning the x-axis on the interval over which we wish to integrate and then choosing from each subinterval in the partition a point known as a tag. For each subinterval, we multiply the functional value at the tag by the width of the subinterval to get an area. A Riemann sum is what results when we add up these areas for each subinterval in the partition. The Riemann integral is defined as the limiting value of these sums as the number of subintervals goes to infinity *for any tagged partition.* If different tagged partitions yield different values, the function is not Riemann integrable.

We will define a few key terms and some notation before giving the formal definition of the Riemann integral.

Definition 3.1. *A* **partition** P *of an interval* $[a, b]$ *is a finite set of points* $\{x_i : 0 \le i \le n\}$ *such that*

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Given a partition P *of* $[a, b]$*, the* **norm** *of the partition, denoted by* $\|P\|$*, is the length of the largest subinterval in the partition, or*

$$\|P\| = \max \{x_i - x_{i-1} : 1 \le i \le n\}$$

A **tagged partition** ${}^tP$ *of an interval* $[a, b]$ *is a partition* P *along with a set of points* $\{t_i : 1 \le i \le n\}$ *such that* $x_{i-1} \le t_i \le x_i$ *for* $1 \le i \le n$*. Often* *this is expressed as* ${}^tP = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$

The **norm of a tagged partition** ${}^tP$*, denoted* $\|{}^tP\|$*, is simply the norm of the partition associated with* ${}^tP$*.*

Given a tagged partition ${}^tP = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ *of* $[a, b]$*, the* **Riemann sum** *of a function* $f : [a, b] \to \mathbb{R}$ *is defined by*
$S(f, {}^tP) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$.

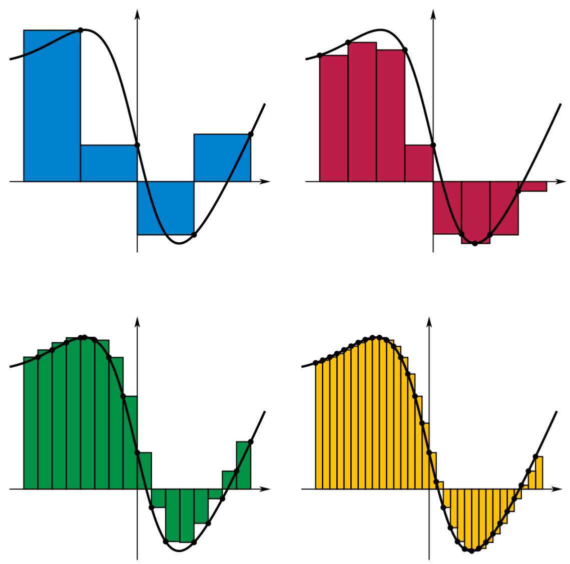

FIGURE 2. Visual representation of four different Riemann sums of a function, using different partitions and tags.
Source: Wikimedia Commons, Riemann Sum

3.2. **Riemann Integrability.** In a typical calculus course, the Riemann integral is often defined as a limit of Riemann sums using explicitly-determined tags, such as right-hand endpoints (see Figure 2). This simplistic approach, while often necessary given the scope of the course, can lead to a distorted perception of the size and nature of the collection of Riemann integrable functions. Rarely are calculus students exposed to the enlightening and pathological counterexamples that to a large extent shaped and motivated the development of analysis into its sophisticated modern incarnation. These sections are intended as a brief survery of the theory of Riemann integration as it relates to measure theory. Ultimately, we will use these tools to construct the particularly problematic counterexample of a function whose derivative is bounded but not Riemann integrable. This presentation is intended to be done in such a way as to accentuate the interconnectedness of the theory and the counterexample that undermines it, as this is a common theme seen throughout mathematics. With that said, let us turn our attention to the Riemann integral.

Definition 3.2. *A function $f : [a, b] \to \mathbb{R}$ is* **Riemann integrable** *on $[a, b]$ if there exists a number $\int_a^b f$ such that for every $\epsilon > 0$ there exists $\delta > 0$ such that $|S(f, {}^tP) - \int_a^b f| < \epsilon$ for all tagged partitions ${}^tP$ of $[a, b]$ that satisfy $\|{}^tP\| < \delta$. The number $\int_a^b f$ is called the integral of f on $[a, b]$.*

This is quite a fearsome looking definition given the number of quantifiers. What it essentially says though is that a function is Riemann integrable if using *any* tagged partition with a sufficiently small norm we can make the value of the Riemann sum as close as we wish to the actual value of the integral.

3.3. **Oscillation and Integrability.** Determining the integrability of functions can often be difficult using the definition since it requires that one already know the value of the integral. Using the notion of the oscillation of a function, however, we are given a very powerful tool in determining integrability. First, we will examine oscillation itself.

Definition 3.3. *The* **oscillation of a function f on an interval** $[a, b]$, *denoted $\omega(f, [a, b])$ is defined as*

$$\omega(f, [a, b]) = \sup \{f(x) : x \in [a, b]\} - \inf \{f(x) : x \in [a, b]\}.$$

The **oscillation of a function f at a point** *x is defined as*

$$\omega(f, x) = \lim_{r \to 0^+} \omega(f, [x - r, x + r]).$$

In other words, oscillation on an interval is the difference between the least-upper and greatest-lower bounds of the function on that interval. In many cases, this is simply the maximum minus the minimum values of the function on the given interval. In regards to oscillation at a point, if f is a bounded function then the limit in question is guaranteed to exist.[7] As mentioned previously, the notion of oscillation gives us access to a very powerful and useful method by which to determine whether or not a function is Riemann integrable. The proof is rather involved, so the interested reader is directed to Gordon [3].

Theorem 3.4. *Let f be a bounded function defined on $[a, b]$. Then f is Riemann integrable on $[a, b]$ if and only if for each $\epsilon > 0$ there exists a partition $P = \{x_i : 0 \le i \le n\}$ of $[a, b]$ such that*

$$\sum_{i=1}^{n} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) < \epsilon.$$

[7]See Gordon [3]

Aside from the fact that one need not know the value of the integral in advance, this method is useful in that it only requires one to find a single partition on which the condition holds, as opposed to using the definition, where the conditions in that case must hold for every tagged partition with a sufficiently small norm.

3.4. **Continuity Almost Everywhere.** While the effectiveness of methods for determining Riemann integrability depend largely on the context in which they are used, determing integrability by way of oscillation (as presented in Theorem 3.4) will be very useful in our quest of constructing a function whose derivative is not Riemann integrable. Prior to that however, the reader should be made aware of the somewhat odd mathematical term "almost everywhere." This is a measure-theoretic term which means that a property holds **almost everywhere** if it holds everywhere on a domain except on a set with measure zero. Mathematicians of the late 19th and early 20th centuries were concerned with pinning down exactly how and why the Riemann integral fails in certain cases, and the answer to that question is today known as **Lebesgue's criterion for Riemann integrability**, which states that a bounded function f is Riemann integrable if and only if it is continuous almost everywhere. That is, it is integrable if and only if its set of discontinuities has measure zero. We will prove one half of this bidirectional result by showing that Riemann integrability implies continuity almsot everywhere.

Theorem 3.5. *If a function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, then f is bounded and continuous almost everywhere on $[a, b]$.*

Proof. A proof of the fact that Riemann integrability implies boundedness can be found in Gordon [3]. To prove continuity almost everywhere, it is sufficient to prove that the set D of discontinuities of f on (a, b) has measure zero. For each positive integer n, let $D_n = \{x \in D : \omega(f, x) \geq 1/n\}$ Since $D = \bigcup_{n=1}^{\infty} D_n$, we need only show that an arbitrary D_n has measure zero, since a countable union of measure zero sets will have measure zero. Therefore, we fix n and let $\epsilon > 0$. Since f is Riemann integrable on $[a, b]$, by Theorem 3.4 there exists a partition $P = \{x_i : 0 \leq i \leq p\}$ of $[a, b]$ such that

$$\sum_{i=1}^{p} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) < \epsilon/2n,$$

Let us then define a set S_0 consisting of the indices of the intervals in the partition that contain members of D_n by

$$S_0 = \{i : D_n \cap (x_{i-1}, x_i) \neq \emptyset\}.$$

Since $\omega(f, [x_{i-1}, x_i]) \geq 1/n$ for each $i \in S_0$ and $S_0 \subseteq [a, b]$, we find that

$$\begin{aligned}\sum_{i \in S_0} (x_i - x_{i-1}) &= n \sum_{i \in S_0} \frac{1}{n} (x_i - x_{i-1}) \\ &\leq n \sum_{i \in S_0} \omega(f, [x_{i-1}, x_i]) (x_i - x_{i-1}) \\ &\leq n \sum_{i=1}^{p} \omega(f, [x_{i-1}, x_i]) (x_i - x_{i-1}) \\ &< n \cdot \epsilon/2n \\ &= \epsilon/2\end{aligned}$$

Notice that when we defined $S_0 = \{i : D_n \cap (x_{i-1}, x_i) \neq \emptyset\}$ we used an open interval, and thereby excluded the endpoints x_{i-1} and x_i for each i. Remember that this proof is concerned with showing D_n to be a subset of a union of open intervals whose total length can be made less than ϵ (this is the definition of measure zero). To account for all the "holes" in this covering set caused by defining S_0 using open intervals (x_{i-1}, x_i), we define a union of open intervals centered around each endpoint by $\bigcup_{i=1}^{p-1} (x_i - \epsilon/4p, x_i + \epsilon/4p)$. In this way, we find that

$$D_n \subseteq \bigcup_{i \in S_0} (x_{i-1}, x_i) \cup \bigcup_{i=1}^{p-1} (x_i - \epsilon/4p, x_i + \epsilon/4p),$$

and the sum of the lengths of all these intervals is

$$\sum_{i \in S_0} (x_i - x_{i-1}) + \sum_{i=1}^{p-1} \frac{\epsilon}{2p} < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus, D_n has measure zero, since it can be covered by a countable union of open intervals whose total length can be made arbitrarily small. Therefore, we conclude that f is continuous almost everywhere on $[a, b]$. □

4. **The Cantor Set**

4.1. **History.** As mentioned previously in Section 3.2, the history of mathematics is dotted with instances of particularly insightful and problematic counterexamples serving as the catalyst for the development of more sophisticated and robust theories. Perhaps the earliest example of this was the legendary discovery of irrational numbers by Pythagoreans in Ancient Greece, which forced them to reevaluate their basic tenet that all measurements were commensurable (i.e. every number is rational). In the same manner, in the late 19^{th} century, three mathematicians independently discovered a family of sets of real numbers, now known as **Cantor-like sets**, which served as the cornerstone of a serious challenge to the efficacy of the Riemann integral.

Although the Cantor-like sets are named after Georg Cantor (1845-1918), he was not the first to discover them. An example of a Cantor-like set was first discovered by the English mathematician Henry J.S. Smith (1826-1883) in 1875.[8] However, few paid attention to results coming from England at this time, as the prestigious research universities of Germany were the center of the mathematical world, so Smith's work went unnoticed. Similar anonymity was shared by Italian physics graduate student Vito Volterra (1860-1940) when in 1881 he published a discovery of a similar set in an obscure Italian journal. In 1883, however, Cantor "discovered" the canonical example of the sets that now bear his name, the famous **Cantor set,** K.[9] While the specifics of the Cantor set will be discussed in subsequent sections, the defining feature of every Cantor-like set is that they are constructed in an infinite number of steps, beginning with the removal of a constant proportion from the middle of the interval $[0,1]$. The second step is to remove that same proportion from the middle of the two subintervals of $[0,1]$ created in the previous step, resulting in four subintervals. Likewise, the third step is to remove that same proportion from the middle of the four subintervals created in step two. This process is repeated *ad infinitum*, and the desired Cantor-like set is the resultant set of this limiting process. Thus, each individual Cantor-like set is distinguished by the proportion that was removed from each subinterval of $[0,1]$ in each step of its construction. For example, the Cantor set is created by successively removing 3^{-n} in the n^{th} step, whereas another variant known as the Smith-Volterra-Cantor set is created by removing 4^{-n} in the n^{th} step. While there are an infinite number of Cantor-like sets, this section is devoted to explicitly constructing these two famous Cantor-like sets and investigating their fascinating properties. To give a bit of a hint of what's to come, the Smith-Volterra-Cantor set will be used in constructing a function with a bounded derivative that is not Riemann integrable.

4.2. **The Cantor Set.** As previously mentioned, the **Cantor set** K is constructed in an infinite number of recursive steps. All Cantor-like sets are constructed by successively removing a constant proportion from the middle of $[0,1]$, and then removing this same proportion from every subinterval in an infinite number of iterations. The Cantor set is created by removing middle thirds. Let K_n denote the sets created in the n^{th} step of the process. Then, the first three terms in the construction of the Cantor set are:

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8 [1]

9 The Cantor set is variously referred to in the literature as the **Cantor ternary set** and **SVC(3)**.

$$K_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$
$$K_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$
$$K_3 = \left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{3}{27}\right] \cup \left[\frac{6}{27}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{9}{27}\right] \cup \left[\frac{18}{27}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{21}{27}\right] \cup \left[\frac{24}{27}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, 1\right].$$

The Cantor set K is what is left over when we continue this casting out of middle thirds indefinitely. Equivalently, K is the intersection of all the K_n terms:

$$K = \bigcap_{n=1}^{\infty} K_n.$$

FIGURE 3. A visual representation of the first five steps in the construction of the Cantor set K, in which we continually remove middle thirds.

This definition of K as an infinite intersection gives us a method to investigate both the measure-theoretic and analytical properties of the Cantor set. As it turns out, K has measure zero, a property that is not in itself too surprising. What is truly fascinating however is the fact that the Cantor set K is also uncountable! While we won't include a full proof of this fact, it is not too difficult to show that the elements of the Cantor set are precisely those numbers between 0 and 1 whose base-3 decimal expansions contain no ones (i.e. they contain only zeroes and twos). With these ternary expansions, we can use Cantor's diagonal argument to prove the uncountability of the Cantor set. We can also prove uncountability by way of the Nested Intervals Theorem. Bressoud [1] and Gordon [3] feature these two proofs, respectively. Despite not explicitly proving the uncountability of the Cantor set, we can in fact easily prove that K has measure zero in two ways: directly, by finding the measure of each K_n and using the limit properties of measure; and indirectly, by subtracting from 1 (the measure of $[0, 1]$) the measure of what we removed in constructing K. First, we will tackle the direct measurement of K.

After the n^{th} step of constructing K, we have 2^n disjoint intervals, each of length of $\left(\frac{1}{3}\right)^n$. To see why this is true, note that after the first step we have two intervals of length $\frac{1}{3}$; after the second, four of length $\frac{1}{9}$; and so forth. Then, the sequence $\{K_n\}$ is a sequence of nested closed intervals, and so we know that for each n, $\mu(K_n) = \ell(K_n) = 2^n \cdot \left(\frac{1}{3}\right)^n = \left(\frac{2}{3}\right)^n$, because the measure of an interval is its length and the measure of a collection of disjoint sets is simply the sum of their individual measures. Since $K = \bigcap_{n=1}^{\infty} K_n$, using Theorem 2.9 and the properties of geometic series, we find the measure of K to be

$$\mu(K) = \lim_{n\to\infty} \mu(K_n) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$$

Thus, the Cantor set K has measure zero, yet is uncountable.

We can confirm that the Cantor set has measure zero indirectly by measuring the total length of what was removed during the construction of K. Note that in the first step, we removed one interval of length $\frac{1}{3}$. In the second, we removed two intervals each of length $\frac{1}{9}$. In the third, four intervals each of length $\frac{1}{27}$. Using this pattern and the properties of geometric series, we find that

$$\begin{aligned}
\mu(K) &= \mu([0,1] \setminus (\mu([0,1] \setminus K))) \\
&= 1 - \left(1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{9} + 3 \cdot \frac{1}{27} + \cdots\right) \\
&= 1 - \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} \\
&= 1 - \frac{\frac{1}{3}}{1 - \frac{2}{3}} \\
&= 1 - 1 \\
&= 0.
\end{aligned}$$

One advantage of this indirect method is that aside from equating measure of an interval with length, it does not rely on measure theory at all, thus allowing us to confirm to ourselves via a practical example that those methods used in the direct method are in fact sound.

The fact that the Cantor set is uncountable yet has measure zero is quite suprising and counterintuitive, since our discussion of measure theory thus far would seem to hint that every measure zero set is countable. The Cantor set shows this is not true. But as was stated above, the Cantor set is but one of an entire family of Cantor-like sets with interesting measure theoretic properties. The next example of such a set is constructed very similarly to the Cantor set, but has positive measure.

5. Cantor-Like Sets

5.1. **The Smith-Volterra-Cantor Set.** The previous section introduced the family of Cantor-like sets through consideration of the eponymous member thereof, the Cantor set. Here, we present another Cantor-like set known as the **Smith-Volterra-Cantor set**[10], or the **SVC** for short. While in

[10]Referred to in some texts, such as Bressoud [1], as SVC(4).

the case of the Cantor set we removed 2^{n-1} intervals of length 3^{-n} from $[0,1]$ to get an uncountable set of measure zero, with the SVC we remove 2^{n-1} intervals of length 4^{-n} to get a set that contains no intervals yet has positive measure. Much like the Cantor set, we begin constructing the SVC by removing the middle fourth of $[0,1]$ and denote what remains as S_1. We then remove 1/16 from the two intervals created in the previous step, denoting what remains as S_2, and continue this casting out until we have an infinite sequence of sets $\{S_n\}$.

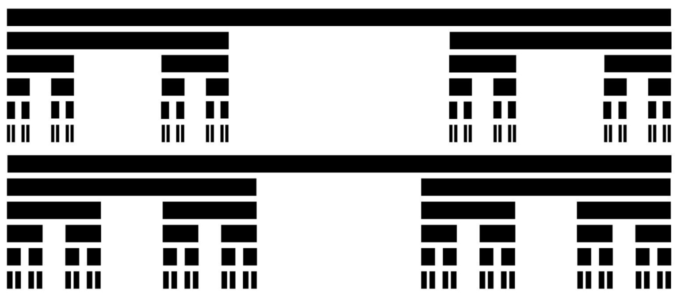

FIGURE 4. A visual representation of $[0,1]$ and the first five steps in the construction of the Cantor set (top) and the SVC (bottom).

We denote the SVC as S and define it as the intersection of the S_ns:

$$S = \bigcap_{n=1}^{\infty} S_n.$$

It can be seen in Figure 4 that less is removed in each step of construction of the SVC than the Cantor set. From this alone, we can at least make a heuristic argument that the measure of the SVC will be larger than that of the Cantor set, which would seem to indicate that the SVC has positive measure. While this is indeed the case (as will be shown shortly), it should be noted that intuition is not always a reliable guide in the field of measure theory. The very fact that the Cantor set is uncountable with measure zero flies in the face of intuition, as does the still-unproven fact that the SVC contains no intervals yet has positive measure. For this reason, we must be very careful to rely primarily on the analytical tools at our disposal rather than clinging to intuition alone. With this in mind, let us begin our investigation of the SVC.

As with the Cantor set, we can measure the SVC both directly and indirectly. To measure it directly, we find the measures of the first few members of S_n and appeal to the principle of mathematical induction to get a general formula for the measure of S_n. Consulting Figure 4 and a bit of algebra yields the following:

$$\{\mu(S_1), \mu(S_2), \mu(S_3), \mu(S_4), \mu(S_5), \mu(S_6), \cdots\}$$
$$= \left\{\frac{3}{4}, \frac{5}{8}, \frac{9}{16}, \frac{17}{32}, \frac{33}{64}, \frac{65}{128}, \cdots\right\}.$$

This would seem to imply that

$$\mu(S_n) = \frac{2^n + 1}{2^{n+1}}.$$

Since for every positive integer n the set S_n is composed of 2^n intervals, then for every n let s_n denote one of those intervals. In order to prove the above equation for the measure of S_n holds, we will prove by induction that

$$\mu(s_n) = \frac{2^n + 1}{2^{2n+1}},$$

so that

$$2^n \cdot \mu(s_n) = 2^n \cdot \frac{2^n + 1}{2^{2n+1}} = \frac{2^n + 1}{2^{n+1}}.$$

First, let $n = 1$ and note that s_1, typical interval in S_1 has measure $\frac{3}{8} = \frac{2^1+1}{2^{2\cdot 1+1}}$. Thus the base case holds. Assume then for some positive integer k that $\mu\left(s_k\right) = \frac{2^k+1}{2^{2k+1}}$. Then to find the measure of s_{k+1} we remove $4^{-(k+1)}$ from the middle of s_k, denote one of the resultant intervals s_{k+1}, and take its length. Thus, by this method and the induction hypothesis, we find

$$\begin{aligned}
\mu(s_{k+1}) &= \frac{1}{2} \cdot \frac{2}{2} \cdot \left(\mu(s_k) - 4^{-(k+1)}\right) \\
&= \frac{1}{2} \cdot \frac{2}{2} \cdot \left(\frac{2^k + 1}{2^{2k+1}} - \frac{1}{2 \cdot 2 \cdot 4^n}\right) \\
&= \frac{1}{4} \cdot \frac{2}{2} \cdot \left(\frac{2^k + 1}{4^k} - \frac{1/2}{4^n}\right) \\
&= \frac{1}{4} \cdot \frac{2}{2} \cdot \frac{2^k + 1/2}{4^k} \\
&= \frac{1}{4} \cdot \frac{2^{k+1} + 1}{2 \cdot 4^k} \\
&= \frac{2^{k+1} + 1}{2^{2(k+1)+1}}
\end{aligned}$$

Thus, we conclude for all positive integers n that

$$\mu(s_n) = \frac{2^n + 1}{2^{2n+1}} \quad \text{and} \quad \mu(S_n) = 2^n \cdot \mu(s_n) = \frac{2^n + 1}{2^{n+1}}.$$

Using this expression and the limit properties of Lebesgue measure, we can find the measure of the SVC:

$$\mu(S) = \lim_{n\to\infty} \mu(S_n) = \lim_{n\to\infty} \frac{2^n + 1}{2^{n+1}} = \frac{1}{2} \cdot \lim_{n\to\infty} \left(1 + \frac{1}{2^n}\right) = \frac{1}{2}.$$

Thus, as we suspected, the SVC has positive measure of $1/2$. To confirm this, we can prove indirectly that the SVC has measure $1/2$ by measuring what we took away from $[0, 1]$ during the construction process. Recall that in the first step, we removed one interval of length 1/4. In the second step, we took away two intervals each of length 1/16. In the third, four of length 1/64, and so on. In this way, we find the measure of $\{[0,1]\setminus S\}$ to be

$$\mu([0,1]\setminus S) = \left(\frac{1}{4} + \frac{2}{16} + \frac{4}{64} + \cdots\right) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{4^n} = \frac{\frac{1}{4}}{1-\frac{1}{2}} = \frac{1}{2}.$$

It was previously mentioned that the SVC has the curious property of having positive measure yet containing no intervals. In the context of the real line, this is equivalent to having the property of being nowhere dense, which is described and proven in the following theorem.

Theorem 5.1. *The Smith-Volterra-Cantor set S is closed and nowhere dense* **nowhere dense**. *That is, it contains no intervals.*

Proof. Since in constructing the SVC out of $[0,1]$ we removed a countable collection of open intervals, which themselves form an open set, we conclude that S is in fact closed. Let x and y be distinct points in S, and assume without loss of generality that $x < y$. We then choose a positive integer n such that $\frac{2^n+1}{2^{2n+1}} < y - x$. Since S_n is comprised of 2^n intervals of total length $\frac{2^n+1}{2^{n+1}}$, then each subinterval in S_n has length $\frac{2^n+1}{2^{2n+1}}$. Furthermore, since the distance between x and y is greater than $\frac{2^n+1}{2^{2n+1}}$, x and y cannot lie in the same subinterval. Because the subintervals that comprise S_n are closed and disjoint, there must be some point $z \notin S$ between x and y. Since the choices of x and y were arbitrary, it follows that between any two points in the SVC there exists a point not in the set. Thus, S cannot contain any intervals. □

While not entirely obvious, without too much trouble one can convince oneself that the points making up the SVC include the endpoints of every interval removed in the construction process. For example, in the first step we removed the interval $\left(\frac{3}{8}, \frac{5}{8}\right)$, and since we will continue removing middles of varying length, the points $\frac{3}{8}$ and $\frac{5}{8}$ will never be cast out as a part of any open interval of length 4^{-n}. Since the SVC contains no intervals and the endpoints of the intervals are never cast out during the construction process, we conclude that the endpoints of every interval removed in construction are elements of the SVC. It should be noted however that there are other points in the SVC also. Stay tuned for the discussion in Section 6.4.

This concludes our exploration of the SVC and its properties. To conclude this section, we will quickly discuss two extreme examples of Cantor-like sets.

5.2. **The Cantor-ϵ Set.** So far, we have discussed two important Cantor-like sets: The Cantor set, which is uncountable with measure zero; and the SVC, which has positive measure yet contains no intervals. A common question when investigating the family of Cantor-like sets is "Can we construct a Cantor-like set with any measure between 0 and 1 that we wish?" Interestingly enough, the answer to this question is yes. Here we present a proof of this result using a method similar to that employed by Bressoud [1].

We will call this generalized set the **Cantor-ϵ set**, and begin by pointing out that it should be clear by now that the measure of a Cantor-like set is completely determined by the fraction removed in each step of its construction. Since removing middle thirds results in a set of measure zero, we will restrict ourselves to considering what happens when we remove a fraction of the form $\frac{1}{3+\epsilon}$, where ϵ is an arbitrary positive number. The Cantor-ϵ set, denoted $\mathcal{E}$, is the set created when we remove $\frac{1}{3+\epsilon}$ from the middle of $[0,1]$ and continually remove the same proportion from the middle of the subsequent intervals, just as we did with the Cantor set and the SVC. The following theorem states and provides proof of the properties of the set $\mathcal{E}$.

Theorem 5.2. *For any $\epsilon > 0$, the Cantor-ϵ set $\mathcal{E}$ is a generalized Cantor-like set with measure $\frac{\epsilon}{1+\epsilon}$. That is, it is possible to construct a Cantor-like set whose measure is any value strictly between 0 and 1.*

Proof. Let $\epsilon > 0$. Since $\mathcal{E}$ is a Cantor-like set, we can find its measure indirectly by finding the total measure of what we removed from $[0,1]$ in constructing $\mathcal{E}$. In the first step, we removed one interval of length $\frac{1}{3+\epsilon}$. In the second step, two intervals each of length $\left(\frac{1}{3+\epsilon}\right)^2$, and so on. It follows that

$$\begin{aligned}
\mu([0,1]\setminus\mathcal{E}) &= \left(1\cdot\left(\frac{1}{3+\epsilon}\right)+2\cdot\left(\frac{1}{3+\epsilon}\right)^2+4\cdot\left(\frac{1}{3+\epsilon}\right)^3+\cdots\right)\\
&= \frac{1}{2}\cdot\sum_{n=1}^{\infty}\left(\frac{2}{3+\epsilon}\right)^n\\
&= \frac{1}{2}\cdot\left(\frac{\frac{2}{3+\epsilon}}{1-\frac{2}{3+\epsilon}}\right)\\
&= \frac{1}{3+\epsilon}\cdot\frac{3+\epsilon}{1+\epsilon}\\
&= \frac{1}{1+\epsilon}.
\end{aligned}$$

Therefore,

$$\mu(\mathcal{E}) = 1-\mu([0,1]\setminus\mathcal{E}) = 1-\frac{1}{1+\epsilon} = \frac{\epsilon}{1+\epsilon}.$$

Also, note that for any positive ϵ,

$$\frac{\epsilon}{1+\epsilon} < 1 \qquad \text{and} \qquad \lim_{\epsilon\to\infty}\frac{\epsilon}{1+\epsilon} = 1.$$

□

Thus, one can use the formula for $\mu(\mathcal{E})$ to choose an appropriate ϵ and use it to construct a Cantor-like set with any measure between 0 and 1. It should also be noted that since the function $f:[0,\infty)\to\mathbb{R}$ defined by $f(x)=\frac{x}{1+x}$ is bijective, we can conclude that no two distinct Cantor-like sets have the same measure.

This concludes our discussion of Cantor-like sets and their measure-theoretic properties. After a brief discussion of some more advanced topics from point-set topology in the next section, we will begin the culmination of this piece: the construction of a differentiable function with a bounded derivative that is not Riemann integrable.

5.3. **Topology and Perfect, Nowhere Dense Sets*.** Thus far, our discussion of Cantor-like sets has been constructive. That is, we have been concerned with the construction and analysis of two specific Cantor-like sets, namely the Cantor set and the SVC. In this section, our exploration will take on a decidedly more theoretical flavor and consist of, among other things, proofs about the properties of what are known as **perfect, nowhere dense sets**, of which Cantor-like sets are a subset. To begin, we will define each of these types of sets in turn.

Definition 5.3. *A set E is* **perfect** *if it is composed entirely of its own limit points. That is, if $E' = E$.*

Definition 5.4. *A set E is* **nowhere dense** *if its closure $\bar{E}$ contains no intervals.*

We can see by definition that if a set is perfect, it contains all of its limit points and is thereby closed. From this, we can conclude that a perfect, nowhere dense set is closed, and since for a closed set E, $\bar{E} = E$, it follows that a perfect nowhere dense set contains no intervals.

Recall that in the previous section we gave a constructive proof of a Cantor-like set that is a subset of $[a, b]$ whose measure is any arbitrary value between 0 and 1. Here, we will prove a more general result that for any measurable set with finite positive measure, we can find both an open and a closed set whose respective measures are arbitrarily close to that of the original set.

Theorem 5.5. *Suppose E is a measurable subset of $[a, b]$. Then,*

(1) For any $\epsilon > 0$ there exists an open set O such that $E \subseteq O$ and $\mu(O \backslash E) < \epsilon$.
(2) For any $\epsilon > 0$ there exists a closed set F such that $F \subseteq E$ and $\mu(E \backslash F) < \epsilon$.

Proof. **(a)** Let $\epsilon > 0$ and let $\{I_n\}$ be a sequence of open intervals with

$$E \subseteq \bigcup_{n=1}^{\infty} I_n \subseteq [a, b] \qquad \text{and} \qquad \sum_{n=1}^{\infty} \ell\left(I_n\right) < \mu(E) + \epsilon.$$

* This section, which presupposes the reader is familiar with basic point-set topology, can be skipped without loss of continuity.

Let $O = \bigcup_{n=1}^{\infty} I_n$. Then O is an open set and $O = E \cup (O \backslash E)$, where E and $(O \backslash E)$ are disjoint. It follows that

$$\mu(O \backslash E) = \mu(O) - \mu(E) \leq \sum_{n=1}^{\infty} \ell\,(I_n) - \mu(E) = \epsilon.$$

(b) We can prove part (b) through the use of the result from part (a) and by taking complements. Using the set E from part (a) defined on the interval $[a, b]$, we then let $\epsilon > 0$ and let G be an open subset of $[a, b]$ such that $[a, b] \backslash E \subseteq G$ and $\mu\big(G \backslash E^{\mathcal{C}}\big) < \epsilon$. If we define $F = [a, b] \backslash G$, then since $[a, b] \backslash E \subseteq G$ implies that $F = [a, b] \backslash G \subseteq E$ it follows that F is closed and that

$$E \backslash F = E \cap F^{\mathcal{C}} = E \cap G = G \backslash E^{\mathcal{C}},$$

and therefore

$$\mu(E \backslash F) = \mu\big(G \backslash E^{\mathcal{C}}\big) < \epsilon.$$

□

It should be clear that while the constructive methods used in the previous section have a certain concreteness to them, this result is much more powerful since it tells us that for *any* set with positive finite measure $\mu(E)$, we can find open and closed sets whose measures are as close to $\mu(E)$ as we wish. Nowhere does the result state that the set has to be Cantor-like or defined on $[0, 1]$.

We showed previously that the SVC contains no intervals. As it turns out, this same result is true for the Cantor set (and indeed for any Cantor-like set). What we have not mentioned however is that every Cantor-like set is nonempty, perfect and nowhere dense. Since Cantor-like sets are closed and we know they contain no intervals, we have actually unwittingly already proven that the SVC is nowhere dense. In the case of the Cantor set, a proof that it is nonempty, perfect, and nowhere dense can be found both in Gordon [2] and [3].

While it might seem that perfect sets with arbitrary measures are a mathematical oddity specific to the Cantor-like sets, it turns out that such sets are actually very common. In fact, for any set with finite positive measure, there exists a perfect set whose measure is arbitrarily close to that of the original set. The following definition and the subsequent theorem will help us in showing this to be the case.

Definition 5.6. *Let E be a set of real numbers. A point x is a* **condensation point** *of E if for any open set O containing x, the set $(O \cap E)$ is uncountable.*

Theorem 5.7. *Any closed set can be expressed as the union of a perfect set and a countable set.*

Proof. Let E be a closed set of real numbers, and let E_c be the set of all condensation points of E. Note that $E = (E \backslash E_c) \cup E_c$. We will prove first that the set $E\backslash E_c$, the set of all points in E that are not condensation points of E, is countable. Let $x \in E\backslash E_c$. Then, define the set

$$E_n = \left\{ x \in E : \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \cap E \text{ is countable} \right\}.$$

Since for every $x \in E\backslash E_C$ there exists a positive integer n such that $(x - 1/n, x + 1/n) \cap E$ is countable, we can conclude that

$$E\backslash E_c = \bigcup_{n=1}^{\infty} E_n.$$

Furthermore, since a countable union of countable sets is itself countable, we need only prove that each E_n is countable. Let us then fix n, and for every integer i define $E_n^i = E_n \cap \left[\frac{i}{n}, \frac{i+1}{n}\right)$. Suppose that $x \in E_n^i$. Then,

$$E_n^i \cap \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \subseteq E \cap \left(x - \frac{1}{n}, x + \frac{1}{n} \right)$$

Since $E \cap \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ is countable, we are led to conclude that ${E_n}^i$ is countable. Furthermore, since

$$E_n = \bigcup_{i=-\infty}^{\infty} {E_n}^i,$$

we conclude that E_n is a countable set, since a countable union of countable sets is itself countable. Since our choice of E_n was arbitrary, it follows that $E\backslash E_C$ is countable.

We will now prove that the set of condensation points of E, E_c, is a perfect set. This will be done in two steps. First, by proving that E_c is closed, and then by showing that any arbitrary point in E_c is a limit point of E_c. We begin by letting x be a limit point of E_c. By the definition of a limit point, any open set O formed around x is such that $O \cap E_c$ contains another distinct point in E_c. Call this other point z. Since z is in $O \cap E_c$, we conclude by the definition of condensation point that $O \cap E$ is uncountable. And since x (the point with which we began) is also in $O \cap E$ (which is uncountable), then by virtue of O being an arbitrary open set, we conclude that x is actually a condensation point of E, and is thereby in E_c. Since our choice of $x \in E_c'$ was also arbitrary, it follows that E_c contains all of its limit points and is therefore a closed set.
To prove that every point in E_c is itself a limit point of E_c, let x be an arbitrary point in E_c. Recall that for any open set O containing x, the set $O \cap E$ must be uncountable. Also, since

$$O \cap E = O \cap ((E\backslash E_c) \cup E_c) = (O \cap (E\backslash E_c)) \cup (O \cap E_c),$$

we know that $(O \cap (E\backslash E_c))$ is countable, so therefore $(O \cap E_c)$ must be uncountable. Then by definition, x must be a limit point of E_c, since $(O \cap E_c)$ contains at least one other (in this case, uncountably many) point in E_c. Thus, E_c is a perfect set, and the proof is complete. $\square$

Thus, by Theorems 5.5 and 5.7, we can conclude that for any set of positive measure E, there exists a perfect set P and a countable set C whose union $P \cup C$ is closed and whose measure is as close to the measure of E as we wish. Since countable sets have measure zero by definition, it follows that for $\epsilon > 0$, $\mu(E \backslash P) < \epsilon$.

Finally, we will prove that the points that make up the SVC (and indeed any Cantor-like set) are the closure of the set of all endpoints of the intervals removed in the construction process. The reason we did not include this result in the sections about Cantor-like sets is that, like the other results in this section, its proof relies on the methods of point-set topology.

Theorem 5.8. *Let E be the set of endpoints of the open intervals removed from $[0, 1]$ in constructing the SVC. The Smith-Volterra-Cantor set S is equal to $\overline{E}$, the closure of E.*

Proof. Since we know that $E \subseteq S$, it then follows that $\overline{E} \subseteq \bar{S} = S$. For the other direction let $x \in S$ and let $\epsilon > 0$. Then since S is perfect, x is a limit point of S. By the definition of a limit point, $B(x, \epsilon) \cap S$ contains y, where $y \neq x$. Furthermore, since S is closed and nowhere dense and therefore contains no intervals, between x and y there must be an interval (a_k, b_k) that was removed in some step of the SVC's construction. Since $y \in B(x, \epsilon)$, it follows that $x \in B(a_k, \epsilon)$ and $x \in B(b_k, \epsilon)$. Therefore, x is a limit point of a_k and b_k. Since (a_k, b_k) was arbitrary, it follows that $x \in \overline{E}$. In addition, since x was arbitrary we conclude that $S \subseteq \overline{E}$. □

6. Volterra's Function and The Pitfalls of the Riemann Integral.

6.1. **Background.** We mentioned in the previous section that Italian graduate student Vito Volterra was one of the first to discover an example of a Cantor-like set 1881. As it turns out, this discovery was made in the context of constructing a function that today is widely used to show one of the theoretical shortcomings of the Riemann integral. This function, known as **Volterra's function** and denoted V, is the subject of this final section. Volterra's function deals a serious blow to the theoretical efficacy of the Riemann integral because V differentiable yet its derivative V' is bounded but not Riemann integrable. This is extremely problematic because it goes against the grain of the Fundamental Theorem of Calculus, which states that one can integrate a function's derivative to recover the original function. Volterra's function cannot be recovered from its derivative via the Riemann integral, and this fact among others was what eventually led to the development of more rigorous integration processes during the early part of the 20^{th} century.

Recall from Section 3.4, particularly the Lebesgue Criterion for Riemann Integrability (Theorem 3.5) that a bounded function f is Riemann integrable if and only if its set of discontinuities has measure zero. Volterra's function is constructed by combining the analytical properties of the function $x^2 \sin(1/x)$ with the measure-theoretic properties of the SVC in such a way that the derivative of Volterra's function, V', is bounded and discontinuous on a set with positive measure, and thereby violates the Lebesgue criterion. With this conceptual framework in our minds, let us begin constructing V.

6.2. **The Analytical Properties of $x^2 \sin(1/x)$.** The function $x^2 \sin(1/x)$ is essential to constructing Volterra's function, V. If we consider the piecewise function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

g is differentiable (and continuous) on $[0, 1]$ but its derivative g' is not continuous at 0. The differentiability of g but the discontinuity of g' is what will keep the derivative of Volterra's function from being Riemann integrable.

Recall that g is differentiable at 0 if the limit

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0}$$

exists. The definition of g seems to indicate that g is in fact differentiable at 0 and that $g'(0) = 0$. To prove this, we note that

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} x \sin(1/x) = 0.$$

Thus, g is differentiable at 0, and thereby continuous as well.

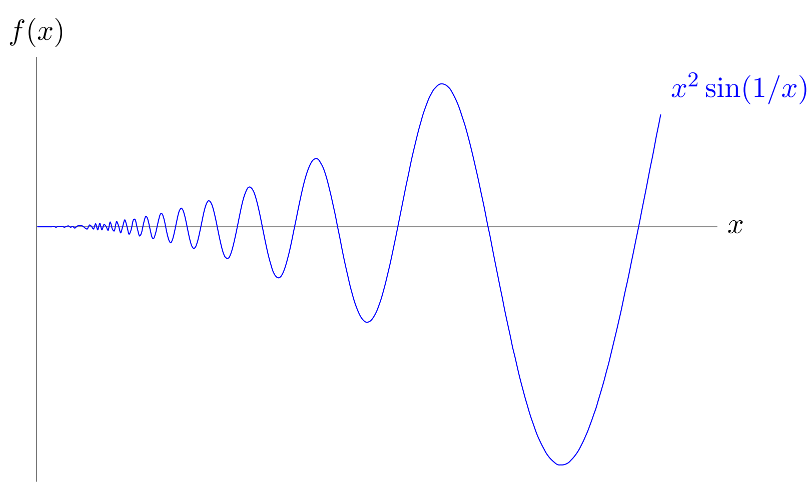

FIGURE 5. The function $x^2 \sin(1/x)$ oscillates wildly as it approaches the origin.

We can also show that the derivative of g, $g' : [0,1] \to \mathbb{R}$, is defined by

$$g'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Furthermore, we can see that g' is not continuous at 0 since the sequence $\{x_n\}$ defined on $[0,1]$ by

$$x_n = \frac{1}{\pi n}$$

converges to 0 but the sequence $\{|g'(x_n)|\}$ does not converge to $|g'(0)| = 0$ since $|g'(x_n)|$ equals 1 for all n. As far as boundedness is concerned, consideration of the equation for g' reveals that g' is bounded on $[0,1]$.

Thus, g is a function whose derivative is bounded and exists everywhere on $[0,1]$, but g' is not continuous at 0. It should also be noted that the function $g'(1-x)$ is bounded but discontinuous at 1. This fact will come in handy in subsequent sections.

6.3. **Constructing and Visualizing Volterra's Function.** We mentioned that the pathological behavior of Volterra's function is a result of combining the discontinuity of the derivative of the sinusoidal function $g(x) = x^2\sin(1/x)$, $x \neq 0$; 0, $x = 0$ with the measure-theoretic properties of the SVC. Recall that in each step of constructing the SVC, we removed a number of intervals. To construct Volterra's function, we will place into every one of the removed intervals a variant of the function g in such a way that this function will be differentiable yet its derivative will have a discontinuity at the endpoints of the interval into which it was placed. Since the SVC contains no intervals, then we will be able to show that the sum of all of those sinusoidal functions is differentiable but has discontinuities at every point of the SVC, which has positive measure. In this way, we will show that the derivative of Volterra's function is not Riemann integrable because it violates the Lebesgue Criterion for Riemann integrability.

It should be mentioned before we begin that our discussion of Volterra's function will come in two distinct parts. First, we will use the method of Bressoud [1] to construct Volterra's function and attempt to visualize it. Second, we will use the methods of Gordon [2] to show that Volterra's function V is differentiable, V' is bounded, but V' is not Riemann integrable. Take note that Bressoud and Gordon go about constructing Volterra's function in slightly different ways, each with their own strengths and weaknesses. Bressoud's method in step one is tailored specifically towards the SVC and allows for better visualizations of the construction process. For step two, Gordon's method is more general (it only requires that we use a perfect, nowhere dense set, of which the SVC is an example) and lends itself to more elegant proofs of the analytical properties of V and V'. We will attempt to use the best parts of both methods to give a more robust treatment of the Volterra's function and its properties.

Our construction of Volterra's function will parallel that of the SVC. In the first step of SVC construction, we removed the interval $\left(\frac{3}{8}, \frac{5}{8}\right)$ from $[0, 1]$. Eventually we are going to place a specially-designed function based on $x^2 \sin 1/x$ into this interval that we removed. First, however, we will begin by constructing that function in the interval $\left(0, \frac{1}{4}\right)$ so that the parallels between constructing Volterra's function and constructing the SVC can be seen in full. At first, our construction of this function will seem arbitrary and strange, but hopefully our reasons and methods will become clear as we proceed further. We will begin by considering the function

$$g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

on the interval $\left(0, \frac{1}{4}\right)$. First, we find the largest x between 0 and $\frac{1}{8}$ such that $g'(x) = 0$ and denote this point as a_1. We will restrict g to the interval $(0, a_1)$, and into $\left[a_1, \frac{1}{4} - a_1\right]$ we will insert the constant function $g(a_1)$. Finally, into the interval $\left(\frac{1}{4} - a_1, \frac{1}{4}\right)$ we place a reflection of g, $g\left(\frac{1}{4} - x\right)$. We will call this piecewise function f_1, defined as

$$f_1(x) = \begin{cases} 0, & x < 0 \\ g(x), & 0 < x < a_1 \\ g(a_1), & a_1 \leq x \leq 1/4 - a_1 \\ g(1/4 - x), & 1/4 - a_1 < x < 1/4 \\ 0, & 1/4 < x. \end{cases}$$

We can see both from the definition of f_1 as well as from its graph in Figure 6 that f_1 is differentiable on $\left(0, \frac{1}{4}\right)$ but its derivative f_1' will be discontinuous at 0 and $\frac{1}{4}$.

Our next step is to place this function into the interval removed in the first step of constructing the SVC. To do this, we define a new function h_1 to be f_1 translated $\frac{3}{8}$ units to the right. Thus, h_1 is defined as:

$$h_1(x) = \begin{cases} 0, & x < 3/8 \\ g(x - 3/8), & 3/8 < x < 3/8 + a_1 \\ g(a_1), & 3/8 + a_1 \leq x \leq 5/8 - a_1 \\ g(5/8 - x), & 5/8 - a_1 < x < 5/8 \\ 0, & 5/8 < x \end{cases}$$

Creating h_1 is the first step in constructing Volterra's function. We will repeat this process and create h_2, h_3, and so on, placing each function into the correct intervals. Note that while h_1 is placed into a single interval, in general h_n will consist of the differentiable sinusoidal function f_n placed into the 2^{n-1} intervals removed in the n^{th} step of the SVC's construction.

To give a better sense of the general procedure for constructing each h_n, we will now construct h_2. Since in the second step of SVC construction we removed the intervals $\left(\frac{5}{32}, \frac{7}{32}\right)$ and $\left(\frac{25}{32}, \frac{27}{32}\right)$, each having length $\frac{1}{16}$, we will

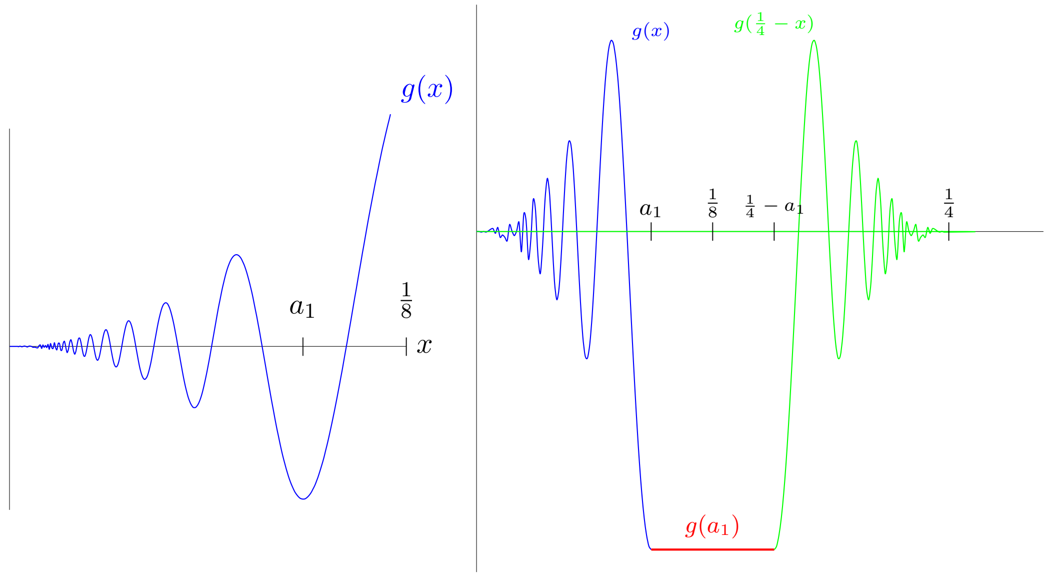

FIGURE 6. **Left:** We define a_1 as the largest x less than $\frac{1}{8}$ such that $g'(x) = 0$. **Right:** The three pieces that comprise the function f_1: the function g, its constant value at a_1, and a reflection of g. We can see that this function f_1 is differentiable (and continuous), but its derivative f_1' will have discontinuities at the endpoints 0 and $\frac{1}{4}$.

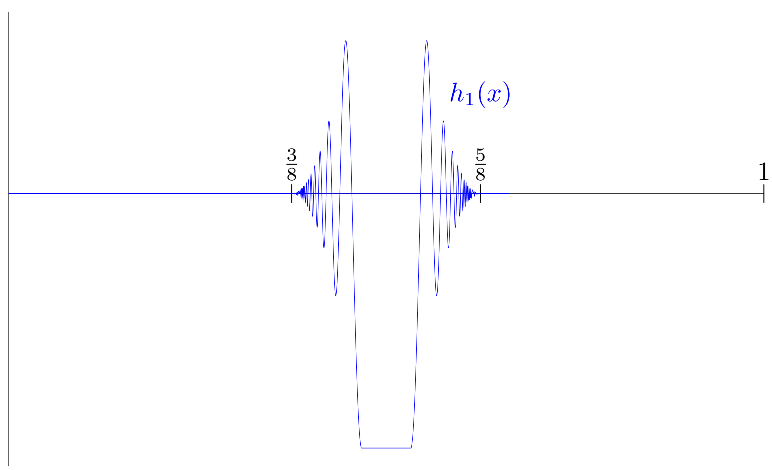

FIGURE 7. Our first step in constructing Volterra's function is to create the function h_1 defined on $\left(\frac{3}{8}, \frac{5}{8}\right)$, the interval removed in the first step of constructing the SVC. Notice how h_1 is simply f_1 translated into the desired interval.

begin by defining a_2 as the largest x less than $\frac{1}{32}$ where $g'(x) = 0$. We then

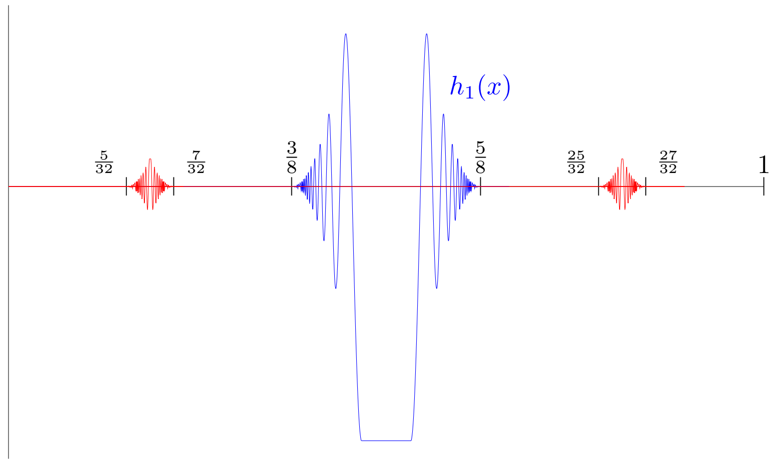

FIGURE 8. A plot of h_1 (blue) and h_2 (red). The derivative of h_1 has two discontinuities, occurring at the points $\frac{3}{8}$ and $\frac{5}{8}$. The derivative of h_2 has four discontinuities, which occur at the points $\frac{5}{32}$, $\frac{7}{32}$, $\frac{25}{32}$, and $\frac{27}{32}$. From this, we can surmise that in general, the derivative of h_n will have discontinuities at 2^n points, which by virtue of being endpoints of the intervals removed in constructing the SVC, are themselves elements of the SVC.

construct the function f_2 on the interval $\left(0, \frac{1}{16}\right)$, defined as

$$f_2(x) = \begin{cases} 0, & x < 0 \\ g(x), & 0 < x < a_2 \\ g(a_2), & a_2 \leq x \leq 1/16 - a_2 \\ g(1/16 - x), & 1/16 - a_2 < x < 1/16 \\ 0, & 1/16 < x. \end{cases}$$

We now have a differentiable function whose derivative is discontinuous at 0 and $\frac{1}{16}$. Our final step is to define h_2 to be two copies of f_2 transplanted into the intervals $\left(\frac{5}{32}, \frac{7}{32}\right)$ and $\left(\frac{25}{32}, \frac{27}{32}\right)$ respectively. Thus, h_2 is differentiable and its derivative h_2' is discontinuous at precisely the four endpoints of the two intervals. Remember that these endpoints are themselves points in the SVC.

We are now ready to define the general functions f_n and h_n. To begin, we note that in the n^{th} step of constructing the SVC, we remove 2^{n-1} open intervals of length 4^{-n}. We then define a_n to be the largest x less than

$\frac{1}{2} \cdot 4^{-n}$ where $g'(x) = 0$. The template function f_n is defined piecewise as

$$f_n(x) = \begin{cases} 0, & x < 0 \\ g(x), & 0 < x < a_n \\ g(a_n), & a_n \leq x \leq 4^{-n} - a_n \\ g(4^{-n} - x), & 4^{-n} - a_n < x < 4^{-n} \\ 0, & 4^{-n} < x. \end{cases}$$

The function h_n is then defined as the piecewise function consisting of f_n transplanted into the 2^{n-1} intervals removed in the n^{th} step of constructing the SVC.

Finally, we can explicitly define Volterra's function V.

Definition 6.1. Volterra's function $V : [0, 1] \to \mathbb{R}$ *is defined as*

$$V(x) = \sum_{n=1}^{\infty} h_n(x),$$

where h_n is the piecewise function consisting of 2^{n-1} copies of the sinusoidal template function f_n placed into the 2^{n-1} intervals of length 4^{-n} removed from $[0, 1]$ in the n^{th} step of constructing the Smith-Volterra-Cantor set.

As was mentioned previously, this section is not intended to prove the analytical properties of Volterra's function, but rather to help visualize it and show how its construction parallels that of the SVC. To actually prove that V is differentiable and that V' is bounded and not Riemann integrable, we will appeal to the more general method employed by Gordon [2].

6.4. **The Properties of Volterra's Function*.** We now seek to actually prove that Volterra's function V is differentiable and that its derivative is both bounded and not Riemann integrable. To show that V is differentiable, it is easiest if we reformulate our definition of V slightly.

Recall that the SVC is a perfect, nowhere dense set (see Section 5.3) comprised of the endpoints of the intervals removed in the construction process and the limit points of those endpoints. Because of this, we can divide the interval $[0, 1]$ into two disjoint sets: the SVC and the open intervals removed in the construction process (i.e. the closed intervals minus the endpoints). Then, we have

$$[0, 1] \setminus S = \bigcup_{k=1}^{\infty} (u_k, v_k).$$

Pick one of these open intervals and denote it (u_n, v_n). Let a_n be a number in $\left(u_n, \frac{u_n + v_n}{2}\right)$ such that $g'(a_n) = 0$. It should be clear that this is a similar point to the a_n's in the previous section. We then define $b_n = u_n + v_n - a_n$, so that $a_n - u_n = v_n - b_n$. Lastly, we define the function $f_n : (u_n, v_n) \to \mathbb{R}$ as

* This section makes use of some basic concepts from topology.

$$f_n(x) = \begin{cases} (x-u_n)^2 \sin\left(\frac{1}{x-u_n}\right), & u_n < x < a_n \\ (a_n-u_n)^2 \sin\left(\frac{1}{a_n-u_n}\right), & a_n \le x \le b_n \\ (v_n-x)^2 \sin\left(\frac{1}{v_n-x}\right), & b_n < x < v_n \\ 0 \text{ elsewhere.} \end{cases}$$

We can see then that:

$$\begin{array}{ll} \text{for } u_n < x < a_n, & |f_n(x)| \le |x-u_n|^2 \le |x-v_n|^2 \\ \text{for } a_n \le x \le b_n, & |f_n(x)| \le |a_n-u_n|^2 \le |x-u_n|^2 \\ \text{for } a_n \le x \le b_n, & |f_n(x)| \le |b_n-v_n|^2 \le |x-v_n|^2 \\ \text{for } b_n < x < v_n, & |f_n(x)| \le |x-v_n|^2 \le |x-u_n|^2 . \end{array}$$

Thus, $|f_n(x)|$ is bounded by both $|x-u_n|^2$ and $|x-v_n|^2$.

Now, since we have $[0,1] \setminus S = \bigcup_{k=1}^{\infty} (u_k, v_k)$, for each interval (u_k, v_k) let f_k be defined as above. We then find that Volterra's function can alternatively be defined as

$$V(x) = \begin{cases} f_k(x), & \text{if } x \in (u_k, v_k) \\ 0, & \text{if } x \in S, \end{cases}$$

where S is the SVC. Given the definition of each f_n we can see that V will be differentiable for any $c \notin S$. Let us then choose a $c \in S$. To prove that V is differentiable at c, we will show that

$$\lim_{x \to c^-} \frac{V(x) - V(c)}{x - c} = 0.$$

The proof can be easily modified for the case of the right hand limit. Let $\epsilon > 0$ and choose $\delta = \epsilon$. Suppose that $x \in (c-\delta, c)$. Note that the result is trivial if $x \in S$, so let x be in (u_n, v_n) for some n. It follows then that

$$\left| \frac{V(x) - V(c)}{x - c} \right| \le \frac{|f_n(x)|}{-(x-v_n)} \le \frac{|x-v_n|^2}{|x-v_n|} = |x-v_n| < \epsilon.$$

Thus, $V'(c) = 0$, so we conclude that V is differentiable at all $c \in S$ and thereby differentiable on $[0,1]$.

To show that V' is not Riemann integrable, we will show that V' has a discontinuity at every $c \in S$, that is, at every point in the SVC. Let $c \in S$. By Theorem 5.8, c is a limit point of the set of the endpoints of the intervals removed to construct the SVC. Then since c is a limit point of that set E, there exists a sequence $\{a_k\}$ of points in E that converges to c. For every n, there exists an integer $q_n > n$ such that

$$\left|V'(x_n)\right| = \left|f'_{k_n}(x_n)\right| = 1 \qquad \text{where} \qquad x_n = a_{k_n} + \frac{1}{q_n \pi}.$$

Since the sequence $\{x_n\}$ converges to c but the sequence $\{V'(x_n)\}$ converges to $1 \ne 0 = V'(c)$, it follows that V' is discontinuous at c. We then conclude that V' is not continuous at any point in S. In other words, V'

takes on the values -1 and 1 arbitrarily close to any point in the SVC, and so is therefore not continuous there. Since the measure of the set of discontinuities of V' is equal to the measure of S and that measure is positive, we conclude by the Lebesgue Criterion for Riemann Integrability that Volterra's function V has a bounded derivative that is not Riemann integrable.

7. Conclusion

The main result of this piece, the failure of Volterra's function to have a Riemann integrable derivative, has a great deal of historical significance in addition to being fascinating in its own right. Here we have a function that in many ways is remarkably well-behaved, but it does not obey the Fundamental Theorem of Calculus. It is differentiable, its derivative is bounded, and yet we cannot recover the original function by integrating its derivative. This troubling yet important result was part of the reason that mathematicians were so concerned in the 19^{th} and early 20^{th} centuries with placing calculus on a firmer theoretical footing, and doing so required a reevaluation of integration theory as a whole. For the purposes of cutting-edge analysis, the Riemann integral was largely abandoned in favor of the Lebesgue integral, which was introduced in 1904 and developed in the following decades. The Lebesgue integral overcomes many of the deficiencies of the Riemann integral. For instance, both the indicator function of the rationals $\chi_{\mathbb{Q}}$ and the derivative of Volterra's function V' are Lebesgue integrable, despite being discontinuous on sets of positive measure.

In a nutshell, the development of both measure and integration theory can be characterized by a striving towards generality, particularly in regards to the Fundamental Theorem of Calculus. Over the years, mathematicians have sought to formulate the Fundamental Theorem with as few extraneous criteria as possible. The leap from the Riemann to the Lebesgue integral loosened the requirement on derivatives from being continuous almost everywhere to simply being bounded in order to be integrable. In fact, in the early to mid-20^{th} century, three integration processes were developed – the Denjoy, Perron, and Henstock integrals – that satisfy an ideal verson of the Fundamental Theorem, stating that if F is differentiable on $[a, b]$, then its derivative F' is integrable on $[a, b]$ and $\int_a^x F' = F(x) - F(a)$ for all $x \in [a, b]$.

It is the hope of the author that this piece has given both a coherrent presentation of the topics discussed, as well as provided a sense of where and how those concepts fit into a larger mathematical narrative. If a reader is to take anything away from this paper, it should be a sense of the exploratory nature of mathematics. We left the comfortable confines of basic calculus and explored the strange new realm of pathological examples such as the Cantor set and Volterra's function. As a result, however, we arrived at a much deeper understanding of the realm we left, that of basic calculus. This method of blazing a trail into unknown territory and returning with new insight and understanding occurs not only in mathematics but throughout science at large. In this way, our investigations into the underpinnings of modern analysis can be seen as emblematic of the thirst for knowledge that underlies all scientific endeavors.

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