Advanced Tests for Convergence

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Abstract

The primary objective of this paper is to discuss advanced tests of convergence for infinite series. Commonly used tests for convergence that are taught to students in early calculus classes, including the Comparison, Root, and Ratio Tests are not sufficient in giving results for more complicated infinite series. These frequently used tests are discussed in the paper, along with examples of infinite series that have interesting properties, in order to effectively examine the more advanced Kummer's and Raabe's Tests. We demonstrate some applications of these more generalized tests through examples where simpler tests fail to yield results. While the main focus of this project is on advanced tests for convergence, we also illustrate connections between the different tests.

Introduction

The study of infinite series began when Archimedes used the first summation of an infinite series to find the area underneath an arc of a parabola. Long after Archimedes wrote the first summation, some of the most famous mathematicians of all time have worked on infinite series, including Gauss, Einstein, Euler, the Bernoulli brothers, and many more. Throughout the centuries, the study of infinite series has led to the proofs of some absolutely astounding results. For example, in 1734, Leonhard Euler proved that the infinite sum of the inverse of squares is equal to $\frac{\pi^2}{6}$. This result was amazing not just because it is an example of how adding up infinitely many numbers can give a single, numerical result but also because the result involves a seemingly unrelated constant, π. This key observation is one of many that demonstrate the complexity and beauty that lies in the world of infinite series.

We begin by proving the commonly used Root and Ratio Tests. It is critical to understand the origins and implications of these tests, especially the Ratio Test, as it is the basis for one of our more complex tests. We also briefly mention the "strength" of a convergence test, and prove the relationship between the Root and Ratio Tests as well as demonstrate this idea of "strength".

An important piece of understanding infinite series is recognizing the connection between sequences and series. We address this connection through some examples of infinite series that have interesting properties. These examples include proofs that show convergence or divergence in a variety of ways, including using the *Cauchy Criterion* for sequences and *"telescoping sums"*. In addition, we use our previous knowledge on *p-series* and *geometric* *series* to make comparisons in order to solve problems. As a resource for understanding these properties of sequences and specific types of series, any background knowledge that we need and all definitions excluding the advanced tests definitions can be found in most Real Analysis textbooks such as *Real Analysis: A First Course* by Russell A. Gordon. Plus, any additional information on the commonly used tests for convergence can be found in most calculus books.

After we have reviewed some common aspects of infinite series, we move on to our more advanced tests of convergence, named after Ernst Eduard Kummer (1810-1893) and Joseph Ludwig Raabe (1801-1859). Kummer was a German mathematician and Raabe was a Swiss mathematician, both of whom did work on infinite series at about the same time in the mid-1800s. Raabe's Test actually preceded Kummer's Test; the latter is just a generalized version of the former. We prove both of these tests as well apply them to some examples and make a connection to one of our better-known tests. We then finish our examination of these advanced tests with examples that employ both an advanced test and other strategies previously mentioned in the first section of the paper in order to tie together all of the main themes of the project.

1 Review and Preliminaries

As previously mentioned, we start with the two convergence tests that are the most common and easy to understand. While these tests and possibly even their proofs may be familiar, it is important to truly understand these tests and where they come from because they are the basis for some of the more advanced tests.

Theorem 1 (Ratio Test). *Let $\sum_{n=1}^{\infty} a_n$ be an infinite series of nonzero terms and let*

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|,$$

where it is assumed that the limit exists. Then

(i) if $0 \le L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges;

(ii) if $1 < L \le \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges;

(iii) if $L = 1$, then the test is inconclusive and the series may either converge or diverge.

Proof. We begin by considering the case where $L < 1$. Let r be a number such that $L < r < 1$. Because $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ and $L < r$, it follows that there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < r \Leftrightarrow |a_{n+1}| < |a_n| r$$

for all $n \geq N$. Now, if we let n be equal to N, $N+1$, $N+2$, and so on, we find that

$$\begin{aligned}
|a_{N+1}| &< |a_N| r \\
|a_{N+2}| &< |a_{N+1}| r < |a_N| r^2 \\
|a_{N+3}| &< |a_{N+2}| r < |a_{N+1}| r^2 < |a_N| r^3 \\
&\vdots
\end{aligned}$$

and, in general,

$$|a_{N+k}| < |a_N| r^k$$

for all positive integers k. The series $\sum_{k=1}^{\infty} |a_N| r^k$ is a geometric series with common ratio r and because $|r| < 1$, the series $\sum_{k=1}^{\infty} |a_N| r^k$ must converge. Thus, by the Comparison Test, the series $\sum_{k=N+1}^{\infty} |a_n|$ must also converge, it follows that $\sum_{n=1}^{\infty} |a_n|$ converges.

Now, consider the case where $L > 1$. Because $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ and $L > 1$, there exists some positive integer K such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1$$

for all $n \geq K$. This implies that

$$|a_{n+1}| > |a_n| > |a_K| > 0$$

for all $n \geq K$. This tells us that $\lim_{n\to\infty} a_n \neq 0$, and we conclude that the series diverges. ■

Theorem 2 (Root Test)**.** *Let* $\sum_{n=1}^{\infty} a_n$ *be an infinite series of real numbers and let*

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

when the limit exists. Then

(i) if $0 \leq \rho < 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *converges;*

(ii) if $1 < \rho \leq \infty$*, then the series* $\sum_{n=1}^{\infty} a_n$ *diverges;*

(iii) if $\rho = 1$*, then the test is inconclusive and* $\sum_{n=1}^{\infty} a_n$ *may either converge or diverge.*

Proof. We first consider the case when $\rho < 1$. Let r be a number such that $\rho < r < 1$. Then there exists some positive integer N_1 such that $\sqrt[n]{|a_n|} < r$ for all $n \geq N_1$. Raising both sides of the equation to the nth power gives us $|a_n| < r^n$ for all $n \geq N_1$. Thus, because $\sum_{n=1}^{\infty} r^n$ is a convergent geometric series, it follows that $\sum_{n=1}^{\infty} |a_n|$ converges by the Comparison Test.

Now suppose that $\rho > 1$ and let r be a number such that $1 < r < \rho$. Using similar reasoning, we can see that there exists some positive integer N_2 such that for all $n \geq N_2$, we have $\sqrt[n]{|a_n|} > r$. Raising both sides to the nth power gives us $|a_n| > r^n > 1$ for all $n \geq N_2$. Because $\sum\limits_{n=1}^{\infty} r^n$ is a divergent geometric series, we find that the series $\sum\limits_{n=1}^{\infty} |a_n|$ diverges by the Comparison Test. ∎

As a preliminary, we consider the "strength" of convergence tests as part of our discussion of the different types of tests. When referring to the "strength" of a test, we are referring to its ability to yield a result. For one convergence test to be "stronger" than another, the stronger test giving a result implies that the weaker test also yields a result. It must also be true that there exists a case where the stronger convergence test works but the weaker test does not.

Theorem 3. *The Root Test is "stronger" than the Ratio Test.*

Proof. Without loss of generality, let $a_n > 0$ for all positive integers n. Essentially, what we want to prove is that if $\lim\limits_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right)$ exists, then $\lim\limits_{n\to\infty} \sqrt[n]{a_n}$ also exists and is equal. This implies that if we are able to use the Ratio Test for a series, then the Root Test works as well.

Let $L = \lim\limits_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right)$ where $0 < L < \infty$ and let $\epsilon > 0$. Then there exists some integer N such that

$$L - \epsilon < \frac{a_{n+1}}{a_n} < L + \epsilon$$

for all $n \geq N$. It follows that

$$\begin{gathered}
(L-\epsilon)a_n < a_{n+1} < (L+\epsilon)a_n \\
(L-\epsilon)^2 a_n < (L-\epsilon)a_{n+1} < a_{n+2} < (L+\epsilon)a_{n+1} < (L-\epsilon)^2 a_n \\
\vdots \\
(L-\epsilon)^i a_n < a_{n+i} < (L+\epsilon)^i a_n.
\end{gathered}$$

Then for $n \geq N$ we have

$$(L-\epsilon)^n \cdot \frac{a_N}{(L-\epsilon)^N} < a_n < (L+\epsilon)^n \cdot \frac{a_N}{(L+\epsilon)^N}$$

and hence

$$L - \epsilon < \sqrt[n]{a_n} < L + \epsilon$$

If the limit L exists, then because $\epsilon > 0$ was arbitrary we can conclude that $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = L$. Therefore, we have shown that the Root Test is at least as strong as the Ratio Test.

We claim that the Root Test is stronger than the Ratio Test and we prove this by giving an example. To demonstrate this, we will show that there exist series such that the Root Test indicates whether the series converges or diverges but the Ratio Test is inconclusive.

Example. Consider the series $\sum_{n=1}^{\infty} a_n$ where

$$a_n = \begin{cases} \frac{1}{2^n} & \text{if } n \text{ is even,} \\ \frac{1}{2^{n+1}} & \text{if } n \text{ is odd.} \end{cases}$$

It follows that the ratio of consecutive terms will either be 1 or $\frac{1}{4}$, thus $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist. However, applying the Root Test gives us

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{2^n}} = \frac{1}{2} \qquad \text{and} \qquad \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{2^{n+1}}} = \frac{1}{2}.$$

Therefore, the series converges by the Root Test while the Ratio Test is inconclusive and thus we have shown that the Root Test is stronger than the Ratio Test. ■

2 Sequences and Series

The purpose of the results and examples in this section are to illustrate some interesting findings related to series and their sequences of partial sums. One important result that will come up often is that if the sequence of partial sums of an infinite series of positive terms is bounded, then the series converges. (Gordon, Theorem 6.4). Therefore, if a sequence of partial sums is not Cauchy, then the corresponding series does not converge.

Our first set of results and examples considers a convergent infinite series and its sequence of partial sums. We demonstrate the use of sequences of partial sums to prove convergence as well as our previous knowledge on the well known geometric and p-series.

Theorem 4. *Let $\sum_{k=1}^{\infty} a_k$ be a convergent series of positive terms and let $t_n = \sum_{k=n}^{\infty} a_k$ for each positive integer n. The series $\sum_{k=1}^{\infty} \frac{a_k}{t_k}$ diverges but the series $\sum_{k=1}^{\infty} \frac{a_k}{\sqrt{t_k}}$ converges.*

Proof. We begin by noting that $\{t_n\}$ is a strictly decreasing sequence that converges to 0. Therefore, given any positive integer m, there exists a positive integer $n > 2m$ such that $t_n < \frac{t_{m+1}}{2}$. Then we have

$$\sum_{k=m+1}^{n} \frac{a_k}{t_k} > \sum_{k=m+1}^{n} \frac{a_k}{t_{m+1}} = \frac{1}{t_{m+1}} \sum_{k=m+1}^{n} (t_k - t_{k+1}) > \frac{1}{t_{m+1}} \sum_{k=m+1}^{n-1} (t_k - t_{k+1}) = \frac{t_{m+1} - t_n}{t_{m+1}} > \frac{1}{2}.$$

Hence, the sequence of partial sums for the series $\sum_{k=1}^{\infty} \frac{a_k}{t_k}$ is not a Cauchy sequence. It follows that $\sum_{k=1}^{\infty} \frac{a_k}{t_k}$ diverges.

To show that the series $\sum_{k=1}^{\infty} \frac{a_k}{\sqrt{t_k}}$ converges, we first note that

$$a_k = t_k - t_{k+1} = \left(\sqrt{t_k} + \sqrt{t_{k+1}}\right)\left(\sqrt{t_k} - \sqrt{t_{k+1}}\right) < 2\sqrt{t_k}\left(\sqrt{t_k} - \sqrt{t_{k+1}}\right)$$

and thus

$$\frac{a_k}{\sqrt{t_k}} < 2\left(\sqrt{t_k} - \sqrt{t_{k+1}}\right)$$

for all positive integers k. Using "telescoping sums", we have

$$\sum_{k=1}^{n} \frac{a_k}{\sqrt{k}} < \sum_{k=1}^{n} 2\left(\sqrt{t_k} - \sqrt{t_{k+1}}\right) = 2\left(\sqrt{t_1} - \sqrt{t_{n+1}}\right) < 2\sqrt{t_1}.$$

Thus, the sequence of partial sums for the series $\sum\limits_{k=1}^{\infty} \frac{a_k}{\sqrt{t_k}}$ is bounded and therefore converges. We conclude that the series $\sum\limits_{k=1}^{\infty} \frac{a_k}{\sqrt{t_k}}$ converges. ■

Example. To illustrate what we have just shown, we consider both of these series for $a_k = 2^{-k}$ and also for $a_k = (k^2+k)^{-1}$.

When $a_k = \frac{1}{2^k}$, the series $\sum\limits_{k=1}^{\infty} a_k$ is a geometric series so we find that $t_k = \frac{2}{2^k}$. This gives us

$$\frac{a_k}{t_k} = \frac{1}{2^k} \cdot \frac{2^k}{2} = \frac{1}{2} \qquad \text{and} \qquad \frac{a_k}{\sqrt{t_k}} = \frac{1}{2^k} \cdot \frac{\sqrt{2^k}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^k$$

Thus the series $\sum\limits_{k=1}^{\infty} \frac{a_k}{t_k} = \sum\limits_{k=1}^{\infty} \frac{1}{2}$ clearly diverges while the series $\sum\limits_{k=1}^{\infty} \frac{a_k}{t_k} = \sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^k$ is a geometric series with $r = \frac{1}{\sqrt{2}} < 1$, hence the series converges.

When $a_k = (k^2+k)^{-1}$, the equation for t_k is a "telescoping sum" so we have

$$t_k = \sum_{i=k}^{\infty} \frac{1}{i(i+1)} = \sum_{i=k}^{\infty} (\frac{1}{i} - \frac{1}{i+1}) = \frac{1}{k}.$$

It follows that

$$\frac{a_k}{t_k} = \frac{1}{k^2+k} \cdot k = \frac{1}{k+1}$$

for all k. Since $\sum\limits_{k=1}^{\infty} \frac{1}{k+1}$ diverges, the series $\sum\limits_{k=1}^{\infty} \frac{a_k}{t_k}$ diverges. Then considering the series $\sum\limits_{k=1}^{\infty} \frac{a_k}{\sqrt{t_k}}$, we have

$$\frac{a_k}{\sqrt{t_k}} = \frac{\sqrt{k}}{k^2+k} = \frac{1}{\sqrt{k}(k+1)} < \frac{1}{k^{\frac{3}{2}}}$$

for all k. We know that $\sum\limits_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}}$ is a convergent p-series since $\frac{3}{2} > 1$. Hence, as expected, the series $\sum\limits_{k=1}^{\infty} \frac{a_k}{\sqrt{t_k}}$ converges.

Now we look at the case where the series $\sum_{n=1}^{\infty} a_k$ is divergent. Note the similarity to the results we have just shown.

Theorem 5. *Let $\sum_{k=1}^{\infty} a_k$ be a divergent series of positive terms and let $\{s_n\}$ be the sequence of partial sums of this series. The series $\sum_{k=1}^{\infty} \frac{a_k}{s_k}$ diverges but the series $\sum_{k=1}^{\infty} \frac{a_k}{s_k^2}$ converges.*

Proof. By the definition of s_n, we see that $a_{n+1} = s_{n+1} - s_n$, and because the series $\sum_{k=1}^{\infty} a_k$ is divergent the sequence $\{s_n\}$ is increasing and unbounded. Hence, given any positive integer m there exists an integer $n > 2m$ such that $s_n > 2s_m$. Thus we have

$$\sum_{k=m+1}^{n} \frac{a_k}{s_k} = \frac{a_{m+1}}{s_{m+1}} + \cdots + \frac{a_n}{s_n} > \frac{a_{m+1} + \cdots + a_n}{s_n} = \frac{s_n - s_m}{s_n} = 1 - \frac{s_m}{s_n} > 1 - \frac{1}{2} = \frac{1}{2}$$

Therefore, the sequence of partial sums is not Cauchy and hence the series $\sum_{k=1}^{\infty} \frac{a_k}{s_k}$ diverges.

To show that the series $\sum_{k=1}^{\infty} \frac{a_k}{s_k^2}$ converges, we use similar reasoning as in previous problems. Noting that that the sum $\sum_{k=2}^{n} \left(\frac{1}{s_{k-1}} - \frac{1}{s_k}\right)$ telescopes to $\frac{1}{s_1} - \frac{1}{s_n}$ we have

$$\sum_{k=1}^{n} \frac{a_k}{s_k^2} < \sum_{k=1}^{n} \frac{a_k}{s_k s_{k-1}} = \sum_{k=1}^{n} \frac{s_k - s_{k-1}}{s_k s_{k-1}} = \sum_{k=1}^{n} \frac{1}{s_{k-1}} - \frac{1}{s_k} = \frac{1}{s_1} - \frac{1}{s_n}$$

for all $k > 1$. Hence, the sequence of partial sums is bounded so the series $\sum_{k=1}^{\infty} \frac{a_k}{s_k^2}$ converges. ∎

Example. Again to display the implications of this result, we consider both series for convergence when $a_k = 1$ and $a_k = \sqrt{k+1} - \sqrt{k}$.

When $a_k = 1$, the equation for s_k is simply $s_k = k$. We then have

$$\sum_{k=1}^{\infty} \frac{a_k}{s_k} = \sum_{k=1}^{\infty} \frac{1}{k} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{a_k}{s_k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2},$$

where $\sum_{k=1}^{\infty} \frac{1}{k}$ is a divergent series and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent series.

Because $\sum_{k=1}^{n} \left(\sqrt{k+1} - \sqrt{k}\right)$ is a "telescoping sum", the equation for s_k is

$$s_k = \sum_{i=1}^{k} \left(\sqrt{i+1} - \sqrt{i}\right) = \sqrt{k+1} - 1.$$

It follows that

$$\begin{aligned}
\frac{a_k}{s_k} &= \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-1} \\
&> \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}} \\
&= \frac{1}{(k+1)+\sqrt{k}\sqrt{k+1}} \\
&> \frac{1}{(k+1)+\sqrt{k+1}\sqrt{k+1}} \\
&= \frac{1}{2(k+1)}
\end{aligned}$$

for all $k \geq 1$. Therefore, since $\sum\limits_{k=1}^{\infty} \frac{1}{2k+2}$ diverges, the series $\sum\limits_{k=1}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-1}$ diverges by the Comparison Test.

Next, we note that

$$\frac{a_k}{s_k^2} = \frac{\sqrt{k+1}-\sqrt{k}}{(\sqrt{k+1}-1)^2} < \frac{\sqrt{k+1}-\sqrt{k}}{\left(\frac{k}{2}\right)} \cdot \frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+1}+\sqrt{k}} = \frac{1}{\left(\frac{k}{2}\right)\left(\sqrt{k+1}+\sqrt{k}\right)} < \frac{1}{\left(\frac{k}{2}\right)\left(2\sqrt{k}\right)} = \frac{1}{k^{\frac{3}{2}}}.$$

We know that the infinite series $\sum\limits_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}}$ converges because it is a p-series where $p = \frac{3}{2}$. Thus, by the Comparison Test, the series $\sum\limits_{k=1}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{(\sqrt{k+1}-1)^2}$ must also converge.

The last result in this section also demonstrates the relationship between sequences and series but instead of starting with a series and using its sequence of partial sums in our proof, we begin with a sequence and make the connection to the corresponding series of the terms of the sequence.

Theorem 6. *Suppose the sequence $\{\sqrt{k} \cdot a_k\}$ converges to a positive number. Then the series $\sum\limits_{k=1}^{\infty} \frac{a_k}{k}$ converges while the series $\sum\limits_{k=1}^{\infty} a_k$ diverges.*

Proof. Because $\{\sqrt{k} \cdot a_k\}$ is a convergent sequence, this implies that it is also bounded. Thus, there exists some number M such that $|\sqrt{k} \cdot a_k| \leq M$ for all $k \geq 1$. Dividing both sides of the inequality by $k^{\frac{3}{2}}$ yields the result

$$\left|\frac{a_k}{k}\right| \leq \frac{M}{k^{\frac{3}{2}}}$$

for all $k \geq 1$. Since $\sum\limits_{k=1}^{\infty} \frac{M}{k^{\frac{3}{2}}}$ is a convergent p-series, it follows that $\sum\limits_{k=1}^{\infty} \frac{a_k}{k}$ converges by the Comparison Test.

Since the sequence $\{\sqrt{k} \cdot a_k\}$ converges to a positive number, we know that it is eventually bounded below by some positive number α. This means that there exists some integer K such that

$$0 < \alpha \leq \sqrt{k} \cdot a_k$$

for all $k \geq K$. Dividing both sides by $\sqrt{k}$ yields the result

$$\frac{\alpha}{\sqrt{k}} \leq a_k$$

for all $k \geq K$. Then, we know that $\sum_{k=1}^{\infty} \frac{\alpha}{\sqrt{k}}$ diverges and thus the series $\sum_{k=K}^{\infty} a_k$ diverges by the Comparison Test. Hence, the series $\sum_{k=1}^{\infty} a_k$ also diverges. ■

3 More Advanced Tests

Now that we have covered some interesting properties of certain series, we examine some more advanced tests for convergence. While we can see that the techniques we have used thus far are useful and can yield intriguing results, the tests and strategies we have just displayed do not always determine convergence and divergence for a series. In fact, it happens often that the tests we mentioned give us no information. To illustrate this, we consider a generic p-series and apply the Ratio and Root Tests:

Let $a_n = \frac{1}{n^p}$ and note that

$$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n} \right) = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^p = 1$$

and

$$\lim_{n\to\infty} \sqrt[n]{a_n} \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^p}} = \lim_{n\to\infty} (n^{-p})^{\frac{1}{n}} = \lim_{n\to\infty} \left(\sqrt[n]{n} \right)^{-p} = 1.$$

We know that a p-series diverges when $p \leq 1$ and converges when $p > 1$. Therefore, we cannot determine whether the series converges or diverges, because both the Ratio and Root Test gives us 1 no matter what p is. In general, these tests fail when they give a result of 1 because these tests compare the series to a geometric series with r equal to the result of the test. When $r = 1$, the terms of a geometric series are constant. Even though the geometric series would diverge in this, case as a comparison we get no information on whether the terms of the series are going off to infinity or converging.

As we can see, the Root and Ratio Test are often not helpful in deciphering the convergence of an infinite series. In the case of the p-series we can often use the Comparison Test, but when we want to examine more complicated series using the Comparison Test can get tricky. This is why we need more advanced tests like the ones that follow.

Theorem 7 (Kummer's Test)**.** *Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms and let $\{B_n\}$ be a be a sequence of positive constants. Let*

$$L = \lim_{n\to\infty} \left[B_n \frac{a_n}{a_{n+1}} - B_{n+1} \right],$$

where it is assumed that the limit exists.

(i) *If* $L > 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *converges.*

(ii) *If* $L < 0$ *and* $\sum_{n=1}^{\infty} \frac{1}{B_n}$ *diverges, then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof. We begin by considering the case where $L > 0$. Choose a number r such that $0 < r < L$. Then there must exist some integer $N > 0$ such that

$$B_n \frac{a_n}{a_{n+1}} - B_{n+1} > r \Leftrightarrow B_n\, a_n - B_{n+1}\, a_{n+1} > r\, a_{n+1}$$

for all $n \geq N$. Given any positive integer m, we have

$$\begin{aligned} B_N\, a_N - B_{N+1}\, a_{N+1} &> r\, a_{N+1} \\ B_{N+1}\, a_{N+1} - B_{N+2}\, a_{N+2} &> r\, a_{N+2} \\ B_{N+2}\, a_{N+2} - B_{N+3}\, a_{N+3} &> r\, a_{N+3} \\ &\vdots \\ B_{N+m-1}\, a_{N+m-1} - B_{N+m}\, a_{N+m} &> r\, a_{N+m} \end{aligned}$$

And then by adding these inequalities together, we get cancellations of all of the terms on the left side of the inequality except for the first and last. Thus we have

$$B_N\, a_N - B_{N+m}\, a_{N+m} > r\,(a_{N+1} + \cdots + a_{N+m}) = r\,(S_{N+m} - S_N),$$

where S_n is the partial sum for $\sum_{k=N+1}^{N+m} a_k$. It follows that

$$rS_{N+m} < r\, S_N + B_N\, a_N - B_{N+m}\, a_{N+m} < r\, S_N + B_N\, a_N.$$

Let C be the constant $\frac{(r\, S_N + B_N\, a_N)}{r}$. The above inequality tells us that $S_n < C$ for all $n \geq N$. Thus, the sequence of partial sums $\{S_n\}$ of the series $\sum_{n=1}^{\infty} a_n$ is bounded and therefore the series converges.

Now consider the case where $L < 0$. This means that there exists an integer $N > 0$ such that $B_n \frac{a_n}{a_{n+1}} - B_{n+1} \leq 0$ for all $n \geq N$. Rearranging this inequality gives us

$$B_n\, a_n \leq B_{n+1}\, a_{n+1}$$

for all $n \geq N$ which then implies that

$$B_N\, a_N \leq B_n\, a_n$$

for all $n \geq N$. Letting $B_N\, a_N$ be a constant C, we have

$$a_n \geq \frac{C}{B_n}$$

for $n \geq N$ and, because $\sum_{n=1}^{\infty} \frac{1}{B_n}$ diverges, the series $\sum_{n=1}^{\infty} a_n$ diverges by the Comparison Test.

■

Remark. Using Kummer's Test, if we let the sequence $\{B_n\} = \{1\}$, then we get

$$L = \lim_{n\to\infty} \left[\frac{a_n}{a_{n+1}} - 1 \right].$$

If $\lim\limits_{n\to\infty} \left(\frac{a_n}{a_{n+1}}\right) > 1$, then it follows that L is positive and thus the series converges by Kummer's Test. This also tells us that the ratio $\lim\limits_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) < 1$, and thus the series converges by the Ratio Test. Similarly, if $\lim\limits_{n\to\infty} \left(\frac{a_n}{a_{n+1}}\right) < 1$ then L is negative and the series diverges by Kummer's Test. From this we can see that the ratio $\lim\limits_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) > 1$ so we conclude that the series also diverges by the Ratio Test. Therefore, we can conclude that the Ratio Test is simply Kummer's Test associated with the sequence $\{B_n\} = \{1\}$

Kummer's Test is inconclusive when $L = 0$, but when this happens it is possible to refine the sequence $\{B_n\}$ to give a result, which is why the case when $L = 0$ is not included in the definition of the test. But because there is no specific strategy for choosing $\{B_n\}$, this can become a tedious and time-consuming task in order to find a sequence that yields a result when used in Kummer's Test. This is where Raabe's Test comes in. Raabe's Test is simply a specific case of Kummer's Test where we let $B_n = n$, therefore we do not have to go through the process of trying to find a suitable B_n.

Theorem 8 (Raabe's Test)**.** *Let* $\sum\limits_{n=1}^{\infty} a_n$ *be a series with positive terms and assume that*

$$\rho = \lim_{n\to\infty} n \left(\frac{a_n}{a_{n+1}} - 1 \right)$$

exists. Then

(i) if $\rho > 1$*, then the series* $\sum\limits_{n=1}^{\infty} a_n$ *converges.*

(ii) if $\rho < 1$*, then the series* $\sum\limits_{n=1}^{\infty} a_n$ *diverges.*

(iii) if $\rho = 1$*, then* $\sum\limits_{n=1}^{\infty} a_n$ *may either converges or diverge and the test is inconclusive.*

Proof. Items (i) and (ii) are consequences of Kummer's Test. To demonstrate this, let $B_n = n$ and compute

$$\begin{aligned} L &= \lim_{n\to\infty} \left(n \frac{a_n}{a_{n+1}} - (n+1) \right) \\ &= \lim_{n\to\infty} \left(n \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right) \\ &= \rho - 1 \end{aligned}$$

If $\rho > 1$, then $L > 0$ and the series $\sum\limits_{n=1}^{\infty} a_n$ converges by Kummer's Test. Similarly, if $\rho < 1$, then $L < 0$ and note that $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges, thus the series $\sum\limits_{n=1}^{\infty} a_n$ diverges by Kummer's Test. If the result of Raabe's Test is $\rho = 1$, then the test is inconclusive. ■

Remark. Applying Raabe's Test to the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ gives us

$$\begin{aligned}
\rho &= \lim_{n\to\infty} n\left(\frac{(n+1)^p}{n^p} - 1\right) \\
&= \lim_{n\to\infty} \left(n\left(\frac{n+1}{n}\right)^p - n\right) \\
&= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^p - 1}{\frac{1}{n}} \\
&= p
\end{aligned}$$

Thus, if $\rho > 1$, then the series converges by Raabe's Test. This implies that $p > 1$ which indicates that the series is a convergent p-series. On the other hand, if $\rho < 1$ then we know the series diverges by Raabe's Test, and this implies that $p < 1$ and therefore the series is a divergent p-series. This demonstrates that Raabe's Test tests a series for convergence by comparing the series to the p-series where $p = \rho$.

Corollary 1 (Of Raabe's Test)**.** *In general, when $\rho > 0$, the sequence $\{a_n\}$ converges to 0.*

Proof. When $\rho > 0$, there must exist some integer M such that

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) > 0$$

for all $n \geq M$. It follows that $a_n > a_{n+1}$ for all $n \geq M$. This tells us that $\{a_n\}$ is eventually decreasing, therefore $\{a_n\}$ converges to some number L.
To show that $L = 0$, we use proof by contradiction and assume that $L > 0$. Similar to our above reasoning, because $\rho > 0$ there exists some integer N and a number $\epsilon > 0$ such that

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) > \epsilon$$

for all $n \geq N$. From this, we see that

$$\frac{a_n}{a_{n+1}} - 1 > \frac{\epsilon}{n} \implies a_n - a_{n+1} > \frac{\epsilon}{n}\, a_{n+1} > \frac{\epsilon\, L}{n}$$

Now, "telescoping" this difference gives us

$$\begin{aligned}
a_n - a_m &= (a_n - a_{n+1}) + (a_{n+1} - a_{n+2}) + \cdots + (a_{m-1} - a_m) \\
&> \epsilon\, L \left(\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{m-1}\right)
\end{aligned}$$

whenever $m > n \geq N$. Since the sequence of partial sums for the harmonic series diverges and we know that $\{a_n\}$ converges, we have reached a contradiction. Therefore, the sequence $\{a_n\}$ converges to 0. ∎

As previously mentioned, Raabe's Test chronilogically preceded Kummer's Test, so instead of considering Raabe's Test to be a refinement of Kummer's Test, we could also consider Kummer's Test to be a generalized version of Raabe's Test. The examples that follow require Raabe's Test to solve as the Root Test would be difficult to calculate because there are no powers of n, the Ratio Test fails, and the unique usage of the *double factorial* makes the Comparison Test difficult to use.

4 Applications of More Advanced Tests

The **double factorial** of an integer n denoted by $n!!$ is the product of all the integers from 1 up to and including n that have the same parity as n. For example, $9!! = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 = 945$.

Example. Test the series $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2n+1}$ for convergence.

Let a_n represent the terms of the series. Since

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left(\frac{(2n+1)!!}{(2n+2)!!(2n+3)} \cdot \frac{(2n)!!(2n+1)}{(2n-1)!!} \right) \\
&= \lim_{n\to\infty} \left(\left(\frac{2n+1}{2n+2} \right) \left(\frac{2n+1}{2n+3} \right) \right) \\
&= 1
\end{aligned}$$

the Ratio Test fails and we turn to Raabe's Test. We find that

$$\begin{aligned}
\lim_{n\to\infty} n \left(\frac{a_n}{a_{n+1}} - 1 \right) &= \lim_{n\to\infty} n \left(\frac{(2n-1)!!(2n+2)!!(2n+3)}{(2n+1)!!(2n)!!(2n+1)} - 1 \right) \\
&= \lim_{n\to\infty} n \left(\frac{(2n+2)(2n+3)}{(2n+1)(2n+1)} - 1 \right) \\
&= \lim_{n\to\infty} n \left(\frac{4n^2+10n+6}{4n^2+4n+1} - 1 \right) \\
&= \lim_{n\to\infty} n \left(\frac{6n+5}{4n^2+4n+1} \right) \\
&= \lim_{n\to\infty} \left(\frac{6n^2+5n}{4n^2+4n+1} \right) \\
&= \frac{3}{2},
\end{aligned}$$

which shows that the series converges by Raabe's Test.

Example. Test the series $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{4n+3}{2n+2}$ for convergence.

Let a_n represent the terms of the series. Using the Ratio Test gives us

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left(\frac{(2n+1)!!(4n+7)}{(2n+2)(2n+4)} \cdot \frac{(2n)!!(2n+2)}{(2n-1)!!(4n+3)} \right) \\
&= \lim_{n\to\infty} \left(\left(\frac{2n+1}{2n+2} \right) \left(\frac{4n+7}{4n+3} \right) \left(\frac{2n+2}{2n+4} \right) \right) \\
&= 1
\end{aligned}$$

so again we turn to Raabe's Test. We then have

$$\begin{aligned}
\lim_{n\to\infty} n \left(\frac{a_n}{a_{n+1}} - 1 \right) &= \lim_{n\to\infty} n \left(\frac{(2n-1)!!(4n+3)}{(2n)!!(2n+2)} \cdot \frac{(2n+2)!!(2n+4)}{(2n+1)!!(4n+7)} - 1 \right) \\
&= \lim_{n\to\infty} n \left(\left(\frac{2n+2}{2n+1} \right) \left(\frac{4n+3}{4n+7)} \right) \left(\frac{2n+4}{2n+2} \right) - 1 \right) \\
&= \lim_{n\to\infty} n \left(\frac{8n^2+22n+12}{8n^2+18n+7} - 1 \right) \\
&= \lim_{n\to\infty} \left(\frac{4n^2+5n}{8n^2+18n+7} \right) \\
&= \frac{1}{2}.
\end{aligned}$$

Thus the series diverges by Raabe's Test.

While it does not have a name, the following version of Kummer's Test, where we let $B_n = n \ln n$, is similar to Raabe's and is helpful when Raabe's Test fails as in the example that follows.

Theorem 9. *Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms and assume that*

$$\delta = \lim_{n\to\infty} \ln n \left[n \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right]$$

exists. Then,

(i) if $\delta > 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

(ii) if $\delta < 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

(iii) if $\delta = 1$, then $\sum_{n=1}^{\infty} a_n$ may either converge or diverge and the test is inconclusive.

Proof. To begin, we refer to Kummer's Test and let $B_n = n \ln n$. Note that the series $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ diverges. Then we have

$$\begin{aligned}
L &= \lim_{n\to\infty} \left[n \ln n \frac{a_n}{a_{n+1}} - (n+1)\ln(n+1) \right] \\
&= \lim_{n\to\infty} \left[n \ln n \frac{a_n}{a_{n+1}} - (n+1)\ln n - (n+1)\ln\left(1+\frac{1}{n}\right) \right] \\
&= \lim_{n\to\infty} \left[\ln n \left(n \frac{a_n}{a_{n+1}} - (n+1) \right) - (n+1)\ln\left(1+\frac{1}{n}\right) \right] \\
&= \lim_{n\to\infty} \left[\ln n \left[n \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right] - n \ln\left(1+\frac{1}{n}\right) - \ln\left(1+\frac{1}{n}\right) \right] \\
&= \lim_{n\to\infty} \left[\ln n \left[n \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right] - \ln\left(1+\frac{1}{n}\right)^n - \ln\left(1+\frac{1}{n}\right) \right]
\end{aligned}$$

Since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, it follows that $\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = 1$ and $\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) = 0$. Thus we can see that $L = \delta - 1$. This tells us that if $\delta > 1$ then $L > 0$ and the series converges by Kummer's Test. Similarly, if $\delta < 1$, then $L < 0$ and the series diverges. If $\delta = 1$ this means that $L = 0$ and the test is inconclusive, therefore a refinement of Kummer's Test is necessary to get a result.

Example. Test the series $\sum_{n=1}^{\infty} \left[\frac{(2n-1)!!}{(2n)!!}\right]^p$ for convergence.

Let a_n represent the terms of the series. The Ratio Test gives us

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left[\frac{(2n+1)!!(2n)!!}{(2n-1)!!(2n+2)!!} \right]^p \\
&= \lim_{n\to\infty} \left(\frac{(2n+1)}{(2n+2)} \right)^p \\
&= 1
\end{aligned}$$

Once again, we turn to Raabe's Test and we find that

$$\lim_{n\to\infty} n \left(\frac{a_n}{a_{n+1}} - 1 \right) = \lim_{n\to\infty} n \left(\left(\frac{2n+2}{2n+1} \right)^p - 1 \right).$$

Letting $x = \frac{1}{2n+1}$, it follows that $n = \frac{1-x}{2x}$ and thus because x approaches 0 as n approaches infinity we have

$$\lim_{n\to\infty} n \left(\left(\frac{2n+2}{2n+1} \right)^p - 1 \right) = \lim_{x\to 0} \left(\frac{1-x}{2x} \cdot \left((1+x)^p - 1 \right) \right) = \lim_{x\to 0} \left(\frac{1-x}{2} \cdot \frac{(1+x)^p - 1}{x} \right)$$

Using the product rule for limits, because both limits exist we can divide up the limit of the product into the product of the limits of each fraction. Then we have

$$\lim_{x\to 0}\left(\frac{1-x}{2}\right)\cdot\lim_{x\to 0}\left(\frac{(1+x)^p-1}{x}\right)=\frac{1}{2}\cdot p=\frac{p}{2}.$$

Therefore we can see that if $p>2$, the series converges and the series diverges when $p<2$. However, when $p=2$ both the Ratio Test and Raabe's Test fail. We use the previously stated refinement of Raabe's Test which gives us

$$\begin{aligned}\lim_{n\to\infty}\ln n\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]&=\lim_{n\to\infty}\ln n\left[n\left[\left(\frac{2n+2}{2n+1}\right)^2-1\right]-1\right]\\&=\lim_{n\to\infty}\ln n\left[n\left(\frac{4n^2+8n+4}{4n^2+4n+1}-1\right)-1\right]\\&=\lim_{n\to\infty}\ln n\left[n\left(\frac{4n+3}{4n^2+4n+1}\right)-1\right]\\&=\lim_{n\to\infty}\ln n\left[\frac{4n^2+3n}{4n^2+4n+1}-1\right]\\&=\lim_{n\to\infty}\ln n\left[\frac{-n-1}{4n^2+4n+1}\right]\end{aligned}$$

and because $\lim_{n\to\infty}\ln n\left[\frac{-n-1}{4n^2+4n+1}\right]=0<1$, the series diverges when $p=2$.

Our final examples will require us to use both Raabe's Test and what we know about some other common types of series; Power Series and Alternating Series.

Example. Find the interval of convergence for the power series $\sum_{k=1}^{\infty}\frac{1\cdot 4\cdot 7\cdot\cdots\cdot(3k-2)}{k!}x^k$.

To begin, we use the Ratio Test to determine the radius of convergence for the series. The Ratio Test yields the result

$$\lim_{k\to\infty}\left|\frac{1\cdot 4\cdot 7\cdot\ \cdots\ \cdot(3k+1)}{(k+1)!}x^{k+1}\cdot\frac{k!}{1\cdot 4\cdot 7\cdot\ \cdots\ \cdot(3k-2)\cdot x^k}\right|=\lim_{k\to\infty}\left|\frac{(3k+1)}{(k+1)}\cdot x\right|=3|x|.$$

This tells us that the series converges when $|3x|<1$. This means that the series converges when the inequality $-\frac{1}{3}<x<\frac{1}{3}$ is true. Now we must test the endpoints of this interval for convergence at these points.

First, we consider when $x=\frac{1}{3}$. This results in

$$\sum_{k=1}^{\infty}\frac{1\cdot 4\cdot 7\cdot\ \cdots\ \cdot(3k-2)}{k!}\cdot\left(\frac{1}{3}\right)^k.$$

Letting $a_k = \frac{1\cdot4\cdot7\cdot\cdots\cdot(3k-2)}{k!\cdot 3^k}$ and applying Raabe's Test we then have

$$\begin{aligned}
\lim_{k\to\infty} k\left(\frac{a_k}{a_{k+1}} - 1\right) &= \lim_{k\to\infty} k\left(\frac{(k+1)}{(3k+1)}\cdot(3) - 1\right)\\
&= \lim_{k\to\infty} k\left(\frac{3k+3}{3k+1} - 1\right)\\
&= \lim_{k\to\infty} k\left(\frac{2}{3k+1}\right)\\
&= \lim_{k\to\infty} \left(\frac{2k}{3k+1}\right)\\
&= \frac{2}{3}.
\end{aligned}$$

Therefore the series diverges when $x = \frac{1}{3}$ by Raabe's Test.

Now we consider the case where $x = -\frac{1}{3}$. The series $\sum_{k=1}^{\infty} a_k\,(-1)^k$ has negative terms, so we cannot use Raabe's Test. We now turn to the Alternating Series Test where a_k is the same as above. The Alternating Series Test tells us that if $\{a_k\}$ is decreasing and the terms go to 0 then the series converges. In our above results, $\rho = \frac{2}{3} > 0$ so by Corollary 1, the sequence $\{a_k\}$ goes to 0. Additionally, a_k is always positive and so the sequence must be decreasing. Therefore, the series $\sum_{k=1}^{\infty} \frac{1\cdot\cdot4\cdot7\cdot\ldots\cdot(3k-2)}{k!}\cdot\left(-\frac{1}{3}\right)^k$ converges by the Alternating Series Test. Hence, the interval of convergence for the power series $\sum_{k=1}^{\infty} \frac{1\cdot4\cdot7\cdot\ldots\cdot(3k-2)}{k!}x^k$ is $\left[-\frac{1}{3}, \frac{1}{3}\right)$.

So far, our discussion on infinite series has only touched on tests that give us information on whether the series converges or not. If we find that a series converges, these tests do not give us a good indication as to what the sum of the series is. However, in our last example we will illustrate how we can use the information that Raabe's Test gives us to find the sum of a series.

Example. Find the sum of the series $\sum_{k=1}^{\infty} \frac{2\cdot4\cdot6\cdot\cdots\cdot(2k)}{5\cdot7\cdot9\cdot\cdots\cdot(2k+5)}$.

We begin by letting

$$a_k = \frac{2\cdot4\cdot6\cdot\cdots\cdot(2k)}{5\cdot7\cdot9\cdot\cdots\cdot(2k+5)} \qquad \text{and} \qquad b_k = \frac{2\cdot4\cdot6\cdot\cdots\cdot(2k)}{3\cdot5\cdot7\cdot\cdots\cdot(2k+3)}.$$

It follows that

$$a_k = \left(\frac{3}{2k+2}\right)b_{k+1} \qquad \text{and} \qquad b_k = \left(\frac{2k+5}{2k+2}\right)b_{k+1}.$$

From here, we apply Raabe's Test to the series $\sum_{k=1}^{\infty} b_k$ which give us

$$\lim_{k\to\infty} k\left(\frac{b_k}{b_{k+1}} - 1\right) = \lim_{k\to\infty} k\left(\frac{(2k+5)}{(2k+2)} - 1\right) = \lim_{k\to\infty}\left(\frac{3k}{2k+2}\right) = \frac{3}{2}.$$

Thus the series $\sum_{k=1}^{\infty} b_k$ converges by Raabe's Test.
Then, we observe that

$$\begin{aligned} b_k - a_k &= \left(\frac{2k+5}{2k+2}\right) b_{k+1} - \left(\frac{3}{2k+2}\right) b_{k+1} \\ &= \left(\frac{2k+2}{2k+2}\right) b_{k+1} \\ &= b_{k+1}. \end{aligned}$$

We can use this result to conclude that

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} (b_k - b_{k+1}) = \lim_{n\to\infty} (b_1 - b_{n+1}) = b_1 = \frac{2}{15}.$$

Thus, the sum of the series $\sum_{k=1}^{\infty} \frac{2\cdot 4\cdot 6\cdot \cdots \cdot(2k)}{5\cdot 7\cdot 9\cdot \cdots \cdot(2k+5)}$ is $\frac{2}{15}$.

Conclusion

The scope of this project merely skims the surface of what has been done in the field of infinite series and tests of convergence. Beyond Kummer's Test and Raabe's Test, there are many more tests including Bertrand's, Gauss's, and Dirichlet's which are all useful in their own way. Tests for convergence are important to the study of infinite series because it is often the case where knowing the property of the series is imperative to a proof or the solution to a problem as shown in our last example.

In mathematics, some of the most useful applications of infinite series are in integration and differential equations. We also use specific infinite series such as Taylor Series to show that calculations involving functions such as e^x and $\sin x$ can be computed using addition, subtraction, multiplication, and division. Aside from mathematics, infinite series are also useful in a wide range of fields including physics, engineering, chemistry, computer science, and finance. In physics they are used to solve the pendulum differential equation while in finance they are used to calculate fiscal multipliers. Infinite series also have a wide range of uses in computer science as they are often used in generating functions. Without infinite series and the tests we use to determine convergence, we would not be able to advance and solve important problems in many areas of study.

In this project we were able to show some very interesting results involving infinite series and their partial sums as well as draw some connections between series and sequences. We introduced two advanced tests of convergence and demonstrated their usefulness as well as their applications and relationships to one another. Further inquiry into this topic would include the examination of other advanced tests such as the ones just mentioned and a more rigorous investigation into the results and implications of the tests we discussed. In conclusion, the study of infinite series and tests of convergence is not only extensive, but it is intriguing and beautiful.

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