

# Universal Totally Ordered Sets

Luke Adams

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## Abstract

In 1874, Georg Cantor published an article in which he proved that the set of algebraic numbers are countable, and the set of real numbers are uncountable. This, at the time controversial article, marked the beginning of modern set theory, and it finally gave mathematicians the notation to explore the infinite. In the intervening century and a half, set theory has blossomed into a central part of mathematics, often acting as a language with which to formalize other branches of mathematics [5].

In this paper, we explore some basic concepts in set theory, and then consider a result that was proven in the Introduction to Higher Mathematics notes: every countable, totally ordered set is embeddable into the rational numbers. We generalize this idea to larger cardinalities and introduce the notion of  $\aleph_\alpha$ -universal sets. We show that  $\eta_\alpha$ -sets are, in fact,  $\aleph_\alpha$ -universal, and conclude by providing a construction of the minimal  $\eta_\alpha$ -set whenever  $\aleph_\alpha$  is a regular ordinal, and the generalized continuum hypothesis holds.

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# 1 Introduction

The study of set theory is due primarily to Georg Cantor in 1874. In this year, Cantor published a paper that proved the existence of a one-to-one correspondence between the set of algebraic numbers and the set of natural numbers. In the same paper, he also showed that no such correspondence exists between the set of natural numbers and the set of real numbers. A few years later, Cantor would coin the terms ‘countable’ and ‘uncountable’ to describe these relationships. Cantor went on to spearhead the development of set theory, creating an entirely new field of study in his wake [5].

Nearly 40 years later, in 1914, Felix Hausdorff introduced the notion of universally ordered sets, which contain copies of all totally ordered sets of a particular cardinality. Hausdorff showed that such universally ordered sets are guaranteed to exist. Further, Hausdorff provided a way to construct these sets in a select number of cases [2].

In this paper we consider the properties of infinite ordered sets. We begin by reviewing some basic notions in set theory, and prove several nonintuitive facts about infinite sets. We then define universally ordered sets, and discuss their implications. We conclude by examining a construction of universally ordered sets that is closely related to the ones that Hausdorff published.

## 1.1 Ordered Sets

The key objects in all of our explorations will be *ordered sets*: a set  $S$  and an order  $\leq$ . We already have many preconceived notions about how an order should behave. For example, we expect that an order should be *transitive* in the same way that an equivalence relation is. That is, if  $a \leq b$  and  $b \leq c$ , then we expect that  $a \leq c$ . We use these expectations to guide our definition of an order.

**Definition 1** ([4]). Let  $S$  be a set, and let  $\leq$  be a relation on  $S$ . Further let  $a, b, c \in S$  be arbitrary elements. We say that  $\leq$  is a *partial order* on  $S$  if it satisfies three properties:

1. *Reflexivity*:  $a \leq a$ .
2. *Transitivity*: If  $a \leq b$  and  $b \leq c$ , then  $a \leq c$ .
3. *Antisymmetry*. If  $a \leq b$  and  $b \leq a$ , then  $a = b$ .

We call the pair  $(S, \leq)$  a *partially ordered set*, or a *POset*. We say that

two elements  $a, b \in S$  are *comparable* if either  $a \leq b$  or  $b \leq a$ . Otherwise, we say that  $a$  and  $b$  are *incomparable*, and we write  $a \parallel b$ .

The relation  $\leq$  is called a *total order* if every pair of elements in  $S$  is comparable. If this extra condition holds, we call the pair  $(S, \leq)$  a *totally ordered set*.

Let  $S$  be a totally ordered set, and let  $A, B \subseteq S$  be subsets. We will write  $A < B$  when  $a < b$  for all elements  $a \in A$  and  $b \in B$ . We say that  $A$  and  $B$  are *neighboring* if there is no element  $s \in S$  such that  $A < s < B$ .

Suppose  $s \in S$  is an element. Then we say that  $s$  is a *successor* element if there is an element  $t \in S$  such that  $t < s$ , and  $t$  and  $s$  are neighboring. If there is no such element, we say that  $s$  is a *lower-limit* element. Dually, we say that  $s$  is a *predecessor* element if there is a  $t$  such that  $s < t$ , and  $s$  and  $t$  are neighboring. If no such  $t$  exists, then  $s$  is an *upper-limit* element.

We say that  $a \in A$  is the *least element* of the subset  $A$  if  $a \leq a'$  for all  $a' \in A$ . There is also the dual notion of the *greatest element* of a subset. If every subset of  $S$  has a least element, then the structure of  $S$  is very limited. These sets are very important, so we give them a special name:

**Definition 2** ([3]). Suppose  $S$  is a totally ordered set with order  $\leq$ . Then we say that  $\leq$  is a *well-order* and  $(S, \leq)$  is a *well-ordered set* if every subset of  $S$  has a least element. Analogously, we say that  $\leq$  is a *reverse well-order* and  $(S, \leq)$  is a *reverse well-ordered set* if every subset of  $S$  has a greatest element.

The structure of a well-ordered set means that *all* of the elements of  $S$  can be listed in order starting with the least element:  $S = \{s_0, s_1, s_2, \dots\}$ .

### 1.1.1 Examples of Ordered Sets

Before continuing, we consider several examples of ordered sets. We encounter ordered sets in all aspects of math, sometimes without realizing. Other examples are more contrived, but they highlight important aspects of ordered set theory. This library of specific examples will allow us to ground abstract arguments in particular examples.

The set of natural numbers,  $\mathbb{N}$ , under the standard ordering is well-ordered<sup>1</sup>, however the integers,  $\mathbb{Z}$ , are only totally ordered because some subsets (including  $\mathbb{Z}$  itself) do not have least elements. The negative integers,

<sup>1</sup>Actually, we must assume that the natural numbers are well-ordered and introduce a new axiom to this effect. But we almost always make this assumption, because it is equivalent to assuming that standard induction is valid [3].

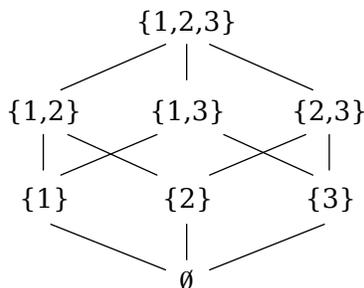


Figure 1: Subset lattice for the set  $\{1, 2, 3\}$ . The lines between sets indicate a subset relationship, with the set that is higher containing the set that is lower.

$\mathbb{Z}^-$ , are reverse well-ordered. Both the rational numbers,  $\mathbb{Q}$ , and the real numbers,  $\mathbb{R}$ , are totally ordered with the standard ordering.

The operations of set theory can also naturally create orders. The *power set* of a set  $S$  is the collection of all subsets of  $S$ . We write this as  $\mathcal{P}(S)$ . Consider the set  $S = \{1, 2, 3\}$ , and observe that

$$\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

The power set of  $S$  is a partial order under set inclusion. That is, if  $A, B \subseteq S$ , then  $A \leq B$  if and only if  $A \subseteq B$ . When the number of elements in a partial set is finite, we can create diagrams to indicate the order structure of the set. Such a diagram for the set  $S$  is shown in Figure 1.

The alphabetic ordering of words forms a total order. To determine which word should come first in an alphabetical listing, we compare the first letter of each word, and the word with an earlier first letter is listed first. If the first letters are the same, we compare the second letters, and so on. For example, consider the two words “love” and “long.” The first two letters of each word are the same, so we compare the third letter of each word, and because ‘n’ < ‘v’, we conclude that “long” < “love.”

Notice that in the alphabetic ordering, we are making use of the underlying order of the letters, and then viewing a word as simply an ordered tuple of letters. This style of ordering is called *lexicographical ordering*, and we can also use it to order other sets consisting of ordered tuples. For example, consider the set  $\mathbb{N} \times \mathbb{N}$ , ordered lexicographically. This set is well-ordered

because every subset has a least element, and thus we can list the elements in order:

$$(1, 1) < (1, 2) < \cdots < (2, 1) < (2, 2) < \cdots < (3, 1) < (3, 2) < \cdots .$$

Notice that the elements  $(1, 1), (2, 1), \dots$  are lower-limit elements, while every other element is a successor. Every element in  $\mathbb{N} \times \mathbb{N}$  is a predecessor element.

### 1.1.2 Isomorphisms and Embeddings

Suppose  $A$  and  $B$  are POsets with orders  $\leq_A$  and  $\leq_B$  respectively. Then we say that  $A$  and  $B$  are *order-isomorphic* and we write  $A \simeq B$  if there is a bijection  $f : A \rightarrow B$  with the property that, for every  $x, y \in A$ ,

$$x \leq_A y \implies f(x) \leq_B f(y).$$

If two sets are order-isomorphic, then they are identical from a set-theoretic point of view. This means that anything we prove about a set  $A$  is also true for every set  $B$  that is order-isomorphic to  $A$ .

Now suppose that, instead of a bijection, the function  $f : A \rightarrow B$  is an order preserving *injection*. Then we say that  $A$  is *embeddable* into  $B$ , and we write  $A \preceq B$ . In other words,  $A$  is embeddable into  $B$  exactly when  $A$  is order-isomorphic to some subset of  $B$ .

The notation that we use for embeddability suggests that that this relation might, in fact, form an order. To determine if this is true, we must check if the  $\preceq$  satisfies the three conditions of a partial order: reflexivity, transitivity, and anti-symmetry.

**Theorem 3.** The relation  $\preceq$  is reflexive and transitive.

*Proof.* Let  $A, B$ , and  $C$  be arbitrary POsets. Note that the identity function  $I_A(a) = a$  is clearly an embedding of  $A$  into  $A$ , and thus  $A \preceq A$ .

Now suppose that  $A \preceq B$  and  $B \preceq C$ . Then it follows that there are embeddings  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . We claim that the composition  $(g \circ f) : A \rightarrow C$  is also an embedding, and thus  $A \preceq C$ .

To see why this is true, suppose  $a, a' \in A$  with  $a \leq a'$ . Then there are images  $b = f(a)$  and  $b' = f(a')$  with the property that  $b < b'$ . Then finally, we have that  $g(b) \leq g(b')$ , and thus  $(g \circ f)(a) \leq (g \circ f)(a')$ , as required.  $\square$

Thus, if  $\preceq$  was anti-symmetric, it would form a partial order on ordered sets. However, this is not true, and it then follows that  $\preceq$  is not a partial

order. We can see the lack of anti-symmetry by considering the two sets  $A = [0, 1]$  and  $B = (0, 1)$ . Clearly these sets have different order properties (particularly at the endpoints), so they are not order-isomorphic.

However, if we consider the embedding functions  $f : A \rightarrow B$  and  $g : B \rightarrow A$  defined by

$$f(x) = \frac{1}{4}x + \frac{1}{4} \quad \text{and} \quad g(x) = x, \quad (1)$$

we can easily verify that  $A \preceq B$  and  $B \preceq A$ . Therefore,  $\preceq$  is not anti-symmetric, and thus not a partial order. We refer to a relation that is reflexive and transitive, but not anti-symmetric as a *quasi-order*.

## 1.2 Cardinal Numbers

Suppose we would like to determine if two sets  $A$  and  $B$  have the same number of elements. If both  $A$  and  $B$  are finite, then we can simply count the number of elements in each set. However, this notion breaks down when the sets become infinite. This was Cantor's dilemma.

To resolve this difficulty, we say that two sets  $A$  and  $B$  have the same *cardinality* if there is an isomorphism  $f : A \rightarrow B$ . In this case, we write  $|A| = |B|$ . For example, in Math 260, we showed that  $|\mathbb{N}| = |\mathbb{Q}|$  and that  $|\mathbb{N}| < |\mathbb{R}|$ .

But what *is*  $|A|$ ? If the set  $A$  is finite, then  $|A|$  is the number of elements in the set, so  $|A| \in \mathbb{W}$ . However, the answer is less obvious if  $A$  is infinite. Clearly the "size" of  $A$  cannot be represented by an integer, so  $|A|$  must be from an even larger class of numbers that is somehow expanded to include infinite numbers.

We call these *cardinal numbers*. The class of cardinal numbers contains all  $\mathbb{W}$ , but it also has an infinite number of other elements, each representing an infinite size. The least infinite cardinal is  $|\mathbb{N}|$  which we typically denote as  $\aleph_0$  or just  $\aleph$ . The next larger cardinal numbers are  $\aleph_1$ ,  $\aleph_2$ , and so on.

The cardinal numbers are well-ordered, so we can list them in ascending order:

$$0 < 1 < 2 < \cdots < \aleph_0 < \aleph_1 < \aleph_2 < \cdots .$$

### 1.2.1 Cardinal Arithmetic

We can use the cardinal numbers to do arithmetic. In fact, in very formal treatments of mathematics, all numbers are actually sets. Before we give the definitions of the arithmetic operations on cardinal numbers, we first need to define some notation.

Let  $X$  and  $Y$  be two sets. We define the *disjoint union* of  $X$  and  $Y$  as

$$X \hat{\cup} Y = (\{0\} \times X) \cup (\{1\} \times Y).$$

Essentially, this disjoint union ensures that, if  $X$  and  $Y$  have some elements in common, then the common elements are “repeated twice” in the union. We also define  $X^Y$  to be the collection of all functions from  $X$  into  $Y$ . With these definitions, we are now ready to define our cardinal arithmetic operators [3].

**Definition 4.** Suppose  $X$  and  $Y$  are sets, then we will define addition, multiplication, and exponentiation of cardinal numbers as

$$|X| + |Y| = |X \hat{\cup} Y|, \quad |X| \cdot |Y| = |X \times Y|, \text{ and } |X|^{|Y|} = |X^Y|.$$

There are a few interesting consequences of these definitions. For example, suppose  $X$  is a set. Then there is a natural bijection between  $\mathcal{P}(X)$  and  $2^X$ , where  $2 = \{0, 1\}$ . In particular, suppose  $A \in \mathcal{P}(X)$ . Then we pair  $A$  with the function  $f : X \rightarrow 2$  defined by

$$f_A(x) = \begin{cases} 1, & x \in A; \\ 0, & x \notin A. \end{cases}$$

This means that  $|\mathcal{P}(X)| = 2^{|X|}$ . As we will see in next section, this relationship can be very useful in some circumstances.

### 1.2.2 The Continuum Hypothesis

Let  $c = |\mathbb{R}|$  denote the cardinality of the real numbers. There is no bijection between the natural numbers and the real numbers, so we know that  $\aleph_0 < c$ . However, we can construct a bijection between the family of subsets of the natural numbers and the real numbers, and thus we have  $|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0} = c$ .

But neither of these facts actually tell us the cardinality of the real numbers. We could have  $c = \aleph_1$ , but we could also have  $c = \aleph_5$ .

It turns that this problem is undecidable. That is, given the standard axioms of set theory, it is impossible to prove the cardinality of the real numbers. Thus, we must make some arbitrary choice, and introduce a new axiom to enforce this choice. One such choice we could make is the Continuum Hypothesis [1].

**Axiom 5.** Continuum Hypothesis (CH) The cardinality of the real numbers is  $c = \aleph_1$ .

There is a order-isomorphism between  $\mathcal{P}(\mathbb{N})$  and  $\mathbb{R}$ , so this means that, assuming the CH,  $\mathcal{P}(\mathbb{N}) = 2^{\aleph_0} = \aleph_1$ . That is, the power set of a countable set is only “one infinity bigger” than the set itself!

Notice that the CH still has not told us anything about the relative sizes of larger  $\aleph$  sets. However, we can use the notion of power sets to create a more general axiom that specifies the relationships between *all* infinite cardinal numbers:

**Axiom 6.** Generalized Continuum Hypothesis (GCH) Suppose  $S$  is a set with  $|S| = \aleph_\alpha$ . Then  $\mathcal{P}(S) = 2^{\aleph_\alpha} = \aleph_{\alpha+1}$ .

In what follows, we will not, in general assume that the CH or the GCH holds. However, there will be several circumstances where we will find it advantageous to assume these axioms to allow us to push our arguments further. In every case where we rely on the CH or GCH, we will make an explicit comment about their use.

### 1.3 Ordinal Numbers

Cardinality allows us to compare the sizes of arbitrary sets, however we can do even better if the sets are *well-ordered*. If two well-ordered sets  $A$  and  $B$  are order isomorphic, we say that  $A$  and  $B$  have the same ordinality, and we write  $||A|| = ||B||$ .

For a finite set  $A$ , there is exactly one well-ordered set (up to order-isomorphism), and thus  $||A|| = |A|$ . That is, the cardinality of  $A$  is equivalent to the ordinality of  $A$ . However, this is not true when  $A$  is infinite. In fact, for a given cardinality, there are an infinite number of well-ordered sets with different ordinalities.

We call these objects *ordinal numbers*, and we typically denote them using lowercase Greek letters (e.g.  $\alpha, \beta, \nu, \mu$ ). In fact, we define the cardinal numbers to be a subset of the ordinal numbers such that  $\aleph_\alpha = \omega_\alpha$  [3].

The ordinal numbers are themselves well-ordered, so we can list the elements, and also indicate the cardinality of each number:

$$0 < 1 < 2 < \dots < \underbrace{\omega_0 < \omega_0 + 1 < \omega_0 + 2 < \dots}_{\aleph_0} < \underbrace{\omega_1 < \omega_1 + 1 < \omega_1 + 2 < \dots}_{\aleph_1} < \dots$$

The ordinal numbers are in fact sets, and their construction gives them the somewhat unusual property that, for all ordinals  $\mu$  and  $\nu$ ,

$$\mu < \nu \iff \mu \in \nu.$$

For a much more detailed discussion of the construction of the ordinal (and cardinal) numbers, see [3].

## 1.4 Order Types

When we defined the ordinal numbers, we only considered order-isomorphisms between well-ordered sets. However, we can relax this restriction to talk about the structure of *all* ordered sets. As we noted earlier, if two sets are order-isomorphic, then they are identical from a set-theoretic perspective. With this in mind, we will organize POsets into order types such that every set within a given order type is order-isomorphic to every other set in the order type.

**Definition 7.** Suppose  $A$  and  $B$  are POsets with  $A \simeq B$ . Then we say that  $A$  and  $B$  share an *order type*, and we write  $\text{tp}(A) = \text{tp}(B)$ . If  $\text{tp}(A) = \alpha$ , then we say that  $A$  is a *realization* of  $\alpha$ .

We refer to some order types so much that we assign them special names. A few are

$$\text{tp}(\mathbb{N}) = \omega, \quad \text{tp}(\mathbb{Q}) = \eta, \quad \text{and} \quad \text{tp}(\mathbb{R}) = \lambda. \quad (2)$$

Suppose  $\alpha$  and  $\beta$  are order types, and  $A$  and  $B$  are realizations of  $\alpha$  and  $\beta$  respectively. We say that  $\alpha$  is *embeddable* in  $\beta$ , and we write  $\alpha \preceq \beta$  if  $A$  is embeddable into  $B$  [1].

## 1.5 Maximal POsets

Suppose  $S$  is some set with a partial order  $\leq$ , and consider the set  $X_{\leq} \in \mathcal{P}(S \times S)$  defined by

$$X_{\leq} = \{(s, t) \mid s, t \in S \text{ and } s \leq t\}. \quad (3)$$

That is,  $X_{\leq}$  is the set of all pairs of elements in  $S$  for which the first element is less than the second element. The set  $X_{\leq}$  is called the *graph* of  $\leq$ . In formal treatments of ordered set theory, the graph of a relation *is* the definition of the relation.

Let's consider the collection  $\mathcal{X} \subseteq \mathcal{P}(S \times S)$  of *all* possible partial orders of  $S$ . This family of sets is partially ordered under inclusion.

**Theorem 8.** The collection  $\mathcal{X}$  has a least element. Namely, the identity relation

$$I_S = \{(s, s) \mid s \in S\}. \quad (4)$$

*Proof.* Suppose  $X \in \mathcal{X}$  is some partial order of  $S$ . Then  $X$  must be reflexive, and thus  $(s, s) \in X$  for all  $s \in S$ . It then follows that  $I_S \subseteq X$ , and thus  $I_S$  is the least element of  $\mathcal{X}$ .  $\square$

We have shown that  $\mathcal{X}$  has a least element, and now we would like to investigate what it means for an order to be *maximal* in  $\mathcal{X}$ . We claim that an order is maximal in  $\mathcal{X}$  if and only if it is a total order. Intuitively, this should make some sense: in a total order, every pair of elements has a defined order, so there are no more pairs of elements that could be added to the graph of  $\leq$  to create a larger graph. Let's formalize this intuition.

**Theorem 9.** Suppose  $S$  is a set and  $\leq$  is a partial order of  $S$  with the graph  $X_{\leq}$ . Then  $\leq$  is a total order exactly when  $X_{\leq}$  is maximal in  $\mathcal{X}$ .

*Proof.* First suppose that  $\leq$  is a total order; we need to show that its graph is maximal in  $\mathcal{X}$ . To that end, suppose  $\leq'$  is another partial order such that  $X_{\leq} \subseteq X_{\leq'}$ . We will show that this is actually an equality.

Let  $(s, t) \in X_{\leq'}$  be a pair of arbitrary, *distinct* elements. If  $(s, t) \in X_{\leq}$ , we are done, so suppose this is not the case. Then  $s \not\leq t$ , and thus we must have  $t \leq s$ . However, this implies that  $t \leq' s$ , and thus  $s = t$ , contradicting our assumption that  $s$  and  $t$  are distinct. We therefore conclude that  $\leq$  is maximal in  $\mathcal{X}$ .

Now suppose that  $\leq$  is *not* a total order; we would like to conclude that  $X_{\leq}$  is *not* maximal in  $\mathcal{X}$  either.

Let  $\preceq$  be some well-ordering of  $S$ . Then we can use this ordering to construct a lexicographic total ordering  $\preceq_L$  on the set  $2^S$ .

Now if  $s \in S$ , we define  $\hat{s} \in 2^S$  as

$$\hat{s}(t) = \begin{cases} 0, & a \not\leq x; \\ 1, & a \leq x. \end{cases}$$

Notice that  $\hat{s}$  is just the indicator function for the subset  $\bar{s} = \{t \in S : t < s\} \subseteq S$ . It follows that  $x \mapsto \hat{x}$  is injective (i.e. if  $x$  and  $y$  map to the same place, then  $x = y$ ). Let  $\hat{A} \subseteq 2^A$  be the image of  $A$  under this mapping.

It is easy to see that, for all  $x, y \in A$ , we have

$$x \leq y \implies \bar{x} \subseteq \bar{y} \implies \hat{x} \preceq_L \hat{y}.$$

Therefore, we can extend  $\leq$  with a new order  $\leq'$  by letting  $x \leq' y$  mean  $\hat{x} \preceq_L \hat{y}$ . Since  $\preceq_L$  is a total order on  $2^A$ , we know that  $\leq'$  is a total order on  $A$ . It follows that  $X_{\leq'}$  is maximal in  $\mathcal{X}$ , and thus  $X_{\leq}$  is *not* maximal.  $\square$

## 1.6 Distinct Order Types

We know that there are an infinite number of POsets of a particular cardinality. This is because, given a particular POset, we can replace some of the symbols

in the POset to produce a new POset. However, these two sets would share the same order type, and thus they would be, for all intents and purpose, identical. In this section, our goal will be to determine how many *unique* order types exist for a particular cardinality.

Let  $T$  be a totally ordered set, and let  $t \in T$  be an element. Then the *initial segment* defined by  $t$  is the set of all  $s \in T$  such that  $s < t$ . Similarly, the *final segment* defined by  $t$  is the set of all  $s \in T$  for which  $t < s$ . We write  $(-\infty, t)$  and  $(t, \infty)$  to indicate the initial and final segments of  $t$  respectively.

We will let  $T[I]$  be the set of elements in  $T$  whose initial segments do not have a greatest element, and we let  $T[F]$  be the set of elements in  $T$  whose final segments do not have a least element. Notice that if  $T$  and  $S$  are two totally ordered sets with  $T \simeq S$ , then  $T[I] = S[I]$  and  $T[F] = S[F]$ .

Consider the set  $A = \aleph_\alpha \times \mathbb{W}$ . The structure of  $A$  is

$$(0, 0) < (0, 1) < \dots < (1, 0) < (1, 1) < \dots < (2, 0) < (2, 1) < \dots .$$

Notice that every element  $(\nu, n) \in A$  has a well defined successor; namely  $(\nu, n+1)$ . Thus every final segment in  $A$  has a least element, and so  $A[F] = \emptyset$ . However, this is not the case for all initial segments. In particular,  $(\nu, 0)$  does not have a predecessor, and thus  $A[I] = \aleph_\alpha \times \{0\}$ . You might be concerned about the element  $(0, 0) \in A[I]$ , but notice that  $(-\infty, (0, 0)) = \emptyset$ , and thus the initial segment does not have a greatest element (i.e. has no elements at all!).

Now we have the notation to answer the key question of this section: how many non-order-isomorphic totally ordered sets with cardinality  $\aleph_\alpha$ ? Consider  $\mathcal{X} = \mathcal{P}(\aleph_\alpha) = \mathcal{P}(\omega_\alpha)$ , the family of all subsets of  $\aleph_\alpha$ . Our goal is to construct, for every  $X \in \mathcal{X}$ , a totally ordered set  $T_X$  with cardinality  $|T_X| = \aleph_\alpha$ , and with the property that  $T_X = T_Y$  if and only if  $X = Y$ . That is, each  $T_X$  should define a *different* order type of cardinality  $\aleph_\alpha$ .

For each  $X \in \mathcal{X}$ , we define  $T_X \subseteq \aleph_\alpha \times \mathbb{Q}$  such that

$$T_X = (\aleph_\alpha \times \mathbb{W}) \cup \left( X \times \left\{ \frac{1}{2}, \frac{1}{4}, \dots \right\} \right).$$

Essentially, every  $T_X$  is the set  $A$  that we considered previously, along with some extra elements that limit down to  $(x, 0)$  for every  $x \in X$ . For example, suppose  $X = \{1\}$ . Then the ordering of  $T_X$  is

$$(0, 0) < (0, 1) < \dots < (1, 0) < \dots < (1, \frac{1}{4}) < (1, \frac{1}{2}) < (1, 1) < \dots < (2, 0) < \dots .$$

Notice that  $|\mathbb{Q}| = \aleph_0$ , and thus  $|\aleph_\alpha \times \mathbb{Q}| = \aleph_\alpha$ . It then clearly follows that  $|T_X| \leq \aleph_\alpha$ .

Observe that  $T_X[I] = \aleph_\alpha \times \{0\}$  just as for our set  $A$ , however, unlike before,  $T_X[F]$  is nonempty because the elements  $(x, 0)$  for  $x \in X$  do not have a successor. It follows that  $T_X[F] = X \times \{0\}$ .

Now suppose  $X, Y \in \mathcal{X}$  are two sets with the property that an order isomorphism  $f : T_X \rightarrow T_Y$  exists. We would like to conclude that  $X = Y$ , and thus none of the  $T_X$  sets are order-isomorphic to each other.

Notice that, by the definition of our set  $T[I]$ , we must have that

$$f(\aleph_\alpha \times \{0\}) = \aleph_\alpha \times \{0\},$$

but this set is well-ordered, so  $f$  must be the identity function for this subset of the elements of  $T_X$  and  $T_Y$ . It then follows that

$$f(X) = Y,$$

but we have already concluded that this subset of  $f$  is the identity function, so we have  $X = Y$ , as required.

We have shown that there are *at least*  $\mathcal{P}(\aleph_\alpha) = 2^{\aleph_\alpha}$  non-order-isomorphic totally ordered sets with cardinality  $\aleph_\alpha$ . Have we found them all, or are there more order types out there? The next theorem tells us that we have indeed found all the possible sets that match our criteria.

**Theorem 10.** For any infinite cardinal  $\aleph_\alpha$ , there are exactly  $2^{\aleph_\alpha}$  order types of totally ordered sets with cardinality  $\aleph_\alpha$ .

*Proof.* Suppose  $S$  is a set with  $|S| = \aleph_\alpha$ , and suppose that  $\leq_1$  and  $\leq_2$  are any two total orders of  $S$ . Then  $(S, \leq_1)$  and  $(S, \leq_2)$  will be order-isomorphic if and only if  $\leq_1$  and  $\leq_2$  have the same graph. However, recall that  $|S \times S| = |S| = \aleph_\alpha$  by our cardinal arithmetic rules, and thus there can be no more than  $2^{\aleph_\alpha}$  subsets.

We have just proven that there are *at most*  $2^{\aleph_\alpha}$  non-order-isomorphic totally ordered set with cardinality  $\aleph_\alpha$ . Previously, we saw that there are *at least* this many sets, so we conclude that there are exactly  $2^{\aleph_\alpha}$  such sets.  $\square$

If we assume the GCH, then we can say even more.

**Corollary 11.** Suppose the GCH holds. Then there are exactly  $\aleph_{\alpha+1}$  distinct order types of cardinality  $\aleph_\alpha$ .

## 2 Countable Totally Ordered Sets

In the final section of the Introduction to Higher Mathematics notes, we proved that every countable, totally ordered set is order-isomorphic to a

subset of the rational numbers [4]. This means that, in some sense, the set of rational numbers  $\mathbb{Q}$  contain every possible countably infinite, totally ordered set. We would like to extend this notion of prototypical sets to larger cardinalities, and use these sets to study the structure of totally ordered sets.

To aid in this generalization, we introduce the notion of universally ordered sets [1].

**Definition 12.** Suppose  $S$  is a totally ordered set. Then  $S$  is  $\aleph_\alpha$ -universal if every totally ordered set  $A$  with  $|A| \leq \aleph_\alpha$  is order-isomorphic to a subset of  $S$ . Equivalently,  $A$  must be embeddable into  $S$ .

In this new language, we rephrase our claim: we proved in Introduction to Higher Mathematics that  $\mathbb{Q}$  is  $\aleph_\alpha$ -universal.

Notice that the every countable, totally ordered set is embeddable in the real numbers as well. That is,  $\mathbb{R}$  is also  $\aleph_0$  universal. But this is a not as nice a result as the first one because the cardinality of  $\mathbb{R}$  is greater than  $\aleph_0$ . In other words,  $\mathbb{R}$  has more elements than needed for this property to hold. With this in mind, we say that a set  $S$  is a *minimal*  $\aleph_\alpha$ -universal set if  $S$  is  $\aleph_\alpha$ -universal, and  $|S| = \aleph_\alpha$ .

Understanding the details of this argument are critical for what follows, so, we state the Introduction to Higher Mathematics claim, and review the proof given in the notes [4].

**Theorem 13.** Every countable, totally ordered set is order-isomorphic to a subset of the rational numbers.

Equivalently, the set of rational numbers is  $\aleph_0$ -universal.

*Proof.* Suppose the set  $A = \{a_1, a_2, a_3, \dots\}$  is a countable, totally ordered set. We will inductively construct an embedding function  $f : A \rightarrow \mathbb{Q}$ .

We assign  $f(a_1)$  be any rational number, say zero. Now suppose that we have defined  $f(a_1), f(a_2), \dots, f(a_n)$  in such a way that the order relations of  $A$  and  $\mathbb{Q}$  have been preserved. We need to define  $f(a_{n+1})$  so that  $f$  continues to preserve order.

To do this, we partition the set  $\{a_1, \dots, a_n\}$  into two subsets:

$$\begin{aligned} A_{n+1}^+ &= \{a_i : i \leq n \text{ and } a_i > a_{n+1}\} \\ A_{n+1}^- &= \{a_i : i \leq n \text{ and } a_i < a_{n+1}\}. \end{aligned}$$

In  $\mathbb{Q}$ , we clearly have that  $f(A_{n+1}^-) < f(a_{n+1}) < f(A_{n+1}^+)$ , and so we can choose a number  $q \in \mathbb{Q}$  such that  $f(A_{n+1}^-) < \{q\} < f(A_{n+1}^+)$ . We then let  $f(a_{n+1}) = q$ , which preserves the order relations of  $A$  and  $\mathbb{Q}$ . When carried out infinitely,

the resulting function will be defined for all of  $A$ , and it will be an order-preserving injective function.

Therefore, we have successfully embedded  $A$  into  $\mathbb{Q}$ , and thus  $\mathbb{Q}$  is  $\aleph_\alpha$ -universal.  $\square$

We know that  $\mathbb{Q}$  is  $\aleph_0$ -universal, but what are the unique properties that make it  $\aleph_0$ -universal? One property that was extremely important in the previous proof was that, given any two rational numbers  $a$  and  $b$  with  $a < b$ , there is always a third rational number  $c$  such that  $a < c < b$ .

**Theorem 14.** Suppose  $T$  is a totally ordered set. The  $T$  will be order isomorphic to  $\mathbb{Q}$  if and only if the following hold:

1.  $|T| = |\mathbb{Q}|$ ;
2. For all pairs of *finite* subsets  $A \subseteq T$  and  $B \subseteq T$ , if  $A < B$ , then there is a  $t \in T$  such that  $A < \{t\} < B$ .

*Proof.* First suppose that  $f : T \rightarrow \mathbb{Q}$  is an order isomorphism. Then clearly  $|T| = |\mathbb{Q}|$ . To verify the second condition, suppose that  $A \subseteq T$  and  $B \subseteq T$  are finite subsets of  $T$  with the property  $A < B$ . Observe that  $f(A) < f(B)$ , and thus there is an element  $q \in \mathbb{Q}$  such that  $f(A) < \{q\} < f(B)$ . Then there is some element  $t \in T$  such that  $f(t) = q$ , and clearly  $A < \{t\} < B$ .

Now suppose that the two conditions hold. We will verify that  $T$  is order isomorphic to  $\mathbb{Q}$  by inductively constructing an order isomorphism  $g : T \rightarrow \mathbb{Q}$ . Both  $T$  and  $\mathbb{Q}$  are countable, so we can enumerate the elements:  $T = \{t_1, t_2, \dots\}$  and  $\mathbb{Q} = \{q_1, q_2, \dots\}$ . We pair the elements  $t_1$  and  $q_1$ , so that  $g(t_1) = q_1$ . Now we repeat the following algorithm:

1. Consider the first element  $t_{n+1} \in T$  that has not yet been assigned an image. Form the sets

$$\begin{aligned} T_{n+1}^+ &= \{t_i \mid i \leq n \text{ and } t_i > t_{n+1}\} \\ T_{n+1}^- &= \{t_i \mid i \leq n \text{ and } t_i < t_{n+1}\}. \end{aligned}$$

In  $\mathbb{Q}$ , it is clear that  $g(T_{n+1}^-) < g(T_{n+1}^+)$ , and so we choose a  $q \in \mathbb{Q}$  such that  $g(T_{n+1}^-) < \{q\} < g(T_{n+1}^+)$ , and assign  $g(t_{n+1}) = q$ . This assignment is consistent with the orders of both  $T$  and  $\mathbb{Q}$ .

2. Now consider the first element  $q_j \in \mathbb{Q}$  that has not been assigned an image. Form the sets

$$\begin{aligned} Q_{n+1}^+ &= \{q_i \mid i \leq n \text{ and } q_i > q_{n+1}\} \\ Q_{n+1}^- &= \{q_i \mid i \leq n \text{ and } q_i < q_{n+1}\}. \end{aligned}$$

In  $T$ , it is clear that  $g^{-1}(Q_{n+1}^-) < g^{-1}(Q_{n+1}^+)$ , and so we choose a  $t \in T$  such that  $g^{-1}(Q_{n+1}^-) < \{t\} < g^{-1}(Q_{n+1}^+)$ , and assign  $g^{-1}(t_{n+1}) = q$ . Once again, this assignment is consistent with the orders of both  $T$  and  $\mathbb{Q}$ .

3. Return to step 1.

Clearly this process will create a mapping that is one-to-one because each element is assigned only once. It will also be onto because every element *must* be assigned within a finite number of steps. Namely,  $t_i$  will certainly be assigned within  $i$  steps (or it may have been assigned earlier in the process). This type of argument is called a back-and-forth argument, and this is not the last time that it will be used.

Also observe that  $g$  preserves the ordering of  $T$ , and so it follows that  $T \simeq \mathbb{Q}$ . □

In the rest of this paper, our goal will be to generalize the following two results to higher cardinalities. We would like to construct minimal  $\aleph_\alpha$ -universal sets for every infinite cardinal  $\aleph_\alpha$ . As we will see, this is almost always possible.

### 3 Cofinality

However, before we can construct larger  $\aleph_\alpha$ -universal sets, we first need to review some more elementary concepts in set theory, beginning with cofinality and coinitality.

These two concepts are duals of each other, so we will mostly focus on cofinality. Keep in mind that everything we prove about cofinal sets is also true for coinital sets, only in reverse.

The notion of cofinality seeks to understand the structure of a totally ordered set. It essentially asks the question, how quickly is a set growing toward infinity, and how many element must be included to bound this growth [1]?

**Definition 15** (Cofinality). Let  $T$  be a totally ordered set.

1. Let  $F \subseteq T$ . We say that  $F$  is *cofinal* in  $T$  if, for every  $t \in T$ , there exists at  $f \in F$  such that  $t \leq f$ .
2. Let  $\nu$  be an ordinal. Then  $\nu$  is *cofinal* in  $T$  if  $S$  has a cofinal subset which is isomorphic to  $\nu$  (i.e. the cofinal subset must be

well-ordered).

3. Let  $\mu$  be the *least* ordinal that is cofinal in  $T$ . Then we say that  $T$  has *cofinality*  $\mu$ , and we write  $\text{cf}(T) = \mu$ .

Note that clearly  $T$  is cofinal in  $T$ , and also that, if  $T$  has a greatest element  $g$ , then the set  $\{g\}$  is cofinal in  $T$ . We might wonder if it is always the case that there will be at least one ordinal that is cofinal in  $T$ . If that was not the case, then the third piece of our definition would not make sense, because we cannot guarantee that such an ordinal exists. However, we need not worry about this, and the next theorem shows [1].

**Theorem 16.** Let  $T$  be a totally ordered set with order  $\leq$ . Then there exists a well-ordered subset  $W \subseteq T$  that is cofinal in  $T$ , and that satisfies  $||W|| \leq |T|$ .

*Proof.* Recall that there is some well-ordering  $\preceq$  of  $S$  with the property that  $S$  has *ordinality*  $|S|$  under the order  $\preceq$ . Then we can write  $S$  as

$$S = \{s_1, s_2, \dots, s_{|S|}\},$$

where the elements  $s_\nu$  are in ascending order under  $\preceq$ . We will use this enumeration to construct our cofinal subset  $W$ , however note that  $W$  must be well-ordered under  $\leq$ , not  $\preceq$ . To solve this problem, we choose a subset of  $S$  where both orders agree. We define our cofinal subset  $W$  as

$$W = \{s_\nu \in S \mid s_\kappa < s_\nu \text{ for all } \kappa < \nu\}.$$

The set  $W$  is ordered under  $\preceq$ , and it is a subset of  $S$ , and thus it is clearly well-ordered. Further, observe that  $||W|| \leq |S|$ .

Finally, we claim that  $W$  is cofinal in  $S$ . To see that this is true, suppose to the contrary that  $s_\nu$  is the least element of  $S$  that satisfies  $\{s_\nu\} > W$ . Then  $s_\kappa < s_\nu$  for all  $\kappa < \nu$ , and thus  $s_\nu \in W$ , a contradiction.  $\square$

Consider the cofinality of an ordinal number  $\nu$ . From Theorem 16, we know that  $\text{cf}(\nu) \leq \nu$ . And we can clearly see  $\nu$  is cofinal in  $\nu$ . Then an interesting question is whether there is some other ordinal  $\mu < \nu$  with the property that  $\mu$  is also cofinal in  $\nu$ .

**Definition 17.** We say that an ordinal  $\nu$  is *regular* if  $\text{cf}(\nu) = \nu$ , and we say that  $\nu$  is *singular* if  $\text{cf}(\nu) < \nu$ . We also set  $\text{cf}(\emptyset) = 0$ .

The ordinals 0 and 1 are regular, however the natural numbers greater than one are singular because they have a greatest element, and thus their cofinality is one. The first infinite ordinal (and the first infinite cardinal)  $\omega_0$  is a regular ordinal because it has cofinality  $\omega_0$ , and the ordinals  $\omega_0 + 1$ ,  $\omega_0 + 2$ ,  $\dots$  are all singular because they have a greatest element.

It appears that the only zero, one, and infinite cardinal numbers can be regular ordinals because any other ordinal will have a greatest element. Does this implication go both ways? That is, are all infinite cardinals regular? We can see easily that this is not the case by considering  $\text{cf}(\omega_\omega)$ . We know that the set  $S = \{\omega_0, \omega_1, \dots\}$  is cofinal in  $\omega_\omega$ , and also that  $S$  is order isomorphic to  $\omega_0$ . But this means that  $\text{cf}(\omega_\omega) \leq \omega_0$ , and thus  $\omega_\omega$  cannot be regular. This leads us to the following theorem.

**Theorem 18.** A regular ordinal is 0, 1, or an infinite cardinal number  $\aleph_\nu$ .

*Proof.* We have already seen that a regular ordinal must be either 0, 1, or infinite. Our claim is that all infinite regular ordinals are actually cardinal numbers.

To that end, suppose  $\mu$  is an infinite regular ordinal with  $|\mu| = \aleph_\alpha = \nu_\alpha$ . It follows that  $\nu_\alpha \leq \mu < \nu_{\alpha+1}$ , and by Theorem 16, there is a well-ordered subset  $W \subseteq \mu$  that is cofinal in  $\mu$  and has cardinality  $|W| \leq \omega_\alpha$ .

We can then conclude that

$$\text{cf}(\mu) \leq |W| \leq \omega_\alpha \leq \mu, \tag{5}$$

and since  $\mu$  is regular, we have that  $\text{cf}(\mu) = \mu$ . It then follows that  $\mu = \omega_\alpha$ .  $\square$

We can use this result to show that the cofinality of any totally ordered set must be either zero, one, or a regular initial number.

Let's recap what we have learned about cofinality: every set  $S$  has a well-ordered subset  $W \subseteq S$  that is cofinal in  $S$ . This cofinal set  $W$  itself is regular, and thus it is 0, 1, or an infinite cardinal number  $\aleph_\nu$  [1].

## 4 $\aleph_\alpha$ -universal sets

Now that we have reviewed cofinality, we are in a position to construct  $\aleph_\alpha$ -universal sets. Recall that a set  $S$  is  $\aleph_\alpha$ -universal if every totally ordered set of cardinality  $\aleph_\alpha$  can be embedded into  $S$ .

In this section, we will study a class of sets called  $\eta_\alpha$ -sets. These sets are, in some sense, a generalization of the rational numbers to larger cardinalities.

In particular, these sets preserve the features that make the rational numbers minimally  $\aleph_0$ -universal. We will show that, if a few conditions hold, a minimal  $\eta_\alpha$ -set is also minimally  $\aleph_\alpha$ -universal [1].

#### 4.1 $\eta_\alpha$ -sets

**Definition 19.** We say that  $S$  is an  $\eta_\alpha$ -set if it has no neighboring subsets which both have a cardinality less than  $\aleph_\alpha$ . The set  $S$  is a *minimal*  $\eta_\alpha$ -set if  $|S| = \aleph_\alpha$ .

This definition sounds quite complicated, but in reality, it just describes how “densely packed” the set  $S$  is. For example, if  $A$  and  $B$  are neighboring subsets in an  $\eta_0$ -set, then either  $A$  or  $B$  has cardinality  $\aleph_0$ . If this is not the case (i.e. both  $A$  and  $B$  are finite), then there is some element between  $A$  and  $B$ .

For example,  $\mathbb{Q}$  and  $\mathbb{R}$  both have this property, and thus they are  $\eta_0$ -sets. If the sets  $A$  and  $B$  were both finite, then  $A$  would have a greatest element  $a$ , and  $B$  would have a least element  $b$ , and so there would be an element between  $A$  and  $B$ , namely  $(a + b)/2$ .

Suppose that we have successfully constructed an  $\eta_\alpha$ -set. Let us first verify that it is indeed  $\aleph_\alpha$  universal.

**Theorem 20.** An  $\eta_\alpha$ -set is  $\aleph_\alpha$ -universal.

*Proof.* Suppose  $S$  is an  $\eta_\alpha$ -set, and  $A$  is a totally ordered set with  $|A| \leq \aleph_\alpha$ . We need to show that  $A \preceq S$ .

To do this, we will construct an order preserving injection  $f : A \rightarrow S$  using transfinite induction. We begin by assigning  $A$  a well-order. We can then use this well-order to list the elements of  $A$ :

$$A = \{a_0, a_1, a_2, \dots, a_{|A|}\}$$

We assign  $f(a_0)$  to any element in  $S$ , and now suppose that  $f$  has been defined in such a way that it preserves the order of the first  $\beta$  elements of  $A$ . We would like to define  $f(a_{\beta+1})$  so that  $f$  is still order preserving.

With this goal in mind, we construct two subsets of  $A$  given by

$$\begin{aligned} A_{\beta+1}^+ &= \{a_\nu \mid \nu \leq \beta \text{ and } a_\nu > a_{\beta+1}\}, \\ A_{\beta+1}^- &= \{a_\nu \mid \nu \leq \beta \text{ and } a_\nu < a_{\beta+1}\}. \end{aligned}$$

We know that  $|A_{\beta+1}^-| \leq \aleph_\alpha$  and that  $|A_{\beta+1}^+| \leq \aleph_\alpha$ , and also that  $f(A_{\beta+1}^-) < f(A_{\beta+1}^+)$ . It then follows that there is an element  $s \in S$  with the property that

$$f(A_{\beta+1}^-) < s < f(A_{\beta+1}^+).$$

Thus we can make the assignment  $f(a_{\beta+1}) = s$  which maintains the order preserving nature of the mapping. Clearly every element in  $S$  is assigned only once, so  $f$  is injective, and thus the function  $f$  embeds  $A$  into  $S$ .  $\square$

We have seen that  $\eta_\alpha$ -sets are  $\aleph_\alpha$ -universal. It clearly follows that a minimal  $\eta_\alpha$ -set will also be minimally  $\aleph_\alpha$ -universal.

**Corollary 21.** A minimal  $\eta_\alpha$ -set is minimally  $\aleph_\alpha$ -universal.

Now we consider the uniqueness of minimal  $\eta_\alpha$ -sets.

**Theorem 22.** If  $A$  and  $B$  are both minimal  $\eta_\alpha$ -sets, then  $A \simeq B$ .

*Proof.* Let  $a_\nu$  and  $b_\mu$  be well-orders of  $A$  and  $B$  respectively. We will construct an order preserving bijection  $f : A \leftrightarrow B$  using a transfinite back-and-forth argument.

Suppose that some of the elements of  $A$  and  $B$  have been assigned images under  $f$  in such a way that the order of these elements is preserved under the mapping. We choose the first element  $a_\beta \in A$  that has not yet been assigned under  $f$ , and construct two sets:

$$\begin{aligned} A_\beta^+ &= \{a_\nu \mid a_\nu \text{ has an image under } f \text{ and } a_\mu > a_\beta\} \\ A_\beta^- &= \{a_\nu \mid a_\nu \text{ has an image under } f \text{ and } a_\mu < a_\beta\}. \end{aligned}$$

Notice that  $A_\beta^-$  and  $A_\beta^+$  have cardinality less than  $\aleph_\alpha$  because they are both subsets of  $A$ . Also observe that  $f(A_\beta^-) < f(A_\beta^+)$  in the set  $B$ . Now, because  $B$  is an  $\eta_\alpha$ -set, we can choose  $b \in B$  such that  $f(A_\beta^-) < b < f(A_\beta^+)$ . We make the assignment  $f(a_\nu) = b$  which continues to preserve the orders of  $A$  and  $B$ .

Now we repeat this process for the first element  $b_\beta \in B$  that has not been assigned an image, and so on. The mapping  $f$  will continue to be order preserving, and it will be injective because each assignment uses elements that have not already been paired. Finally,  $f$  is surjective because each element is guaranteed to be paired after some (transfinite) number of iterations.

Therefore  $A \simeq B$ , and thus minimal  $\eta_\alpha$ -sets are unique.  $\square$

To summarize, we know that  $\eta_\alpha$ -sets are  $\aleph_\alpha$ -universal, and that minimal  $\eta_\alpha$ -sets are unique.

Another reasonable question we might ask is are minimal  $\aleph_\alpha$ -universal sets unique? The answer is no. For example, both  $\mathbb{Q}$ , and  $\mathbb{Q}^+ \cup \{0\}$ , the set of non-negative rational numbers are minimally  $\aleph_0$ -universal, however the two sets are clearly not order-isomorphic because one has a least element and the other does not.

## 4.2 Constructing a minimal $\eta_\alpha$ -set

Now we turn our attention to the construction of minimal  $\eta_\alpha$ -sets. We will see that such a set can be constructed whenever  $\aleph_\alpha$  is regular, and a relaxed statement of the GCH holds.

Consider the set  $E = \{-1, 0, 1\}$  with the standard ordering. Now let  $Q_\alpha$  be the collection of all functions  $f : \aleph_\alpha \rightarrow E$  with the property that for some  $\mu < \aleph_\alpha$ , we have  $f(\nu) = 0$  for all  $\mu \leq \nu < \aleph_\alpha$ . That is,  $Q_\alpha$  is the set of all functions from  $\aleph_\alpha$  to  $E$  that are eventually always zero.

We can also view the elements of  $Q_\alpha$  as tuples of length  $\aleph_\alpha$ . Each entry of each tuple can take on one of the three values in  $E$ , and each tuple must eventually become all zeros.

Suppose that  $f, g \in Q_\alpha$ , and that the first element  $\aleph_\alpha$  for which  $f$  and  $g$  disagree is  $\nu$ . Then we will say that  $f < g$  if and only if  $f(\nu) < g(\nu)$ .

**Theorem 23.** Suppose  $\aleph_\alpha$  is a regular cardinal. Then the set  $Q_\alpha$  is an  $\eta_\alpha$ -set.

*Proof.* Suppose  $A$  and  $B$  with  $A < B$  are neighboring subsets in  $Q_\alpha$ . Further, suppose that  $\text{cf}(A) < \aleph_\alpha$ . Then we need to show that  $B$  has a coinital subset of order type  $\aleph_\alpha$ . Later, we will argue that the definition of  $Q_\alpha$  is symmetric, and thus all of the arguments we make to construct coinital subsets in  $B$  can also be flipped to construct cofinal subsets in  $A$ .

We consider three cases:  $\text{cf}(A) = 0$ ,  $\text{cf}(A) = 1$ , and  $\text{cf}(A) = \kappa$  for some regular infinite ordinal  $\kappa$ . First suppose that  $A = \emptyset$ . Then, for each  $\beta < \aleph_\alpha$ , we define  $h_\beta$  as

$$h_\beta(\nu) = \begin{cases} -1 & \text{if } \nu < \beta, \\ 0 & \text{if } \nu \geq \beta. \end{cases}$$

Clearly the set  $H = \{h_\beta\}_{\beta \in \aleph_\alpha}$  is well-ordered, and has cardinality  $\aleph_\alpha$ . We know that  $\aleph_\alpha$  is regular, so the cofinality of  $B$  is  $\aleph_\alpha$ , as we had hoped. All that remains is to show that  $H$  is coinital in  $B$ .

Suppose  $f \in B$  is some function, and let  $\nu$  be the first ordinal for which  $f(\nu) \neq -1$  (we know such a  $\nu$  exists because  $f$  must eventually be always zero). Then  $h_\nu < f$ , as required.

Now suppose that the set  $A$  has a greatest element  $g$  (i.e.  $\text{cf}(A) = 1$ ). Let  $\mu$  be the ordinal for which  $g(\nu) = 0$  for all  $\mu \leq \nu \leq \aleph_\alpha$ . Notice that the first  $\mu$  entries of each element in  $B$  must be equal to  $g$ .

Then for each  $\beta < \aleph_\alpha$ , we define a function

$$h_\beta(\nu) = \begin{cases} g(\nu) & \text{if } \nu < \mu \\ -1 & \text{if } \mu \leq \nu < \mu + \beta \\ 0 & \text{if } \mu + \beta \leq \nu. \end{cases}$$

That is, the function  $h_\beta$  is equivalent to  $g$  while  $g$  is nonzero, followed by  $\beta$  copies of  $-1$ , and then finally all zeros. Once again, the set  $H = \{h_\beta\}_{\beta < \aleph_\alpha}$  has cardinality  $\aleph_\alpha$ , so if we can conclude that  $H$  is cointial in  $B$ , we will be done.

Suppose  $f \in B$  is some function. Let  $\gamma$  be the first entry of  $f$  after  $\mu$  that is not equal to  $-1$ . Then  $h_\gamma < f$ , as required.

Finally, suppose that  $\text{cf}(A) = \kappa < \aleph_\alpha$ . It follows that  $A$  has a cofinal subset  $\{f_\delta\}_{\delta < \kappa}$  where  $f_\delta < f_\epsilon$  if and only if  $\delta < \epsilon$ .

Our goal is to construct the supremum of the set  $A$ . If we can do this, then the same family of functions that we used in the previous case will be a cofinal in  $B$ .

To this end, we would like to define two strictly increasing sequences for all  $\nu < \kappa$  that are bounded by  $\kappa$  and  $\aleph_\alpha$  respectively. We will denote the first sequence as  $\delta_\nu < \kappa$ , and the second sequence as  $\beta_\nu < \aleph_\alpha$ . Additionally, we require that these sequences satisfy, for all  $\nu < \kappa$ ,

1.  $f_{\delta_\nu}(\gamma) = f_\epsilon(\gamma)$  for all  $\epsilon > \delta_\nu$  and for all  $\gamma < \beta_\nu$  and
2.  $f_{\delta_\nu}(\beta_\nu) < f_\epsilon(\beta_\nu)$  for some  $\epsilon > \delta_\nu$ .

We construct these sequences using transfinite induction. Suppose we have constructed such a pair of sequences for all  $\nu < \mu$ . We need to construct  $\delta_\mu$  and  $\beta_\mu$ .

Let  $\bar{\beta} = \sup\{\beta_\nu\}_{\nu < \mu}$ , and let  $\bar{\delta} = \sup\{\delta_\nu\}_{\nu < \mu}$ . We choose  $\delta_\mu$  so that  $\bar{\delta} \leq \delta_\mu < \kappa$ , and

$$f_{\delta_\mu}(\bar{\beta}) = \max\{f_\epsilon(\bar{\beta}) : \bar{\delta} \leq \epsilon < \kappa\} \in E.$$

We also choose  $\bar{\beta} \leq \beta_\mu < \aleph_\alpha$  such that

$$\beta_\mu = \min\{\gamma \geq \bar{\beta} : f_{\delta_\mu}(\gamma) < f_\epsilon(\gamma) \text{ for some } \delta_\mu < \epsilon < \aleph_\alpha\}.$$

Now we must verify that  $\delta_\mu$  and  $\beta_\mu$  actually satisfy our two conditions. However, this is easy because, by construction,  $\beta_\mu$  is the *least* ordinal for which  $f_{\delta_\mu}$  is different than a function that is greater than it, and thus both conditions are satisfied.

Now we will actually find the supremum of the set  $A$ . Let  $\theta = \sup\{\beta_\nu : \nu < \kappa\}$ . This means that  $\theta$  is a limit ordinal, and, since  $\aleph_\alpha$  is regular, we have that  $\theta < \aleph_\alpha$ .

We define the function  $g : \theta \rightarrow E$  as follows: for each  $\gamma \in \theta$ , choose a  $\nu < \kappa$  such that  $\gamma < \beta_\nu$ , and let  $g(\gamma) = f_{\delta_\nu}(\gamma)$ . Because of the construction of our two sequences, this does not depend on which  $\nu$  we choose.

We then define the same family of functions as in the previous case. In particular, we have the functions  $h_\beta$  such that

$$h_\beta(\nu) = \begin{cases} g(\nu) & \text{if } \nu < \theta \\ -1 & \text{if } \kappa \leq \nu < \theta + \beta \\ 0 & \text{if } \theta + \beta \leq \nu. \end{cases}$$

Clearly this set has cardinality  $\aleph_\alpha$ , so we need to show that it is coinital in  $B$ .

Notice that every element in  $B$ , when restricted to  $\theta$ , must be greater than  $g$ , and so for the same reason as the previous case, the family  $h_\beta$  will be coinital in  $B$ .

We have shown that, if  $\text{cf}(A) < \aleph_\alpha$ , then we must have  $\text{coin}(B) \geq \aleph_\alpha$ . Now we need to show that the reverse is also true. However, notice that the function  $m : E \rightarrow E$  given by  $m(e) = -e$  induces an order-reversing bijection on  $Q_\alpha$ , and thus all of the work we have just done can just as easily be applied to this new case.

Therefore,  $Q_\alpha$  is indeed an  $\eta_\alpha$ -set. □

We have just shown that  $Q_\alpha$  is an  $\eta_\alpha$ -set, however, our goal in this section was to construct a *minimal*  $\eta_\alpha$ -set. We claim that, provided a slightly less restrictive version of the GHC holds,  $Q_\alpha$  is *the* minimal  $\eta_\alpha$  set.

**Theorem 24.** Suppose  $\aleph_\alpha$  is a regular ordinal, and that for every ordinal  $\kappa < \aleph_\alpha$ , we have  $2^\kappa \leq \aleph_\alpha$ . Then the totally ordered set  $Q_\alpha$  is the minimal  $\eta_\alpha$ -set.

*Proof.* We have already seen that, provided that  $\aleph_\alpha$  is a regular ordinal,  $Q_\alpha$  is an  $\eta_\alpha$ -set. All that remains is to verify that  $|Q_\alpha| = \aleph_\alpha$ . Let  $Q_\alpha[\nu]$  be the set of functions in  $Q_\alpha$  which are always zero after  $\nu$ . That is

$$Q_\alpha[\nu] = \{f \in Q_\alpha \mid f(\mu) = 0 \text{ for all } \mu > \nu\}.$$

Observe that  $|Q_\alpha[\nu]| = 3^\nu \leq \aleph_\alpha$  by our hypothesis, and also observe that

$$Q_\alpha = \bigcup_{\nu \in \aleph_\alpha} Q_\alpha[\nu].$$

It follows that  $Q_\alpha$  is the union of  $\aleph_\alpha$  sets, each with cardinality less than or equal to  $\aleph_\alpha$ , and thus  $|Q_\alpha| = \aleph_\alpha$ , as required.  $\square$

An obvious corollary follows from the fact that  $\aleph_1$  is a regular ordinal.

**Corollary 25.** Suppose the continuum hypothesis holds. Then there is a minimal  $\eta_1$ -set.

We have shown that we can construct minimal  $\aleph_\alpha$ -universal sets in some circumstances. Now we consider the slightly broader question: can we always construct a (not necessarily minimal)  $\aleph_\alpha$ -universal set?

The answer is yes. To see why, notice that an  $\eta_\beta$ -set is also an  $\eta_\alpha$ -set for all  $\beta \leq \alpha$ . This means that, as long as the GCH holds, we can always construct an  $\eta_\beta$ -set that is also an  $\eta_\alpha$ -set, as we desired.

## 5 Conclusion

In this paper, we have considered some of the foundations of set theory. In particular, we have focused on developing a sufficient amount of mathematical machinery so that we could explore some interesting results about universally ordered sets. We proved that minimal  $\aleph_\alpha$ -universal sets exist provided  $\aleph_\alpha$  is regular, and the GCH holds. We also showed that minimal  $\eta_\alpha$ -sets are unique up to order-isomorphism, and that it is always possible to construct an  $\aleph_\alpha$ -universal set, as long as the GCH holds.

This is an extremely rich subfield of set theory, and there are many directions in which this work could be taken in the future. For example, the construction of  $\eta_\alpha$ -sets is related to the surreal numbers, and alternative construction of the real numbers. Additionally, there are still many interesting questions remaining about minimal  $\aleph_\alpha$ -universal sets; unlike minimal  $\eta_\alpha$ -sets, these sets are not unique, however, we might conjecture that all minimal  $\aleph_\alpha$ -universal sets share some common core structure. As always in mathematics, more thought is needed.

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