The Median Value of a Function on an Interval

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1 Introduction

This paper is a review of the median value of a function on an interval. Motivated by *The Median Value Of A Continuous Function* by Irl C. Bivens and Benjamin G. Klein, we consider the median values of various continuous functions, and focus in particular, we focus on minimization properties of the median value. We attempt to build an intuitive connection between the median of a set of discrete numbers and the median of a function on an interval. To help the reader see the characteristics of the median value, we have created a number of interactive examples that demonstrate the minimization properties of the median value and published these on the website GeoGebraTube. The reader will find links to relevant examples embeded sequentially throughout the paper, and is encouraged to interact with these examples at her or his own pace to acheive a concrete view of the phenomena described. This will help the reader to build profeciency in predicting the median of a function based on the geometric properties of that function's graph.

We go on to investigate discontinuous functions and consider how the median values of these functions differ from the median values of continuous functions.Once we relax the condition of continuity, we introduce a new definition for the median value of a function on an interval. This definition is motivated by theorems and defintions from Bivens and Klein, highlighting the minimization properties that are the focus of this paper and streamlining discussion of these properties. Finally, we give some examples to which we can apply the logic of the median value, including functions of more than one variable.

2 From Discrete Points to an Interval

2.1 The Average

The initial motivation for this subject is the simple process by which the concept of the average can be applied to a continuous function. We begin by outlining that process so that the reader may see analogous steps regarding the median value. Start by considering a set of discrete numbers

$$U = \{x_1, x_2, x_3, ..., x_n\}.$$

Then

$$\text{ave}(U) = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

is the the average of the values in U.This leads naturally to the following definition of the average value of a continuous function.

Definition 1. *Suppose $f(x)$ is continuous on $[a, b]$. Partition $[a, b]$ into n equal subintervals and call the midpoints of these subintervals $x_1^*, x_2^*, \ldots, x_n^*$. Let $\text{ave}_f(n)$ be the average value of the n function values $f(x_1^*), f(x_2^*), \ldots, f(x_n^*)$.*

Then the average value of f on $[a, b]$ is

$$f_{ave} = \lim_{n\to\infty} \underset{f}{\text{ave}}(n)$$

This definition is typically accompanied by the fact that

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx,$$

which looks rather like the average of a set of discrete points, with the sum replaced by an integral and the number of values averaged replaced by the length of an interval. This demonstration helps convince a calculus student of the sensible link between averages of points and averages of functions, and helps her or him build an intuition for the average of a function by relating it to a discrete average.

2.2 The Median

In discussing the average, the median value of a set is often forgotten. There are many aspects of the median value that have analogs in the more common discussion of the average. We fowllow this analogy while introducing the median in order to help illustrate the median as an approachable and useful quantity.

Definition 2. *Suppose the entries of the set U, $x_1, x_2, x_3, ..., x_n$ are in increasing order. Then the median value of U is*

$$\text{med}(U) = \begin{cases} x_{(n+1)/2}, & n \textit{ is odd} \\ \frac{1}{2}(x_{n/2} + x_{1+n/2}), & n \textit{ is even} \end{cases}$$

In practice, this value is located by arranging the entries of U in increasing order, then taking the middle entry on the list. The middle two are averaged if there is an even number of entries. Note that only the middle entry, or middle two entries, of U determine the value of med(U). This is what makes the median a so-called robust statistic: any of the smallest entries of U could be made arbitrarily small without affecting the median, and any of the largest entries of U could be made arbitrarily large without affecting the median. That is, outliers of U have little effect on its median value.

Another consequence of this definition is that at most half of the entries of U are greater than med(U) and at most half are less than med(U). From this insight a useful minimization theorem follows.

Theorem 1. *Suppose U is a list of real numbers, $x_1, x_2, x_3, ..., x_n$. Then for any real number t,*

$$\sum_{k=1}^{n} |x_k - \text{med}(U)| \leq \sum_{k=1}^{n} |x_k - t|.$$

[1]

We explain this theorem with an appeal to the reader's intuition. Let t be a real number and suppose that more than half the entries of U are greater than t. Then $t \neq med(A)$, and increasing t will decrease $|x_k - t|$ for more than half of the n terms of $\sum_{k=1}^{n} |x_k - t|$, decreasing the sum. Next, suppose that more than half the entries of U are less than t. Then $t \neq med(A)$. Now decreasing t will decrease $|x_k - t|$ for more than half of the n terms of $\sum_{k=1}^{n} |x_k - t|$, decreasing the sum. In this way we can further decrease $\sum_{k=1}^{n} |x_k - t|$ as long as more than half the entries of A are less than or greater than t. Because at most half of the entries of U are greater than $med(A)$ and at most half are less than $med(U)$, if $t = med(A)$, then the sum $|x_k - t|$ is at a minimum. This argument is more convinving with a visual demonstration. At this point we introduce the first of our examples published on the website GeoGebraTube.

Example 1. *The reader may use this demonstration in order to better visualize this sum. The example consists of a set of seven points, with the x coordinates of these points representing the points $x_1, x_2, x_3, ..., x_7$ in $\mathbb{R}$. The orange point on the x axis represents the number t, and the horizontal, black lines show the quantities $|x_k - t|$, the distances from each point to t. The reader may drag the orange point to manipulate the value of t and note how the distances to the points in the set and the sum of these distances change in response.*

In a similar procces to that which we have applied to the average, we now define the median of a continuous function using a limit of discrete medians.

Definition 3. *Suppose $f(x)$ is continuous on $[a, b]$. Partition $[a, b]$ into n equal subintervals and call the midpoints of these subintervals $x_1^*, x_2^*, \ldots, x_n^*$. Let $med_f(n)$ be the median value of the n function values $f(x_1^*), f(x_2^*), \ldots, f(x_n^*)$.*

Then the median value of f on $[a, b]$ is

$$f_{med} = \lim_{n \to \infty} med_f(n)$$

[1] For the average, we made the transition from the quantity

$$\text{ave}(U) = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

to

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)dx. \tag{1}$$

In an analagous shift, we turn the sum over the entries of U,

$$\sum_{k=1}^{n} |x_k - t|, \tag{2}$$

into an integral over the domain $[a, b]$,

$$A(t) = \int_a^b |f(x) - t| dx.$$

These quantities do not explicitly define the median as Equations 2 and 1 define the average. But, we now know that 1 implictly gives the median of U by way of minimization, and we see that $A(t)$ is a quantity similar to $\sum_{k=1}^{n} |x_k - t|$, with the entries of U replaced by the function values of f, and the integral giving a continuous sum over $[a, b]$. We can therefore expect that $f(x) < f_{med}$ on at most half the interval $[a, b]$, and $f(x) > f_{med}$ on at most half the interval $[a, b]$.

We use the familiar sine function as an example to demonstrate this property. This page in Geogebra shows the area function for f(x)=sin(x) on the interval $[0, 2\pi]$. The slider in the upper left allows the reader to manipulate the value of t, and the resulting value $A(t)$ is shown so that she or he may note how this manipulation affects the total area. The length of the interval between the points C and C', on which $f(x) \geq t$, also is shown as "LengthAbove." Similarly, the combined length of the intervals on either side, on which $f(x) \leq t$, is shown as "LengthBelow." The reader should note that the minimum value of $A(t)$ coincides with equality between *LengthAbove* and *LengthBelow*.

2.3 Graphs with Symmmetry

One simple result of equally dividing the sets of function values "above" and "below" t is that, for any f monotonic on $[a, b]$, $f_{med} = f(\frac{a+b}{2})$. In fact, it is straightforward to see that $f_{med} = f(\frac{a+b}{2})$ if f has one of various types of symmetry:

1. f lies beneath the line $y = f(\frac{a+b}{2})$ for $x \leq \frac{a+b}{2}$ and above $y = f(\frac{a+b}{2})$ for $x \geq \frac{a+b}{2}$
2. f lies above the line $y = f(\frac{a+b}{2})$ for $x \leq \frac{a+b}{2}$ and beneath $y = f(\frac{a+b}{2})$ for $x \geq \frac{a+b}{2}$
3. The graph of f has rotational symmetry of order 2 about the point $(\frac{a+b}{2}, f(\frac{a+b}{2}))$

As described for discrete sets, it is significant that the median value only depends directly on the function value at one point, $\frac{a+b}{2}$. This is why we can make these generalizations about some median values by noting the general behavior of the function on either side of the center of the interval $[a, b]$. In particular, if $f(x) > f_{med}$ at any point of $[a, b]$, then it does not alter the median to increase arbitrarily the function value at that point.

3 Using Measure

To specify the notion of "half the interval," we will introduce some basic concepts of measure theory. In particular, if the set U is an interval or a collection of intervals, then the Lebesgue measure, which we will simply call the "measure," of U is the sum of the lengths of the intervals in U. The measure of a single point is 0. We will let $m[U]$ denote the measure of U.

Now, for f defined on $[a, b]$, define the sets

$$Above(t) = f^{-1}((t, \infty)) = x \in (a, b) | f(x) > t$$

$$Below(t) = f^{-1}((-\infty, t)) = x \in (a, b) | f(x) < t.$$

In the example of the sine function, $Above(t)$ is the two intervals identified by the orange lines . When we discuss the set of x values for which $f(x) \geq t$, we are discussing the set $Above(t)$.

With Theorem 2, Bivens and Klein use the concept of measure to provide insight and a useful criterion for a median value.

Theorem 2. *Suppose f is continuous on $[a, b]$. A real number t_m is a median of f on $[a, b]$ if and only if*

$$m[Below(t_m)] \leq \frac{b-a}{2} \quad \text{and} \quad m[Above(t_m)] \leq \frac{b-a}{2}.$$

Note that $m[Below(t_m)] < \frac{b-a}{2}$ or $m[Above(t_m)] < \frac{b-a}{2}$ only if $f(x) = f_{med}$ on a set of positive measure. In Theorem 1, if $x_k \geq t$ for more than $\frac{n}{2}$ entries, increasing t lowers the sum $\sum_{k=1}^{n} |x_k - t|$ by shrinking more terms than it expands. This is analagous to the case where $f(x) > t$ on more than half of the interval $[a, b]$, which is the same as saying $m[Above(t)] > \frac{a+b}{2}$. Thus we argue by way of extension from the discrete sum $\sum_{k=1}^{n} |x_k - t|$ to $A(t)$, that the median value of a function should minimize $A(t)$ for that function by implying the inequalities of Theorem 2. Bivens and Klein confirm this conjecture with their central theorem regarding the continuous median.

Theorem 3. *The median value f_{med} exists for any continuous function f on a bounded interval. Furthermore, if f is absolutely integrable on the interval, then $t = t_m = f_{med}$ is the unique parameter value that minimizes the area function $A(t)$.*

This absolute value function shows that, if $t \neq fmed$, moving t closer to f_{med} decreases the integral $A(t) = \int_a^b |f(x) - t| dx$ by subtracting more area than it adds. The horizontal, red line is $y = t$, the blue section of the x axis is the set $Below(t)$, and the orange section of the x axis is the set $Above(t)$. The dark shaded regions directly above and below $y = t$ represent the area removed from $A(t)$ by increasing or decreasing t by a small increment, respectively. Note that, until $Above(t)$ and $Below(t)$ each represent half the interval $[a, b]$, we can continue to decrease $A(t)$, so $A(t)$ is not at a minimum.

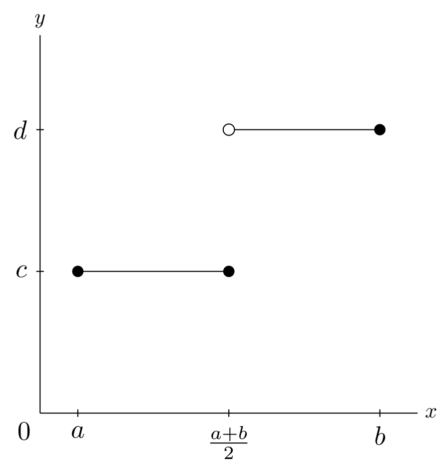

Figure 1: Step function

Bivens and Klein give a proof of Theorem 3. Rather than investigating this proof, we will move our discussion forward with more examples.

4 Functions With Discontinuities

We now seek to further understand the median by examining what happens when we weaken the hypotheses of Theorem 3. We begin with a simple example, graphed in Figure 1, where we have abandoned the condition of continuity.

Example 2.

$$f = \begin{cases} c & \text{if} \quad a \leq x \leq \frac{a+b}{2} \\ d & \text{if} \quad \frac{a+b}{2} < x \leq b \end{cases}$$

f is not continuous and Definition 3 therefore does not apply. However, for the sake of investigation, we can still apply the process described, say that $f_{med} = \frac{c+d}{2}$, and recognize that if we discuss the median of a discontinuous function, we are applying the process of Definition 3 without satisfying all of its hypotheses. We find by this process that many discontinuous functions still have sensible median values. Where we find a difference compared to continuous functions is in the uniqueness result of Theorem 3. The median value of a discontinuous function still minimizes $A(t)$, but that median value may not be the unique minimizer of $A(t)$.

In this representation, it is visually apparent that, as long as $2 \leq t \leq 4$, altering the value of t adds the same amount of shaded area to $A(t)$ that it subtracts. We use the derivative of $A(t)$ to show that this is the case for any similar function, with generic constants c and d given as function values instead of 2 and 4. this in a simple proof involving the derivative of f.

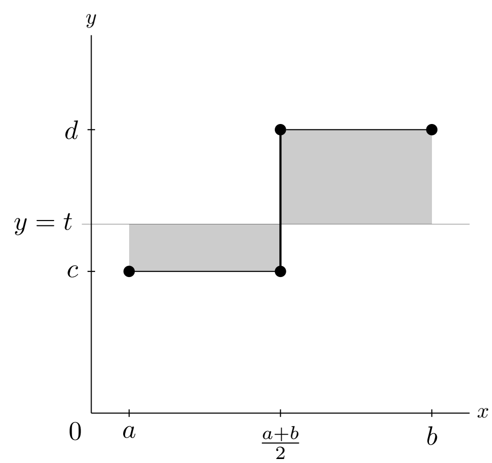

Figure 2: Continuous graph

Proof.

$$
\begin{aligned}
A(t) &= \int_a^b |f(x)-t|dx \\
A'(t) &= -\int_a^b \frac{|f(x)-t|}{f(x)-t}dx \\
A'(t) &= -\int_a^{\frac{a+b}{2}} \frac{t-c}{c-t}dx - \int_{\frac{a+b}{2}}^b \frac{d-t}{d-t}dx \\
&= (\frac{a+b}{2}-a)-(b-\frac{a+b}{2}) \\
&= 0
\end{aligned}
$$

□

$A(t) = A(t_m)$ for any $c \leq t \leq d$, so every t value in that range is a minimizer, and t_m is not unique. We can see, consistent with Theorem 3, that this is not possible for a continuous function. As in Figure 2, connecting the two present steps with a vertical line is the only way to create a continuous graph with a constant area for $c \leq t \leq d$. This is impossible, of course, because then f is not well defined on $[a,b]$.

There are many different types of discontinuity to consider in relation to the median of a fuction.

1. For monotonic f with a single jump discontinuity, if the discontinuity is not at $\frac{a+b}{2}$ then the median of f exists and is the unique minimizer of $A(t)$. There is an interval of t values for which $A(t)$ is constant, but that interval does not include the minimizer f_{med} so that minimizer is still unique.

2. If f has a removable discontinuity, the medain of f is unchanged from the case where the discontinuity is removed to make f continuous.

3. If the function has an asymptote at one of the end points, then the median exists and it is still easy to apply Definition 3.

4. If the function has an asymptote or is otherwise undefined at $c \in (a, b)$, then the ease with which we apply the limit definition depends on the value of c. The process of taking $\lim_{n\to\infty} med_f(n)$ involves the value of the function at the midpoints of every regular partition, and therefore at every rational number in the interval $[a, b]$. So, if c is rational, then at some point the limit process will run into an undefined value.

5 More Discontinuities

We now investigate three functions that each have something to show regarding their median values.

Example 3. *Consider the graph of $f(x) = \frac{1}{\sqrt{x}}$ on the interval $[-2, 2]$. The reader can note the various symmetries about the origin and the two coordinate axes, and conclude that $f_{med} = 0$. Manipulating the value of t in Geogebra confirms that $t = 0$ does in fact minimize $A(t)$. But, any value in the range $\frac{\sqrt{2}}{2} \le t \le \frac{\sqrt{2}}{2}$ returns the same value $A(0)$, so the median of f is not the unique minimizer of $A(t)$ and we see once again a discontinuity that sacrifices the uniqueness result of Theorem 3.*

We use Example 3 as a starting point to consider what aspect of a discontinuous function may preclude uniqueness for the minimizer of $A(t)$. With a visual approach, we see that the t values that minimize $A(t)$ for each of these two functions are the gaps that contain no function values. Thus, we close the "gap" in the graphs of these functions in this example where the two branches of $f(x) = \frac{1}{\sqrt{x}}$ are shifted up and down, respectively. The branches intersect the x axis at $x = -2$ and $x = 2$. The graph still has the same symmetries that the un-shifted function $\frac{1}{\sqrt{x}}$ did, so we find that the median value is still $t_m = 0$. Though the function is still discontinuous, we have eliminated the gap between the function values, and it appears that the sets $Above(t)$ and $Below(t)$ will change if we change t. This is borne out in GeoGebra: $t_m = 0$ minimizes $A(t)$ and is now the unique minimizing value.

Motivated by this shift, we can gain some insight by altering the step function of our first example. Consider this piecewise linear function, where we turn the horizontal steps into linear segments with negative slope. We have changed relatively little from the step function; in particular, the median is still $t_m = 3$ and the jump discontinuity between the two branches is still glaringly present. Yet, as in the case where we shifted the branches of $\frac{1}{sqrtx}$, we see that this minimizer is now unique. We clearly see that continuity is not necessary for a median value to be a unique minimizer of $A(t)$, so what aspect of a function does confer uniqueness?

6 Conditions for Uniqueness

We have worked thus far with Definition 3, provided by Bivens and Klein, defining the median by a limiting process. Yet, there is little about this limiting process that we find compelling for its own sake. Our discussion focuses mainly on the minimizing property of the median value with respect to the area function $A(t) = \int_a^b |f(x) - t|dx$. We therefore frame our remaining discussion by using this minimizing characteristic to define the median value of a function.

Definition 4. *Suppose f is absolutely integrable on $[a, b]$. A real number is a median value of f on $[a, b]$ if it is a minimizing value of*

$$A(t) = \int_a^b |f(x) - t|dx.$$

This is little more that a semantic issue: we will now say that a function has a unique median rather than say that a function's median is a unique minimizer of $A(t)$. The limit definition provided an intuitive bridge to the median of a function from the discrete median, but we will proceed using Definition 4 in order to streamline our descriptions and highlight the characteristic of the median value that we are most interested in. For continuous functions, the two definitions are equivalent. We know this because Theorem 3 states that a median according to Definition 3 minimizes $A(t)$ and therefore satisfies Definition 4. Additionally, a median according to Definition 3 is the unique minimizer of $A(t)$, so if a value is a median of a continuous function according to Definition 4, it must be the median identified by Definition 3.

We can now frame our guiding question as: Under what conditions does a function on $[a, b]$ have a unique median value? Harkening back to the examples of the inverse square root function, the shifted inverse square root function, the step function, and the piecewise linear function, we note that the median values of these functions gain the characteristic of uniqueness when we close the empty space in the graph, making f a function onto an interval. This suggests the conjecture that the median of $f : [a, b] \to \mathbb{R}$ is unique if the range $f([a, b])$ is an interval. This condition is restrictive enough that we certainly expect it is sufficient for a unique median, but a one may quickly outline the counterexample of Figure 3 to demonstrate that it certainly is not necessary.

Figure 3 reiterates that, if a discontinuity exists away from the median value, it does not affect uniqueness of that median. So, given that our previous hypothsis was stronger than necessary but that an onto map seems to be helpful in guaranteeing uniqueness, we suggest a theorem that localizes the condition of the previous conjecture.

Theorem 4. *Suppose f is absolutely integrable on $[a, b]$. If f maps onto an open interval that contains a median value t_m of f, then the median t_m is unique.*

The counter example of Figure 4 shows that this condition is not necessary for uniqueness.

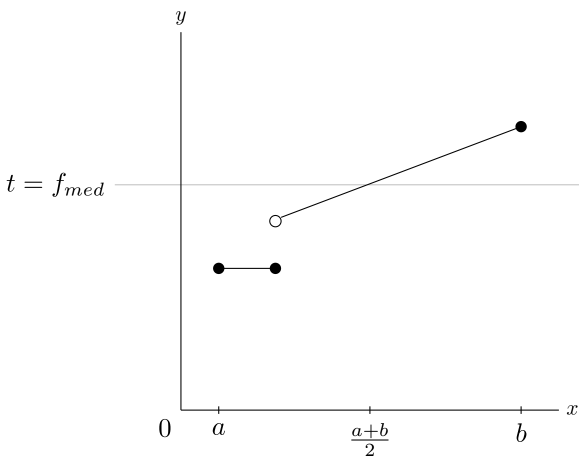

Figure 3: f not onto, with a unique median

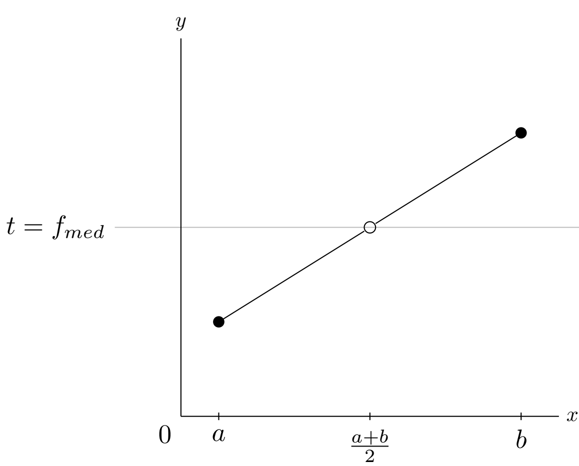

Figure 4: Removable discontinuity

We are eventually unable to prove the result for an arbitrary, absolutely integrable function. There is a class of functions, however, defined previous to our work, that satisfies exactly the condition that we hoped for. This class of functions is said to satisfy Luzin's Condition (N).

Definition 5 (Luzin's Condition (N))**.** *A function $f : G \to \mathbb{R}$ satisfies Luzin's Condition (N) if*

$$m[B] > 0 \quad \textit{whenever} \quad B \subseteq G \quad \textit{and} \quad m[f(B)] > 0$$

So, our theorem now imposes slightly more strict hypotheses.

Theorem 5. *Suppose f is absolutely integrable on $[a, b]$ and satisfies Luzin's Condition (N). If f maps onto an open interval that contains a median value t_m of f, then the median t_m is unique.*

[2]
We begin with a new definition and a lemma.

Definition 6. *convex A function $f : \mathbb{R} \to \mathbb{R}$ is* ***convex*** *on an interval $[a, b]$ if the line segment connecting any two points on the graph of f lies entirely on or above the graph of f.*

Bivens and Klein show that $A(t)$ is a convex function, a fact that we use in this lemma.

Lemma 1. *Suppose that t_m is a median value of f. If there exists another median t'_m of f, then every point in the open interval with endpoints t_m and t'_m is also a median value of f.*

Proof. Suppose that $t_m < t'_m$ and let $t^* \in (t_m, t'_m)$.

Because t_m and t'_m both minimize $A(t)$, $A(t_m) = A(t'_m)$. We know that $A(t)$ is convex, so because $t_m < t^* < t'_m$,

$$A(t^*) \leq A(t_m) \quad \text{and} \quad A(t^*) \leq A(t'_m).$$

Because t_m and t'_m both minimize $A(t)$,

$$A(t^*) \geq A(t_m) \quad \text{and} \quad A(t^*) \geq A(t'_m)$$

as well, so

$$A(t_m) = A(t^*) = A(t'_m)$$

and t^* is a median of f.

□

We now return to prove Theorem 5.

Proof. Suppose that f maps onto an open interval that contains a median value t_m of f, and there exists another median $t' > t_m$. The proof for $t' < t_m$ is similar.

Let I be the open interval in the range of f that contains t_m. Then, for some endpoint d,

$$(t_m, d) = (t_m, t') \cap I,$$

so f maps onto (t_m, d) and every point in (t_m, d) is a median of f.

Let $Above(t_m) \setminus Above(z) = \Omega$.

Then for all x such that $f(x) \in (t_m, d)$, $x \in Above(t_m)$ and $x \notin Above(d)$. Therefore, because $m[(t_m, d)] = d - t_m > 0$, Lusin's Condition (N) implies that

$$m[\Omega] > 0.$$

Now let $z \in (t_m, d)$.

Then

$$A(z) - A(t_m) = \int_{\Omega} |z - t| dx > 0,$$

so z is not a median of f. □

7 An Example and an Extension

7.1 Decaying Oscilations

Example 4. *An instructive example is $f(x) = x \sin(\frac{1}{x})$ on the interval $[0, \frac{1}{\pi}]$, as shown in Figure 5. Becausethis function is continuous, Definition 3 guarantees a unique minimizer of $A(t)$. This process is computationally tedious, however, and offers little intuitive insight. The other method available is to use Theorem 2, which does not give the median value for a function explicitly but gives a condition that we can use in an approximating process.*

We might guess, based on its oscilation around the "center" $y = 0$, that $t = 0$ is the median of this function. We can use Theorem 2 to disprove this conjecture and show that $f_{med} < 0$.

Proof. $\sin(n\pi) = 0$, so $f(\frac{1}{n\pi}) = 0$. This means that $Below(t)$ is the union of every interval

$$(\frac{1}{2i\pi}, \frac{1}{2i\pi + 1}), \ i \geq 0,$$

and

$$\begin{aligned} m[Below(t)] &= \sum_{i=0}^{\infty} (\frac{1}{2i\pi + 1} - \frac{1}{2i\pi}) \\ &= \sum_{i=0}^{\infty} \frac{1}{(4i^2 + 2i)\pi}. \end{aligned}$$

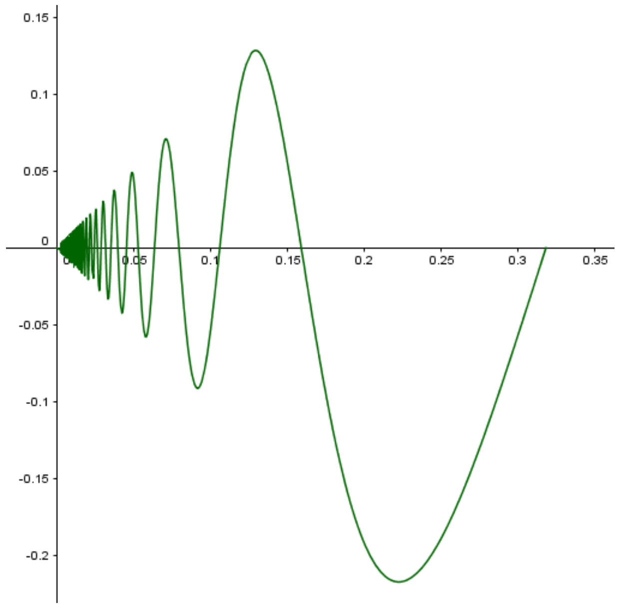

Figure 5: $f(x) = x\sin(\frac{1}{x})$

This is a simple convergent series; the sum must be finite and less than $\frac{1}{\pi}$ because each of the intervals is within $[0, \frac{1}{\pi}]$. Evaluating the sum we find that

$$\begin{aligned} m[Below(t)] &= \sum_{i=0}^{\infty} \frac{1}{(4i^2+2i)\pi} \\ &= \frac{\ln(2)}{\pi} \approx 0.221 \\ &> \frac{1}{2\pi} \approx 0.1592. \end{aligned}$$

So, $t = 0$ cannot be a median value of f, and because $m[Below(t)] \geq m[Below(0)]$ for $t > 0$, we know that $f_{med} < 0$.

□

Just as we calculated $m[Below(0)]$ using the roots of f, we can calculate $m[Below(t)]$ for a given value of t by solving for the intersections where $x\sin(\frac{1}{x}) = t$ for $t \neq 0$. We have written a program that finds these intersections, takes the difference of adjacent intersections, and then takes the sum of these differences, thereby computing the length of the intervals in $Below(t)$. If $m[Below(t)] > \frac{1}{2\pi} \approx 0.1592$, we know that t is too high and must lower it to approach f_{med}. Conversely, if $m[Above(t)] < \frac{1}{2\pi}$, we know that t is too low and we must raise it. By repeating this process, we find that $m[Below(-0.4587)] = 0.15915$ so, accurate to four decimal places, $f_{med} = -0.4587$.

7.2 Three Dimensions

We can extend the logic of Theorem 3 and Theorem 2 to realize that there is a close analogy to the two dimensional median in three dimensions. we now consider the median values of a function of two variables.

Example 5. *Let S be the surface of the hemisphere of radius 1, with its base on the xy plane, centered at the origin.*

We define the volume function

$$V(t) = \int_{\Omega} |f - t| dA$$

as an analog to the area function $A(t)$. In this case, $f(x, y) = \sqrt{1 - x^2 - y^2}$ and the region Ω is the circle of radius 1 in the xy plane centered at the origin. $V(t)$ gives the volume in between the surface S and the horizontal plane $z = t$, above the region Ω. For $t \in [0, 1)$, the intersection of S and $z = t$ is a ring of radius equal to or less than 1. The portion of the xy plane inside this ring is the set $Above(t)$, and the annulus between this ring and the ring of radius 1 is the set $Below(t)$. We argue that, if $m[Above(t)] > \frac{m[\Omega]}{2}$, then increasing t removes from $V(t)$ a volume of equal height and greater area compared to the volume that it adds. Similarly, if $m[Below(t)] > \frac{m[\Omega]}{2}$, then decreasing t removes from $V(t)$ a volume of equal height and greater area compared to the volume that it adds. Therefore, as in the two dimensional case, we can continue decreasing $V(t)$ until

$$m[Above(t)] \leq \frac{m[\Omega]}{2} \text{ and } m[Below(t)] \leq \frac{m[\Omega]}{2}.$$

At this t value, $V(t)$ is at a minimum. We call this value the median of f.

In this diagram, the black circle on the xy plane is $Above(t)$, and the annulus surrounding that circle is $Below(t)$. We also display the ratio $\frac{Above(t)}{Below(t)}$, which equals $frac12$ for $t = f_{med}$. Letting r be the radius of the inner circle, we find that $m[Above(t)] = m[Below(t)]$ when $\pi r^2 = \frac{\pi}{2}$, or $r = \frac{\sqrt{2}}{2}$. The reader can confirm this by observing the ratio of the two areas in relation to the quantity $V(t)$.

8 Conclusion

The aim of this paper has been to illustrate a strategy of spatial reasoning that illuminates various concepts of the median value. The reasoning that shows the discrete or continuous medians' minimizing properties offers insight into diverse situations once the reader constructs an analogy to familiar quantities, as illustrated by our concept of a three dimenisional median. The initial motivation for this paper was the definition of the median value for continuous functions, and the objective of broadening that definition. Similarly, though we have imposed conditions on the functions and sets discussed here, one may investigate the consequences of abandoning some hypotheses. A similar line of reasoning can offer insight into functions, sets, and other mathematical objects beyond our discussion. The reader may consider what the median may represent in four dimensions or more, or in physical situations. We hope these ideas convince the reader that the median is not just a number to be studied in grade school, but a broad concept with satisfying and unexpected applications.

References

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