Even Famous Mathematicians Make Mistakes!

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Contents

1 Abstract

A Graeco-Latin Square of order n is an $n \times n$ array of unique ordered pairs whose entries are numbers between 1 and n such that every number appears in each row and column once for each coordinate of the ordered pair [8]. In 1782, Euler predicts that Graeco-Latin Squares of order n exist if and only if $n \geq 3$ and $n \not\equiv 2 \pmod 4$. This prediction, known as Euler's conjecture, is eventually proved almost entirely wrong in the following 180 years. In this paper we will explore the mathematics and history behind the conjecture.

2 Introduction

Leonhard Euler (1707-83), a Swiss mathematician, is well known for his contributions to the field of mathematics as well as to physics. Most students who have taken a math course at the calculus level or above have heard of him, and he is considered one of the most prolific mathematicians to this day. He is perhaps most famous for producing Euler's Identity $e^{i\pi} + 1 = 0$, which uses the constant e that is also referred to as Euler's number [10]. Much less famous is Euler's work with Graeco-Latin Squares, which are also known as Euler Squares as Euler is the one who introduces them. Most research before Euler's deals with Latin Squares, which can be used to construct Graeco-Latin Squares if the Latin Squares are orthogonal. We will save the details of this relationship for a later section.

In this paper we will first go over some of the properties of Latin Squares in Section 3. Once we have built up sufficient background, we will begin our investigation of what sizes of Graeco-Latin Squares exist in Section 4. After proving some preliminary results we will introduce Euler's Conjecture that a Graeco-Latin Square of order n exists if and only if $n \geq 3$ and $n \not\equiv 2 \pmod 4$. In Section 5 we will cover the history of how this conjecture is gradually resolved over the course of almost 180 years thanks to the collective work of several mathematicians.

3 Latin Squares and MOLS

A **Latin Square** is an $n \times n$ arrangement of integers 1 through n such that no number occurs more than once in any row or column. An $n \times n$ Latin Square is also called a Latin Square of order n [3]. A standard Sudoku puzzle is an example of a 9×9 Latin Square, though a Sudoku puzzle has the additional restriction that the nine 3×3 subsquares also contain the digits 1 through 9.

We start this section with a demonstration of how to use Latin Squares to reduce the bias from extraneous variables in an experiment, as the focus is on the statistical analysis rather than the Latin Square itself. Once we have worked through this application, we will begin to discuss the more complicated properties of Latin Squares.

3.1 Growing Strawberries

Suppose a fertilizer company wants to compare five different varieties of strawberry fertilizer, labelled as fertilizer A,B,C,D, and E. They want to test the fertilizers using five different tracts of land (1,2,3,4 and 5) and they want to test the fertilizers over five growing seasons (I,II,III,IV,and V). An easy way this experiment could be arranged is given by Table 3.1. But as is often the case, the easiest way is not the best way. If the experiment is carried out according to Table 3.1, the individual differences between the growing locations could result in a huge bias in the outcome. There would be a similar problem if we assigned each fertilizer to one growing season and used it in all growing locations during that season. The presence of bias in the outcome is highly undesirable, so we need to take care in our assignment of fertilizer. If we use a Latin Square arrangement for the five fertilizer varieties such as the one in Table 3.2, the experiment is minimally affected by differences between growing seasons and locations since each fertilizer is used during every growing season and is used at all locations exactly once. [16]

	1	2	3	4	5
I	A	B	C	D	E
II	A	B	C	D	E
III	A	B	C	D	E
IV	A	B	C	D	E
V	A	B	C	D	E

Table 3.1: A poor assignment of fertilizers [16]

	1	2	3	4	5
I	A 24	B 20	C 19	D 24	E 24
II	B 17	C 24	D 30	E 27	A 36
III	C 18	D 38	E 26	A 27	B 21
IV	D 26	E 31	A 26	B 23	C 22
V	E 22	A 30	B 20	C 29	D 31

Table 3.2: A good assignment of fertilizers [16]

Let's say the company decides to use the fertilizer schedule described in Table 3.2. Let the numbers next to each letter represent the measured value of our variable of interest, say, the number of berries produced (in thousands).

Before we begin analysis of the results, we need to normalize the results by subtracting the average value from each entry so the average of the data is now 0. The result is the matrix (n) depicted in Table 3.3. Our next task is to split (n) into four components (all of which are 5×5 matrices) to isolate the effects of the different growing seasons, growing locations, and fertilizer varieties. The first component is a matrix where each entry is the row average in (n) of the row containing that entry. The second component has the column average in (n) for each entry, and the third component has the average in (n) of each type of fertilizer (or the average of all values with the same background color). To see what these look like, refer to (a),(b),and (c) respectively from Figure 3.1[9].

The last component is the matrix made from subtracting the cooresponding entries of (a),(b),and (c) from (n). The result is depicted as (d) in Figure 3.1. Through this process, we have isolated the differences between the fertilizers and the errors that cannot be accounted for by the growing season or location, represented by (c) and (d), respectively. In this experiment we assume the null hypothesis, that is, we assume that the variety of fertilizer has no impact on the number of berries produced. Under this hypothesis, the data in (c) and (d) should follow a normal distribution. [9]

	1	2	3	4	5
I	-1.4	-5.4	-6.4	-1.4	-1.4
II	-8.4	-1.4	4.6	1.6	10.6
III	-7.4	12.6	0.6	1.6	-4.4
IV	0.6	5.6	0.6	-2.4	-3.4
V	-3.4	4.6	-5.4	3.6	5.6

Table 3.3: The normalized matrix (n) [9]

-3.2	-3.2	-3.2	-3.2	-3.2
1.4	1.4	1.4	1.4	1.4
0.6	0.6	0.6	0.6	0.6
0.2	0.2	0.2	0.2	0.2
1	1	1	1	1

(a)

-4	3.2	-1.2	0.6	1.4
-4	3.2	-1.2	0.6	1.4
-4	3.2	-1.2	0.6	1.4
-4	3.2	-1.2	0.6	1.4
-4	3.2	-1.2	0.6	1.4

(b)

3.2	-5.2	-3	4.4	0.6
-5.2	-3	4.4	0.6	3.2
-3	4.4	0.6	3.2	-5.2
4.4	0.6	3.2	-5.2	-3
0.6	3.2	-5.2	-3	4.4

(c)

2.6	-0.2	1	-3.2	-0.2
-0.6	-0.3	0	-1	4.6
-1	4.4	0.6	-2.8	-1.2
0	1.6	-1.6	2	-2
-1	-2.8	0	5	-1.2

(d)

Figure 3.1: The four components of the original normalized matrix (n), with the row and column labels omitted. [9]

To analyze this data, we need the sample variances of (c) and (d), calculated by dividing the sum of the squares of the entries by the number of degrees of freedom. The sample variance of (c) is 82.5, while the sample variance of (d) is only 10.66. The ratio of two sample variances of a normal distribution follow the $F_{n,k}$ distribution, or the "Variance Ratio" distribution, where n and k are the degrees of freedom of the two samples ((c) and (d)). Under this distribution, we find that the probability of observing a sample variance ratio greater than or equal to 82.5/10.66=7.74 is only 0.28%. This means that under our null hypothesis, the probability of observing a ratio of the sample variances of (c) and (d) at least as high as what we observed is 0.28% [9]. Therefore it is highly likely that the null hypothesis is incorrect and that there is at least some difference in berry growth based on the type of fertilizer. However, further analysis would be needed to determine the magnitude of the differences between fertilizers; our work to this point has only shown that (it is extremely likely that) some difference exists.

3.2 MOLS

Now that we've seen an application of Latin Squares, we will move on to the more theoretical properties of Latin Squares, starting with orthogonality.

Definition 3.1. *Consider an $n \times n$ Latin Square superimposed onto another $n \times n$ Latin Square, creating an ordered pair for each entry of the new $n \times n$ matrix. The two original Latin Squares are said to be mutually orthogonal Latin Squares (MOLS) if no ordered pair of the superimposed matrix appears more than once. Three or more Latin Squares are all mutually orthogonal if the superimposed matrix of any two Latin Squares in the set contains all unique ordered pairs. Two or more MOLS can also be referred to as orthogonal mates. [3]*

For example, given the the two 3×3 Latin Squares

$$L_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad L_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix},$$

the superimposed matrix would be

$$\begin{pmatrix} (1,1) & (2,2) & (3,3) \\ (2,3) & (3,1) & (1,2) \\ (3,2) & (1,3) & (2,1) \end{pmatrix}.$$

We can see that every ordered pair occurs exactly once, making L_1 and L_2 MOLS of order 3.

3.3 Transversals

Another relevant concept that goes with MOLS and orthogonality is the transversal of a Latin Square, defined below.

Definition 3.2. *A transversal of a given $n \times n$ Latin Square is a set of n entries such that each row and column has one entry and the set of entries contains the numbers 1 through n. [16]*

For example, for the Latin Square

$$\begin{pmatrix} \mathbf{1} & \underline{2} & 3 \\ 2 & \mathbf{3} & \underline{1} \\ \underline{3} & 1 & \mathbf{2} \end{pmatrix},$$

the bolded set of entries form a transversal, as do the set of underlined entries, as do the set of remaining entries. This brings us to an important theorem regarding transversals.

Theorem 3.1. *A Latin Square of order $n \geq 2$ has an orthogonal mate if and only if it contains n disjoint transversals. [16]*

Proof. We will only prove the backward direction, as the forward direction is essentially the same proof in the reverse order. Assume an $n \times n$ Latin Square has n disjoint transversals. Assign each transversal a number between 1 and n, then change every entry in that transversal to that number [16]. For instance, if we apply this procedure to the earlier Latin Square example, the result is

$$\begin{pmatrix} \mathbf{1} & \underline{2} & 3 \\ 2 & \mathbf{3} & \underline{1} \\ \underline{3} & 1 & \mathbf{2} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}.$$

This result will always be a new Latin Square, as every row and column contains one entry of each transversal (and hence every number from 1 to n). We will now prove that this new Latin Square and the original Latin Square are orthogonal mates.

Let t_i denote the set of positions of entries from the transversal whose entries were all changed to i to create the new Latin Square. Next we superimpose the two Latin Squares to create ordered pairs letting the first coordinate refer to the entry in the new Latin Square and the second to the original Latin Square. Consider the ordered pairs in t_i: $(i,1),(i,2),...,(i,n)$ (though not necessarily in that order). Note that each of these ordered pairs will differ from one another by the second coordinate. By design they will be the only ordered pairs with an i as the first coordinate, so they will also be different from all other ordered pairs in the matrix. The same logic can be applied to all $t_1...t_n$, which means that all entries of the superimposed matrix will be unique. Therefore the new and original Latin Squares are orthogonal mates. □

An additional perk we get out of this proof is a method for finding orthogonal mates. For people like me who don't want to create MOLS candidates and check that each of the ordered pairs is unique, only to find a duplicate pair, it can be easier to find n disjoint transversals and produce an orthogonal mate from there.

Another note worth mentioning is that Theorem 3.1 did not originally require that $n \geq 2$. I added this condition, as without it the case where $n = 1$ becomes ambiguous. This is because there is only one Latin Square of order 1, the Latin Square (1), so there is no possible orthogonal mate. However, the arrangement (1) is itself a transversal so according to the theorem it has an orthogonal mate, itself (which is not allowed). This led me to question why the multiple sources I read that used this theorem did not include this condition, so I have decided to add the condition to Theorem 3.1, as well as to Definition 3.3, which also uses transversals.

While we're on the topic of trivial cases, we'll look at the other trivial case, when $n = 2$. Latin Squares of order 2 also don't have an orthogonal mate, as there are only two,

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

and they clearly aren't mutually orthogonal. [3]

We can also refer to Latin Squares of order 1 or 2 as **bachelor Latin Squares**, defined below. I've also included the definition of **monogamous Latin Squares**, as both definitions pertain to MOLS.

Definition 3.3. *A bachelor Latin Square is a Latin Square with no orthogonal mate. An alternative definition is a Latin Square that cannot be partitioned into disjoint transversals (for $n \geq 2$) [18]. See Figure 3.2 for an example. [6]*

1	0	2	3	4
0	2	1	4	3
2	3	4	0	1
3	4	0	1	2
4	1	3	2	0

Figure 3.2: A bachelor Latin Square of order 5 [19]

Definition 3.4. *A monogamous Latin Square has an orthogonal mate but is never in a set of three MOLS. [6]*

I was surprised that these Latin Squares are categorized based on their romantic relationship tendencies, though I should have taken the term "orthogonal mates" as a hint to this possible eventuality. Admittedly, this paper only uses the term "bachelor Latin Square," but I felt that monogamous Latin Squares should be introduced together with the bachelors so I included it.

3.4 How Many Orthogonal Mates Can a Latin Square Have?

Now that we understand the basics of orthogonality and MOLS, we can begin to explore the number of Latin Squares that can be mutually orthogonal, starting with the upper bound.

Theorem 3.2. *There can be at most $n-1$ Latin Squares of size n in a set of MOLS. [3]*

Proof. Consider an arbitrarily large set of $n \times n$ Latin Squares that are potentially MOLS. Without loss of generality, assume the first row of each Latin Square is $1, 2, ..., n$. This means that the ordered pairs in the first row of the superimposed matrix of any two of the Latin Squares has the form (x, x), where $x = 1, 2, ..., n$. Next, we'll examine the first entry of the second row. This entry can't be a 1 as that would mean the matrix is not a Latin Square. In addition, no two MOLS candidates can have the same value for this entry as that would make another ordered pair of the form (x, x), so the two Latin Squares would not be orthogonal. This means that each Latin Square in the set of MOLS must have a different element for the first entry of the second row and that entry must not be 1, leaving $n-1$ options and therefore a maximum of $n-1$ Latin Squares can be MOLS. [3]

□

After establishing this upper bound for the number of MOLS, the next course of action is exploring the possibility of achieving this upper bound and what conditions might be necessary. We will denote the maximum number of MOLS of size n as $N(n)$. To clarify, a set of $n-1$ MOLS of size n does not exist for all n values, so $N(n) \leq n-1$ is the largest number of MOLS of size n that exist. To start our exploration of $N(n)$ we'll look at prime numbers.

Theorem 3.3. *A set of $n-1$ MOLS exists if n is a prime number. That is, $N(n) = n-1$ if n is prime. [16]*

Proof. We will prove this by constructing a set of $p-1$ MOLS of size p, where p is any prime number. Instead of using the numbers 1 through p, we will use the numbers 0 through $p-1$ as the element names. To follow along this process with an example, refer to Figure 3.3. We'll start by putting the numbers 0 through $p-1$ in a row above a $p \times p$ grid such that there is one number above each column. For simplicity, we will assume they are in the order $0, 1, ..., p-1$. Repeat this procedure for the rows, so there is an extra column to the left of the grid containing the numbers $0, 1, ..., p-1$.

Next, let each entry b_{ij} of the $p \times p$ grid be equal to $ax + y \pmod{p}$, where $a = 1$, x equals the number on the ith row in the column left of the grid, and y equals the number in the jth column of the row above the grid [16]. This will produce a Latin Square: for each row, the entries are the y values $0, 1, 2, ..., p-1$ shifted by a constant ax (then calculated in mod p), so each entry is still unique. The column entries are the values $0, a, 2a, ..., (p-1)a$ all shifted by a constant y, so the entries are all distinct if and only if $0, a, 2a, ..., (p-1)a$ (mod p) are distinct. From the properties of modular arithmetic, we know that $ab \equiv ac$ (mod p) implies that either $a \equiv 0$ or $b \equiv c$ (mod p). By design $a \not\equiv 0$ (mod p) and for any $b, c \in \{0, 1, ..., p-1\}$ and $b \neq c$, $b \not\equiv c$ (mod p). Therefore the $0, a, 2a, ..., (p-1)a$ (mod p) are distinct so the construction is a Latin Square. To construct the other $p-2$ Latin Squares, simply let a equal $2, 3, ..., p-1$.

	0	1	2	3	4
0	$\underline{0}$	1	2	3	4
1	1	2	3	4	$\underline{0}$
2	2	3	4	$\underline{0}$	1
3	3	4	$\underline{0}$	1	2
4	4	$\underline{0}$	1	2	3

	0	1	2	3	4
0	$\underline{0}$	1	2	3	4
1	2	3	4	0	$\underline{1}$
2	4	0	1	$\underline{2}$	3
3	1	2	$\underline{3}$	4	0
4	3	$\underline{4}$	0	1	2

	0	1	2	3	4
0	$\underline{0}$	1	2	3	4
1	3	4	0	1	$\underline{2}$
2	1	2	3	$\underline{4}$	0
3	4	0	$\underline{1}$	2	3
4	2	$\underline{3}$	4	0	1

	0	1	2	3	4
0	$\underline{0}$	1	2	3	4
1	4	0	1	2	$\underline{3}$
2	3	4	0	$\underline{1}$	2
3	2	3	$\underline{4}$	0	1
4	1	$\underline{2}$	3	4	0

Figure 3.3: The four MOLS constructed for $p = 5$, where $a = 1$, $a = 2$, $a = 3$, and $a = 4$ respectively. We can easily check that these Latin Squares are mutually orthogonal, as the set of elements that have the same entry for one Latin Square correspond to a transversal for all of the other Latin Squares. For example, the entries in the underlined spaces are all 0 in the first Latin Square, while they contain all five elements in each of the other Latin Squares.

Now we've reached the million dollar question: are these $p-1$ Latin Squares MOLS? Naturally we wouldn't bother constructing them if that weren't the case, but let's prove this to remove all doubt. We'll start by supposing that (at least) two of the Latin Squares are not orthogonal. That is, there are (at least) two ordered pairs in the superimposed matrix, $(a_i x_1 + y_1, a_j x_1 + y_1)$ and $(a_i x_2 + y_2, a_j x_2 + y_2)$ that have the same value for some $a_i, a_j \in 1, 2, ..., p-1$ where $a_i \neq a_j$, and for some $x_1, x_2, y_1, y_2 \in 0, 1, ..., p-1$. This means that

$$a_i x_1 + y_1 = a_i x_2 + y_2 \qquad \text{mod } p,$$
$$a_j x_1 + y_1 = a_j x_2 + y_2 \qquad \text{mod } p.$$

Rearranging these inequalities, we get

$$a_i(x_1 - x_2) = y_2 - y_1 \qquad \text{mod } p,$$
$$a_j(x_1 - x_2) = y_2 - y_1 \qquad \text{mod } p.$$

Therefore,

$$a_i(x_1 - x_2) = a_j(x_1 - x_2) \qquad \text{mod } p.$$

From this equation we can see that either $a_i = a_j$ (mod p) or $x_1 - x_2 = 0$ (mod p). We know the former of the two is not true, so $x_1 = x_2$ (mod p). Since x_1 and x_2 are less than p (and non-negative), we can conclude that $x_1 = x_2$. Plugging this back into one of our original equalities, we have that

$$a_i x_1 + y_1 = a_i x_1 + y_2 \qquad \mod p,$$

so $y_1 = y_2 \pmod p$. Once again, both y_1 and y_2 are non-negative and less than p so we may conclude $y_1 = y_2$. Therefore for two ordered pairs to be the same in any superimposed matrix of the Latin Squares we constructed, those two ordered pairs must be the same entry of the matrix. This means that none of the superimposed matrices will have a duplicate ordered pair, so the set of $p-1$ Latin Squares are MOLS.

□

In addition to prime numbers, it turns out that any power of a prime will achieve the upper bound of $n-1$ MOLS of order n. The proof of this found in [3] bears a strong resemblance to the proof of Theorem 3.3 except that instead of integers, x and y are elements of the Galois Field $GF(p^\alpha)$ (when n is the prime power p^α). Because the introduction of fields makes the proof more complicated and less intuitive, will state the theorem without proof. [3]

Theorem 3.4. *There exists a set of $n-1$ MOLS of order n if $n \geq 3$ and n is a power of a prime. [3]*

With this we draw our analysis of the upper bounds of N(n) to a close as we transition to our investigation of the lower bounds of N(n).

3.5 How MOLS Beget Larger MOLS

In this section we will introduce a theorem that will be useful especially for Latin Squares not of a prime power order. We will start by defining an important Latin Square construction method that uses smaller Latin Squares.

Definition 3.5. *The Kronecker Product of two Latin Squares A and B of order m and n respectively is defined as the $mn \times mn$ array*

$$\begin{pmatrix} (a_{11}, B) & (a_{12}, B) & ... & (a_{1m}, B) \\ (a_{21}, B) & (a_{22}, B) & ... & (a_{2m}, B) \\ \vdots & \vdots & & \vdots \\ (a_{m1}, B) & (a_{2m}, B) & ... & (a_{mm}, B) \end{pmatrix},$$

$$\text{where } (a_{ij}, B) = \begin{pmatrix} (a_{ij}, b_{11}) & (a_{ij}, b_{12}) & ... & (a_{ij}, b_{n1}) \\ (a_{ij}, b_{21}) & (a_{ij}, b_{22}) & ... & (a_{ij}, b_{n2}) \\ \vdots & \vdots & & \vdots \\ (a_{ij}, b_{n1}) & (a_{ij}, b_{n2}) & ... & (a_{ij}, b_{nn}) \end{pmatrix} [16].$$

To build some intuition for this operation, let's work through an example. For instance the Kronecker Product of

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

is equal to

$$\begin{pmatrix} (1,0) & (1,1) & (1,2) & (0,0) & (0,1) & (0,2) \\ (1,2) & (1,0) & (1,1) & (0,2) & (0,0) & (0,1) \\ (1,1) & (1,2) & (1,0) & (0,1) & (0,2) & (0,0) \\ (0,0) & (0,1) & (0,2) & (1,0) & (1,1) & (1,2) \\ (0,2) & (0,0) & (0,1) & (1,2) & (1,0) & (1,1) \\ (0,1) & (0,2) & (0,0) & (1,1) & (1,2) & (1,0) \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 6 & 4 & 5 & 3 & 1 & 2 \\ 5 & 6 & 4 & 2 & 3 & 1 \end{pmatrix} [16].$$

To translate the Latin Square produced from the Kronecker Product to a more conventional notation for Latin Squares, we start by noting that there will be 6 different ordered pairs in the Latin Square. We can assign each ordered pair a number label, and rename each entry to yield the final Latin Square in standard form. For instance, the ordered pair (1,0) is translated to 1 and (0,0) is translated to 4.

One very useful property of the Kronecker Product is that it preserves orthogonality, as illustrated in the following theorem.

Theorem 3.5. *If there is a set of r MOLS of order m and a set of r MOLS of order n, then there is a set of r MOLS of order mn. That is,*

$$N(mn) \geq \min\{N(m), N(n)\}.[3]$$

Proof. Let $A_1, A_2, ..., A_r$ be a set of MOLS of order m and $B_1, B_2, ..., B_r$ be a set of MOLS of order n. Consider the set $C_1, C_2, ..., C_r$ of Kronecker Products such that C_l is the Kronecker Product of A_l and B_l. Suppose that C_e is not orthogonal to C_f. Then at least one ordered pair of the superimposed matrix occurs twice: for some i, j, k, l, p, q, s, and t,

$$\left(a_{ij}^{(e)}, b_{kl}^{(e)}\right) = \left(a_{pq}^{(e)}, b_{st}^{(e)}\right) \qquad \text{and} \qquad \left(a_{ij}^{(f)}, b_{kl}^{(f)}\right) = \left(a_{pq}^{(f)}, b_{st}^{(f)}\right),$$

where $a_{ij}^{(e)}$ denotes the entry in the ith row and jth column of A_e and so on. This means that

$$a_{ij}^{(e)} = a_{pq}^{(e)}, \qquad b_{kl}^{(e)} = b_{st}^{(e)}$$

and

$$a_{ij}^{(f)} = a_{pq}^{(f)}, \qquad b_{kl}^{(f)} = b_{st}^{(f)}.$$

Because A_e and A_f are MOLS, the first and third equality imply that $i = p$ and $j = q$. Similarly because B_e and B_f are MOLS, the second and fourth equality imply that $k = s$ and $l = t$. This is a contradiction to the original claim that C_e and C_f are not orthogonal. [3] □

Using this theorem, we can produce a lower bound for $N(n)$ for any n value based on the prime factorization of n.

Theorem 3.6. *Suppose the prime factorization of n is $p_1^{a_1}p_2^{a_2}...p_k^{a_k}$, where the p_i are distinct primes and the a_i are positive integers. Then $N(n) \geq r$, where $r = \min\{p_1^{a_1}, p_2^{a_2}, ..., p_k^{a_k}\} - 1$. We will denote the lower bound of $N(n)$ calculated through this method as*

$$n(n) = \min\{p_1^{a_1}, p_2^{a_2}, ..., p_k^{a_k}\} - 1.[3]$$

Proof. We know from Theorem 3.5 that there exists a set of $p_1^{a_1}-1$ MOLS of order $p_1^{a_1}$, a set of $p_2^{a_2}-1$ MOLS of order $p_2^{a_2}$... and $p_k^{a_k}-1$ MOLS of order $p_k^{a_k}$. Since $r \leq p_1^{a_1}-1, p_2^{a_2}-1, ..., p_k^{a_k}-1$, we may simply select r MOLS of each size $p_1^{a_1}, p_2^{a_2}, ..., p_k^{a_k}$ and use the Kronecker Product to create r MOLS of order $p_1^{a_1}p_2^{a_2}...p_k^{a_k} = n$. [3] □

Now we have finally built up enough background for the main part of this paper, the investigation of Graeco-Latin Squares and Euler's Conjecture.

4 Euler's Conjecture

To discuss Euler's Conjecture, we first need to understand the idea of a Graeco-Latin Square, so we'll go over the definition and give a brief introduction.

4.1 Graeco-Latin Squares

Definition 4.1. *A Graeco-Latin Square of order n is an $n \times n$ array of ordered pairs such that each entry of the ordered pair is one of n different symbols, each symbol appears in each row and column once for each coordinate of the ordered pair, and each ordered pair is unique. [8]*

The name Graeco-Latin Square comes from the original notation used by Euler, as the first term of the ordered pair is traditionally a Latin letter while the second entry is a Greek letter. However this notation is inconvenient so he soon abandons it and switches to using integers and exponents, where the first entry of the ordered pair is the base while the second is the exponent [8]. I am a fan of using integer ordered pairs so we will use that notation in this paper.

You might have noticed that the definition of a Graeco-Latin Square resembles the definition of MOLS. This is because we can construct a Graeco-Latin Square by superimposing two MOLS, as the definition of orthogonality implies that each ordered pair in the resulting matrix will be unique and hence the superimposed matrix is a Graeco-Latin Square. This means that proving the existence of two MOLS of order n (i.e. that $N(n) \geq 2$) is equivalent to proving the existence of a Graeco-Latin Square of order n. For example, if we superimpose the two MOLS

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix},$$

they yield the array

$$\begin{pmatrix} (1,1) & (2,2) & (3,3) \\ (2,3) & (3,1) & (1,2) \\ (3,2) & (1,3) & (2,1) \end{pmatrix},$$

which is a Graeco-Latin Square of order three.

4.2 The Conjecture

Almost all research regarding Graeco-Latin Squares aims to answer one question: what sizes of Graeco-Latin Squares exist? The one who sparks the interest in this subject is Euler, who first starts working with Graeco-Latin Squares in the 1780s. Soon after, he publishes a one hundred page paper introducing his conjecture, outlined below.

Conjecture 4.1. *There is a Graeco-Latin Square of order n if and only if $n \geq 3$ and $n \not\equiv 2 \pmod 4$. [8]*

Euler is unable to prove his conjecture, but his paper discusses extensively why it is highly likely that the conjecture is true. In fact he is able to prove the forward direction of the conjecture, that there is a Graeco-Latin Square of order n if $n \not\equiv 2 \pmod 4$, as we will prove shortly. However, the backwards direction of the conjecture (that there is no Graeco-Latin Square of order n if $n \equiv 2 \pmod 4$) is left unresolved for almost 180 years and turns out to be almost entirely incorrect. Before we get into that, we'll prove the forward direction, as it conveniently builds off of what we proved in the MOLS section of this paper. [8]

Theorem 4.1. *There is a Graeco-Latin Square of order n if $n \geq 3$ and $n \not\equiv 2 \pmod 4$. [3]*

Proof. We can see from Theorem 3.6 that if $n(n) \geq 2$, then $N(n) \geq 2$, so there would indeed exist a Graeco-Latin Square of order n. The single case where $p_i^{a_i} - 1 < 2$ for any p_i and a_i is when $p_i = 2$ and $a_i = 1$. Therefore for any $n \geq 3$ value that doesn't contain 2^1 in its prime factorization, i.e. if $n \not\equiv 2 \pmod 4$, $n(n) \geq 2$. [3] □

4.3 The search for the 6×6 Graeco-Latin Square

One challenge that first leads Euler to make his conjecture is the famous 36 officer problem: given 6 regiments that each contain 6 officers who each have a different rank, is it possible to arrange the officers in a 6×6 grid such that each rank and regiment occurs exactly once in each row and column? Finding the solution to this question is equivalent to finding a Graeco-Latin Square of order 6 [8]. Euler is unable to find such a solution, and much of his paper is dedicated to outlining his argument that the non-existence of a 6×6 Graeco-Latin Square (and all other n such that $n \equiv 2 \pmod 4$-sized Graeco-Latin Squares) is at least plausible. [8]

It's worth mentioning that Euler doesn't write a one hundred page paper without at least some results. He proves that several styles of 6×6 Latin Squares can not have an orthogonal mate, and many of the methods used to obtain these results are applicable to Latin Squares of other orders as well. For instance, he proves that a Single-Step Latin Square, as defined below, can only have an orthogonal mate if n is odd.

Definition 4.2. *A Single-Step Latin Square is a Latin Square of order n whose first row is the sequence $1, 2, ..., n$ and second row is the first row shifted one position to the left, the third row is the first row shifted two to the left, and so on. The general form of a Single-Step Latin Square is*

$$\begin{pmatrix} 1 & 2 & 3 & ... & n-1 & n \\ 2 & 3 & ... & n-1 & n & 1 \\ 3 & ... & n-1 & n & 1 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n & 1 & 2 & ... & n-2 & n-1 \end{pmatrix} .[8]$$

Admittedly this technically only proves that one specific 6×6 Latin Square is a bachelor. But permuting the symbols of a Latin Square does not affect its orthogonality properties, so we can say that we have effectively proved that $6! = 720$ specific Latin Squares are bachelors [16]. Euler uses several other "clever transformations" to let one Latin Square represent many Latin Squares and then prove that the representative Latin Square is a bachelor. Unfortunately, even with this strategy, Euler is only able to cover a fraction of the Latin Squares of size 6.

4.4 Tarry's Proof

In the year 1900, Gaston Tarry is the first to confirm the nonexistence of a Graeco-Latin Square of order 6, the smallest n such that $n \geq 3$ and $n \equiv 2 \pmod 4$. Because 6 is relatively small, Tarry is able to use a comparatively brute force method that considers specific cases, whereas the proofs regarding larger $n \equiv 2 \pmod 4$ cases such as 10, 14, 18 etc. will need to find other methods due to the exponential increase in the number of cases.

I say that Tarry's proof is brute force, but his proof is far more than simply pairing up Latin Squares and testing if they are orthogonal; in fact this method would be essentially impossible (especially without the aid of computers) due to the sheer volume of Latin Squares of order 6. According to the On-Line Encyclopedia of Integer Sequences, there are 812,851,200 unique Latin Squares of order 6, and superimposing every pair would result in over 3×10^{17} separate cases to check for orthogonality. [1]

Tarry first makes the number of cases a bit more manageable by only considering **reduced Latin Squares**, or Latin Squares of order n whose entries of the first row and column are the sequence $1, 2, ..., n$ [17]. Refer to Figure 4.1 for an example. Any Latin Square can be transformed into its reduced form by some permutation of its rows and columns. These operations do not affect the existence of an orthogonal mate, so Latin Squares that have the same reduced form should have all have orthogonal mates or all be bachelors [16]. There are only 9408 unique reduced Latin Squares, so each of these reduced Latin Squares represents 86,400 Latin Squares, as opposed to representing just 720 Latin Squares by permuting the symbols. [8]

Other papers that cover Tarry's proof usually stop here without going into more detail, so I read Tarry's original paper to get more details about the process. The paper is about

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 5 & 1 & 6 & 4 & 2 \\ 4 & 3 & 6 & 5 & 2 & 1 \\ 5 & 6 & 2 & 1 & 3 & 4 \\ 6 & 4 & 5 & 2 & 1 & 3 \end{pmatrix}$$

Figure 4.1: A reduced Latin Square of order 6 [17]

30 pages long and is entirely in French, so I have only included the information that I am fairly confident that I interpreted correctly.

The first third of the proof defines 17 different families of Latin Squares, the second third proves that every reduced Latin Squares of order 6 is in one of these families, and the last third proves that any reduced Latin Squares in the 17 families cannot be mutually orthogonal [17]. We will go over the first third in some detail, but we will only give a brief summary of the remainder of the paper.

Determining which family a Latin Square belongs to can done in a few basic steps, starting with converting the Latin Square into a **symbol matrix**. The symbols in this matrix, denoted as α, β, γ, and δ, represent four different permutations of six elements such that no element is mapped to its original location. These permutations can be interpreted as the relationship between any two rows or any two columns of a Latin Square of order 6 [17]. We know from permutation group theory that for a permutation group of order 6 can only contain cycles of size 2, 3, 4, or 6. Looking at the four permutations in Figure 4.2, we can see that these are the only possible cycle combinations. The location of the cycle and the actual permutation within a cycle do not have to match the ones in Figure 4.2, rather, the important part is the size and quantity of cycles. For example, the permutations

$$\left(\begin{array}{cc|cccc} A & B & C & D & E & F \\ B & A & E & C & F & D \end{array}\right) \quad \text{and} \quad \begin{pmatrix} A & B & C & D & E & F \\ B & F & D & C & A & E \end{pmatrix}$$

would be considered β permutations because they each have a cycle of size 4 and a cycle of size 2.

$$\overset{\alpha}{\left(\begin{array}{cc|cc|cc} A & B & C & D & E & F \\ B & A & D & C & F & E \end{array}\right)} \qquad \overset{\beta}{\left(\begin{array}{cc|cccc} A & B & C & D & E & F \\ B & A & D & E & F & C \end{array}\right)}$$

$$\overset{\gamma}{\left(\begin{array}{ccc|ccc} A & B & C & D & E & F \\ B & C & A & E & F & D \end{array}\right)} \qquad \overset{\delta}{\begin{pmatrix} A & B & C & D & E & F \\ B & C & D & E & F & A \end{pmatrix}}$$

Figure 4.2: The four symbols and their corresponding permutations. Vertical lines have been added to separate the different cycles. [17]

To convert a Latin Square of order 6 into its corresponding 6×6 symbol matrix, simply let the entry of the ith row and jth column be the permutation relationship between row

i and row j. Refer to Figure 4.3 to see the symbol matrix of our example reduced Latin Square from Figure 4.1. Notice that the symbol matrix is symmetrical along the diagonal and the diagonal entries are blank, as the permutation between row i and row i does not match any symbol's permutation as all entries are mapped to themselves. Since the diagonal entries are empty, we can condense the symbol matrix into a 6×5 matrix without losing any information (also demonstrated in Figure 4.3). [17]

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 5 & 1 & 6 & 4 & 2 \\ 4 & 3 & 6 & 5 & 2 & 1 \\ 5 & 6 & 2 & 1 & 3 & 4 \\ 6 & 4 & 5 & 2 & 1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} - & \alpha & \beta & \delta & \delta & \beta \\ \alpha & - & \delta & \gamma & \gamma & \delta \\ \beta & \delta & - & \delta & \delta & \gamma \\ \delta & \gamma & \delta & - & \gamma & \delta \\ \delta & \gamma & \delta & \gamma & - & \delta \\ \beta & \delta & \gamma & \delta & \delta & - \end{pmatrix} \longrightarrow \begin{pmatrix} \alpha & \beta & \delta & \delta & \beta \\ \alpha & \delta & \gamma & \gamma & \delta \\ \beta & \delta & \delta & \delta & \gamma \\ \delta & \gamma & \delta & \gamma & \delta \\ \delta & \gamma & \delta & \gamma & \delta \\ \beta & \delta & \gamma & \delta & \delta \end{pmatrix}$$

Figure 4.3: Step 1, Calculating the symbol matrix and condensing it. [17]

Once we've found the condensed symbol matrix, the second step is quite simple. In this step we reorder the entries of the matrix by putting the entries of each row into alphabetical order (α first, β second, γ third, and δ last) and then permuting the rows so the first column is in alphabetical order [17]. Refer to Figure 4.4 for an example.

$$\begin{pmatrix} \alpha & \beta & \delta & \delta & \beta \\ \alpha & \delta & \gamma & \gamma & \delta \\ \beta & \delta & \delta & \delta & \gamma \\ \delta & \gamma & \delta & \gamma & \delta \\ \delta & \gamma & \delta & \gamma & \delta \\ \beta & \delta & \gamma & \delta & \delta \end{pmatrix} \longrightarrow \begin{pmatrix} \alpha & \beta & \beta & \delta & \delta \\ \alpha & \gamma & \gamma & \delta & \delta \\ \beta & \gamma & \delta & \delta & \delta \\ \beta & \gamma & \delta & \delta & \delta \\ \gamma & \gamma & \delta & \delta & \delta \\ \gamma & \gamma & \delta & \delta & \delta \end{pmatrix}$$

Figure 4.4: Step 2, alphabetizing the entries. [17]

The third step is to repeat steps one and two, except this time we will let the entries of the symbol matrix correspond to the permutation relation between columns instead of rows. Tarry provides a list of sixteen alphabetized symbol matrices, each identified by a number between 1 and 16, and he claims that the alphabetized row and column symbol matrices of any reduced Latin Square will be in this set [17]. He then provides a set of sixteen pairs of numbers between 1 and 16, each paired together by a curly brace underneath. These are **crests**, and as with the symbol matrices, each one is identified by a number between 1 and 16, and every reduced Latin Square can be matched with a crest.

To see how crests work, let's look at our example in Figure 4.5. The left and right matrices in the figure are the row and column symbol matrices, respectively, of our original reduced Latin Square example. If we were to look at Tarry's list of symbol matrices (which is not included in this paper), we would see that the row symbol matrix is matrix number 10 and the column symbol matrix is number 14. Combining these, we have the crest $\underbrace{10+14}$, which is crest number 10 in the list of crests provided in Figure 4.6. [17]

$$\underbrace{\begin{pmatrix} \alpha & \beta & \beta & \delta & \delta \\ \alpha & \gamma & \gamma & \delta & \delta \\ \beta & \gamma & \delta & \delta & \delta \\ \beta & \gamma & \delta & \delta & \delta \\ \gamma & \gamma & \delta & \delta & \delta \\ \gamma & \gamma & \delta & \delta & \delta \end{pmatrix} + \begin{pmatrix} \beta & \beta & \gamma & \delta & \delta \\ \beta & \beta & \gamma & \delta & \delta \\ \beta & \beta & \gamma & \delta & \delta \\ \beta & \beta & \gamma & \delta & \delta \\ \beta & \delta & \delta & \delta & \delta \\ \beta & \delta & \delta & \delta & \delta \end{pmatrix}} \quad = \quad \underbrace{10+14}$$

Figure 4.5: Matching the symbol matrices to a crest.

$$\overset{\mathbf{1}}{\underbrace{1+1}} \quad \overset{\mathbf{2}}{\underbrace{2+2}} \quad \overset{\mathbf{3}}{\underbrace{3+11}} \quad \overset{\mathbf{4}}{\underbrace{4+5}} \quad \overset{\mathbf{5}}{\underbrace{5+5}} \quad \overset{\mathbf{6}}{\underbrace{6+12}} \quad \overset{\mathbf{7}}{\underbrace{7+7}} \quad \overset{\mathbf{8}}{\underbrace{8+8}}$$

$$\overset{\mathbf{9}}{\underbrace{9+13}} \quad \overset{\mathbf{10}}{\underbrace{10+14}} \quad \overset{\mathbf{11}}{\underbrace{11+11}} \quad \overset{\mathbf{12}}{\underbrace{12+12}} \quad \overset{\mathbf{13}}{\underbrace{13+13}} \quad \overset{\mathbf{14}}{\underbrace{14+14}} \quad \overset{\mathbf{5}}{\underbrace{15+15}} \quad \overset{\mathbf{16}}{\underbrace{16+16}}$$

Figure 4.6: The list of crests. The bolded number above each crest is the number label of that crest. When the entries in the crest are distinct, either entry can be the row symbol matrix number as long as the other is the column symbol matrix number. [17]

From here Tarry shows that reduced Latin Squares that belong to the same crest are of the same family and labels that family with the number of the crest. This is often done by introducing several families and then paring down the number of families that are actually distinct. For instance, Tarry identifies the families 16, 16a, and 16b, and then proves they are all actually the same family, so they are combined under the name "family 16." The one exception is crest 7, which can only be narrowed down to two families, labelled 7 and 7a. Therefore all Latin Squares are one of 16+1=17 families. [17]

Once Tarry establishes that there are only 17 families, he defines the terms magic networks, magic groups, and magic numbers which do not exist outside of his paper. He uses these to analyze the families; he proves that none of the families have a magic network and therefore there is no Graeco-Latin Square of order 6 [17].

Soon after Tarry's proof is published, more concise proofs of the 36 officer problem are published, but they are eventually proved to be incorrect [8]. According to Kalei Titcomb's paper, it is almost one hundred years after Tarry's proof is published that the first valid concise proof is published [7]. Written by mathematician D. R. Stinson in 1984, the paper is a four-page, non-computer proof of the 36 officer problem that uses transversal designs [15].

5 Disproof of Euler's Conjecture

1959, almost sixty years after Tarry's proof is published, is the momentous year that Euler's Conjecture is finally resolved by a series of results found over the course of about six months. The spark that sets off this chain of new discoveries is the first counterexample of Euler's Conjecture: a Graeco-Latin Square of order 22, constructed by mathematicians R. C. Bose and S. S. Shrikhande [4]. Inspired by Bose and Shrikhande's findings, mathematician E. T. Parker then constructs a a Graeco-Latin Square of order 10, which is the smallest

$n \equiv 2 \pmod 4$ value larger than 6 [13]. After Parker publishes his paper, he, Bose, and Shrikhande join forces to prove the existence of a Graeco-Latin Square of all other orders where $n \equiv 2 \pmod 4$ [5].

In this section we will go over material from the three papers that contain these important results in chronological order. For each paper, we'll start with background information followed by a discussion of the main proof. But before that, we'll take a brief look at group-based Latin Squares and their role in the search for Graeco-Latin Squares.

5.1 Group-Based Latin Squares

One of the reasons for the lull in progress after Tarry's publication is the nature of the constructions of (orthogonal) Latin Squares that have been used until this point, for instance the construction in the proof of Theorem 3.3 and the construction of Latin Squares using the Kronecker Product. In 1942, Henry Mann makes the important observation that all of these constructions are group-based [8]. To better understand what a **group-based Latin Square** is, consider a reduced Latin Square like the one in Figure 5.1.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix} \begin{matrix} \longrightarrow (1)(2)(3)(4)(5) \\ \longrightarrow (12345) \\ \longrightarrow (13524) \\ \longrightarrow (14253) \\ \longrightarrow (15432) \end{matrix}$$

Figure 5.1: A reduced Latin Square of order 5 and the corresponding row permutations [8]

We want to find a permutation for each row, and the first row's permutation is always (1)(2)(3)(4)(5) since the first row is 12345 (because it's a reduced Latin Square). Next, to get the permutation for row i, permute row $i-1$ by the permutation for row $i-1$. For example, to get the third row permutation for Figure 5.1, we calculate (23451)(12345) = (13524). Also, note that the row five calculation is (51234)(15432)=(1)(2)(3)(4)(5), yielding the first row's permutation. Finally we look at the five permutations, and if they are a group then the Latin Square is group-based. [8]

The fact that the known constructions are group-based is significant because of the fact that Mann proves that the equality $N(n) = \min\{p_i^{\alpha_i}\} - 1$ holds (given n has the prime factorization $p_1^{a_1}, p_2^{a_2}, ..., p_l^{a_l}$) for all group-based Latin Squares. This means that any Latin Squares of order n where $n \equiv 2 \pmod 4$ that are produced by these constructions would have $N(n) = 1$. Therefore to confirm or disprove Euler's conjecture, we would need to prove that all MOLS are group-based, or find new MOLS construction methods. Mann later proves that there are indeed MOLS that aren't group based, so the latter method is needed to make further progress. This is a very important result and it takes another 17 years before the another way to construct MOLS is found. [8]

5.2 BIBDs and Pairwise designs

Written by Bose and Shrikhande, the first paper we will cover features the first counterexample to Euler's conjecture [4]. Before we go over the proof, we need to build up a bit of background information about **balanced incomplete block designs** and **pairwise designs**.

Definition 5.1. *Given a set S of v elements (or "varieties"), a balanced incomplete block design, or BIBD is a set of b subsets (or "blocks") of S such that for some $k < v$ and $\lambda > 0$, each subset has k elements and any pair of elements of S appear in exactly λ subsets.[2]*

The notation for a BIBD is a (b, v, r, k, λ)-configuration, where r is the number of blocks that each variety appears in, and the rest of the parameters are as described in the definition. For example, the BIBD

$$\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 3\}$$

is an example of a (7,7,3,3,1)-configuration. [2]

Another example we will mention here is one of the most commonly-used BIBD story problems: supposing fifteen girls walk to school every day for a week in groups of three, how could one arrange their groups such that no pair of girls walk together twice? This is known as Kirkman's schoolgirl problem, and the solution is equivalent to finding a $(35, 15, 7, 3, 1)$-configuration [2]. Another important property of this problem is that the solution is an example of a **resolvable** BIBD, that is, a BIBD that can be partitioned into sets of blocks such that each variety v occurs in each set exactly once. To see how this works, let each day be one of these sets: each girl walks exactly once each day, so the design is resolvable.[4]

Next we will introduce a special version of a BIBD, where $\lambda = 1$ and k can take on more than one value.

Definition 5.2. *A pairwise balanced design of index unity and type $(v; k_1, k_2, ..., k_m)$ is an arrangement of v varieties into b blocks such that each block is of size k_i for some $1 \leq i \leq m$ and every pair of varieties occurs together in a block exactly once. Note that $k_i \leq v$ for all $1 \leq i \leq m$ and that the k_i are all distincy, though there can be multiple blocks of size k_i. [4]*

To give a couple of examples,

$$\{1, 4\}\{2, 4\}\{3, 4\}\{1, 2, 3\}$$

is a pairwise balanced design of index unity and type (4:2,3), and

$$\{1, 4\}\{1, 5\}\{1, 6\}\{2, 4\}\{2, 5\}\{2, 6\}\{3, 4\}\{3, 5\}\{3, 6\}\{1, 2, 3\}\{4, 5, 6\},$$

is a pairwise balanced design of index unity and type (6:2,3). In addition, any BIBD where $\lambda = 1$ is a pairwise balanced design of index unity and type $(v; k)$, so we will denote such BIBDs as BIBD $(v; k)$.[4]

Now that we understand the basics of these terms, we'll look at a couple of theorems that will set up the framework for the main proof. For the first theorem, we will simply state it without proof as the proof is fairly complicated and lengthy.

Theorem 5.1. *If a pairwise balanced design of index unity and type* $(v; k_1, k_2, ..., k_m)$ *exists, then* $N(v) \geq \min\{N(k_i)\} - 1$*,* $1 \leq i \leq m$*. [4]*

Changing the notation to cater to the cases when a pairwise balanced design of index is a BIBD $(v; k)$, we can adapt Theorem 5.1 so that it applies specifically to a BIBD $(v; k)$:

Theorem 5.2. *If there exists a BIBD* (v, k)*, then* $N(v) \geq N(k) - 1$*. [4]*

The last bit of information we need to cover before the main proof is the existence of a set of BIBDs with specific parameters.

Lemma 5.1. *There exists a resolvable BIBD with the parameters*

$$b = (8m + 5)(12m + 7),\ v = 24m + 15,\ r = 12m + 7,\ k = 3,\ \text{and}\ \lambda = 1$$

for any non-negative integer m*. [4]*

Proof. These parameters follow the basic format of Kirkman's schoolgirl problem that was introduced earlier. When $m = 0$, the parameters in Lemma 5.1 are identical to the schoolgirl problem; for other m values, we can consider it to be equivalent to having $24m + 15$ girls walking every day for $12m + 7$ days. [4]

The schoolgirl problem is known to have a solution for any $v = 6t + 3$ (for a positive integer t) number of schoolgirls, and since we are looking for cases when $t \equiv 2 \pmod 4$, we can say that $t = 4m + 2$ (for a non-negative integer m). Therefore, we know that there is a solution for $v = 6(4m + 2) + 3 = 24m + 12 + 3 = 24m + 15$. This proves the existence of a solution for all m, and every solution is resolvable by the design of the schoolgirl problem: the blocks can be partitioned into $12m + 7$ subsets that are the $8m + 5$ groups that are on the same day for each of the $12m + 7$ days, as each girl walks in exactly one group each day. [4]

Also, there is some special case when $12m + 7$ is a prime power. Bose and Shrikhande use Galois fields to address this, but due to the length and complexity of that section of the proof, we will not include it in this paper. [4] □

5.3 The 22×22 Case

Now that we've built up a sufficient framework, we'll begin the main proof of this report: the proof that there exist infinitely many Graeco-Latin Square of order n where $n \equiv 2 \pmod 4$.

Theorem 5.3. *There exist at least two MOLS of order* $v = 36m + 22$ *where* m *is any non-negative integer. [4]*

Proof. Consider a resolvable BIBD with the parameter values from Lemma 5.1 (for any m value). We can introduce $12m + 7$ new varieties $\alpha_1, \alpha_2, ..., \alpha_{12m+7}$ into the design by adding α_i to each of the $8m+5$ blocks in the ith day and one entirely new block $\{\alpha_1, \alpha_2, ..., \alpha_{12m+7}\}$. Because the original BIBD is resolvable, the result is a pairwise balanced design of index unity and type $(v; k_1, k_2)$, where $v = 36m + 22$, $k_1 = 4$, and $k_2 = 12m + 7$. [4]

With this information we can use Theorem 5.1 to find a lower bound for $N(v = 36m+22)$. First, we'll calculate $N(k_1)$ and $N(k_2)$. Since $k_1 = 4$ which is a prime power, we know $N(k_1) = 4 - 1 = 3$.

Finding $N(k_2)$ is a bit more tricky since $12m + 7$ has multiple values. First, recall that for any k with a prime factorization $p_1^{a_1}, p_2^{a_2}, ..., p_l^{a_l}$,

$$N(k) \geq \min\{p_i^{a_i}\} - 1, \qquad 1 \leq i \leq l.$$

Looking at the quantity $12m + 7$, it cannot be divisible by 2, 3, or 4 for any value of m, so any $p_i^{a_i}$ in the prime factorization must be ≥ 5. Therefore $N(12m + 7) \geq 5 - 1 = 4$. This implies $N(36m + 22) \geq \min\{3, 4\} - 1 = 2$. [4] □

This proof is quite significant since Euler's Conjecture had gone unconfirmed for the better part of 200 years. But the end of the proof marks only the halfway point in Bose and Shrikhande's paper. They go on to prove that there are $n \equiv 2 \pmod 4$ cases where the number of MOLS of order n is much higher than two. They also prove there are Graeco-Latin Squares of order n for about two thirds of the n values of the form $4t + 2$ that are between 22 and 150 [4]. This information will be important in our discussion of the third and final paper, so we will hold off from further explanation until then.

5.4 Orthogonal Arrays

Shortly after Bose and Shrikhande's paper is published, Parker writes a proof of the existence of a 10×10 Graeco-Latin Square (as well as infinitely many other sizes of Graeco-Latin Square that are equivalent to 2 (mod 4)). Even though the proof covers material similar to that in Bose and Shrikhande's paper, Parker's proof is also significant as it uses a very different method to construct MOLS and it covers the case when $n = 10$, the smallest n where $n > 6$ and $n \equiv 2 \pmod 4$. [13]

The key term we'll need to understand Parker's proof is orthogonal arrays, so we'll start with a brief introduction of the topic. I thought that Klyve and Stemkoski's paper [8] provided the best detailed explanation of orthogonal arrays, so I've modeled my explanation after the information in [8].

Definition 5.3. *An orthogonal array of order n is a $k \times n^2$ matrix (for some $k \leq N(n)$) with entries that are in the set $\{1, 2, ..., n\}$ such that any two rows have all n^2 ordered pairs containing $\{1, 2, ..., n\}$ if the ith entry of each row is treated as an ordered pair for each i in $\{1,2,...,n\}$. [8]*

You might have noticed that parts of this definition resemble the definition of MOLS . It's not a coincidence, as it turns out that finding an orthogonal array of order n is equivalent to finding $k-2$ MOLS of order n. To see how this works, we'll use the example from Figure 5.2. We'll let the first row represent the row number, the second represent the column number, the third the entries of the first Latin Square, and the fourth the entries of the second Latin Square. For example, consider the fourth column of the orthogonal array, (2,1,2,3). This means when producing the two MOLS from the array, we would let the entries in the second row and first column of the two Latin Squares be 2 and 3 respectively. Repeating this procedure for all nine columns of the orthogonal array will yield the two MOLS shown in Figure 5.2. Another property of orthogonal arrays worth mentioning is that we could have picked any two rows to represent the Latin Square coordinates (making the remaining two rows represent the entries) and the Latin Squares produced would still be orthogonal [8]. For orthogonal arrays where $k > 4$, the MOLS would be constructed as before with the first four rows, then the fifth row would represent the entries of the 3rd Latin Square, the sixth row the 4th Latin Square, and so on.

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 & 3 & 1 & 3 & 1 & 2 \\ 1 & 2 & 3 & 3 & 1 & 2 & 2 & 3 & 1 \end{pmatrix}$$

$$\swarrow \qquad\qquad \searrow$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

Figure 5.2: Two MOLS of order 3 derived from an orthogonal array where $k = 4$ [8]

5.5 The 10×10 Case

In Parker's proof of the existance of a 10×10 Graeco-Latin Square, he doesn't use the term orthogonal array but instead uses "a set of n^2 quadruples $(a_{1i}, a_{2i}, a_{3i}, a_{4i}), i = 1, 2, ..., n^2$" [13]. This setup has the same properties of an orthogonal array of order n with $k = 4$ when the quadruples are lined up such that each is a column of a $n^2 \times 4$ matrix.

Parker produces a construction of an orthogonal array (with $k = 4$) that proves the theorem below. Unfortunately, the construction and the explanation is somewhat complex and uses Galois fields, so I will state the theorem without proof.

Theorem 5.4. *There exists a Graeco-Latin Square of order n when $n = (3q - 1)/2$, q is a prime power, and $q \equiv 3 \pmod 4$. [13]*

As an additional note, Parker's construction in the proof does not work for the case $q = 3$, but when $q = 3$, $n = 4$ and we know $N(4) = 4 - 1 = 3 > 2$ since 4 is a prime power. So the theorem still holds for all prime powers q [13].

Parker uses this theorem not only to construct a 10×10 Graeco-Latin Square but also to guarantee infinitely many Graeco-Latin Squares of order m such that $m \equiv 2 \pmod 4$. There might be some overlap, but in general the Graeco-Latin Square sizes found by Parker are separate from the ones previously found by Bose and Shrikhande in [4].

To construct two MOLS of order 10 (i.e. a 10×10 Graeco-Latin Square), Parker simply uses the construction method in the proof of Theorem 5.4 for the case $q = 7$ to derive an orthogonal array of order 10 [13]. From there we can produce two MOLS using the process outlined earlier that was used for Figure 5.2. The construction produced in Parker's paper is shown in Figure 5.3.

$$
\left(\begin{array}{ccccccc|ccc}
0 & 4 & 1 & 7 & 2 & 9 & 8 & 3 & 6 & 5 \\
8 & 1 & 5 & 2 & 7 & 3 & 9 & 4 & 0 & 6 \\
9 & 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 & 0 \\
5 & 9 & 8 & 3 & 0 & 4 & 7 & 6 & 2 & 1 \\
7 & 6 & 9 & 8 & 4 & 1 & 5 & 0 & 3 & 2 \\
6 & 7 & 0 & 9 & 8 & 5 & 2 & 1 & 4 & 3 \\
3 & 0 & 7 & 1 & 9 & 8 & 6 & 2 & 5 & 4 \\
\hline
1 & 2 & 3 & 4 & 5 & 6 & 0 & 7 & 8 & 9 \\
2 & 3 & 4 & 5 & 6 & 0 & 1 & 8 & 9 & 7 \\
4 & 5 & 6 & 0 & 1 & 2 & 3 & 9 & 7 & 8
\end{array}\right)
\left(\begin{array}{ccccccc|ccc}
0 & 7 & 8 & 6 & 9 & 3 & 5 & 4 & 1 & 2 \\
6 & 1 & 7 & 8 & 0 & 9 & 4 & 5 & 2 & 3 \\
5 & 0 & 2 & 7 & 8 & 1 & 9 & 6 & 3 & 4 \\
9 & 6 & 1 & 3 & 7 & 8 & 2 & 0 & 4 & 5 \\
3 & 9 & 0 & 2 & 4 & 7 & 8 & 1 & 5 & 6 \\
8 & 4 & 9 & 1 & 3 & 5 & 7 & 2 & 6 & 0 \\
7 & 8 & 5 & 9 & 2 & 4 & 6 & 3 & 0 & 1 \\
\hline
4 & 5 & 6 & 0 & 1 & 2 & 3 & 7 & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 & 0 & 9 & 7 & 8 \\
2 & 3 & 4 & 5 & 6 & 0 & 1 & 8 & 9 & 7
\end{array}\right)
$$

Figure 5.3: Two MOLS of order 10. Because $q = 7$, a break is included after the 7th row and column to make patterns in the Latin Squares more visible. Notice that the bottom right 3×3 entries form two (orthogonal) Latin Squares of order 3. [13]

As for the existence of infinitely many Graeco-Latin Squares of order m such that $m \equiv 2 \pmod 4$, we only need Dirichlet's theorem that states there exist infinitely many prime powers q where $q \equiv 7 \pmod 8$, and a bit of modular arithmetic. If q is a prime power and $q \equiv 7 \pmod 8$ (and therefore $q \equiv 3 \pmod 4$), then there exist at least two MOLS of order $n = (3q-1)/2$. Rewriting q as $8l+7$ for some integer l, we get that $n = 12l+10 = 4(3l+2)+2$, so $n \equiv 2 \pmod 4$. Therefore there are infinitely many Graeco-Latin Squares of order m such that $m \equiv 2 \pmod 4$ [13]. As I mentioned before, this is not new, but it's another way to prove it.

In addition to the Graeco-Latin Squares of order $n = (3q-1)/2$ that we just found, we also know that there are Graeco-Latin Squares of any odd multiples of n when $n = (3q-1)/2$ (and $q \equiv 3 \pmod 4$ and q is a prime power). This is because there exist at least two MOLS of any odd order a, and from Theorem 3.5 we know $N(an) \geq min\{N(a), N(n)\} \geq 2$. [13]

With this we conclude the discussion of Parker's paper and are one step closer to the answer to Euler's conjecture.

5.6 Outline of the Remaining Material

Once Parker publishes his paper, he, Bose, and Shrikhande decide to team up to resolve Euler's Conjecture once and for all. In the end, Euler's Conjecture is only correct in the forwards direction; there is a Graeco-Latin Square of every order n where $n \geq 3$ and $n \not\equiv 2 \pmod 4$, but other than the case where n=6, there is also a Greco-Latin Square of every order n where $n \geq 3$ and $n \equiv 2 \pmod 4$. [5]

The third paper is about 40 pages long, so as before I will simplify a lot of material and sometimes take some theorems/results as given and not prove them. Keeping that in mind, I've done my best to make a manageably short but representative coverage of the proof that Graeco-Latin Squares exist for $v \neq 6$.

There are three main stages for the proof from the paper: proving $N(v) \geq 2$ when $6 < v < 158$, then for when $158 \leq v \leq 726$, and finally proving $N(v) \geq 2$ when $v > 726$.

The interval cutoffs may seem arbitrary, but there have been many previous instances where the specific numbers or formulas seem random, such as the parameters in Lemma 5.1 and formula for n in Theorem 5.4. But just as with these numbers, I imagine the cutoffs are carefully chosen to suit the formulas the authors use in their proofs.

5.7 For v Values below 158

As a preliminary note, so I don't need to write it out every time, we'll assume from now on that mentioning Graeco-Latin Squares of order v, we're only concerned with the values where $v \equiv 2 \pmod 4$. This means the statements $v < 158$ and $v \leq 154$ are equivalent, as we already know that $N(v) \geq 2$ for $v = 155, 156$, and 157 as they are $\not\equiv 2 \pmod 4$ and so we are no longer concerned with those v values.

The first phase of this proof focuses on v values where $v \leq 154$. I mentioned earlier that Bose and Shrikhande show in their first paper that $N(v) \geq 2$ for about two-thirds of the v values ≤ 150. Each $N(v)$ has been calculated using one of the many other theorems that appear in [4] but are not used for the main proof. In general these theorems follow the general format of Theorem 5.6 that is included in a later section of this paper. Several more variations of these theorems are derived and used in Bose et al.'s second paper to show through case-by-case calculation that $N(v) \geq 2$ for the rest of the v values for $v \leq 154$ [5]. I can only imagine the amount of time and effort the authors have put into deriving these theorems and figuring out which one to use for each v value. I would venture to guess that this is one of the reasons that the case by case calculation ends at $v = 154$, as it becomes necessary to use more broad-sweeping methods if the authors hope to actually address all v values larger than 6. Just writing out and proving the theorems that are applied to individual cases accounts for about half of each paper, so we will be taking those results as a given, allowing us to focus on the second and third stages of the proof. So thanks to the patience and toil of Bose et al., we are able to claim that $N(v) \geq 2$ for all $6 < v \leq 154$. [5]

5.8 Group Divisble Designs

In the next section we'll prove that $N(v) \geq 2$ for $158 \leq v \leq 726$. But before that we need to go over a few new terms. Fortunately we've already gone over the terminology for pairwise balanced designs of index unity, so we're already halfway there. To start, we'll introduce another cousin of BIBDs known as **group divisible designs**. Consider an arrangement of v varieties into b blocks that each contains k distinct varieties. The design is a group divisible design, denoted as GD$(v; k, m; \lambda_1, \lambda_2)$, if the varieties can be partitioned into l groups of size m such that any two varieties in the same group appear together in exactly λ_1 blocks while any two varieties in different groups appear in λ_2 blocks [5]. For example, consider a set of four varieties {1,2,3,4} partitioned into two groups of two, {1,2} and {3,4}. Then the arrangement

$$\{1, 2\}, \{3, 4\}$$

is a GD(4,2,2,1,0) and the arrangement

$$\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}$$

is a GD(4,2,2,0,1).

Group divisible designs can be categorized as one of three different classes, Regular (R), Semi-regular (SR), and Singular (S). For this proof we're only interested in Semi-regular group divisible designs, so we'll skip the definition for the other two.

Definition 5.4. *A group divisible design is Semi-regular if* $(rk - \lambda_2 v) = 0$ *and* $(r - \lambda_1) > 0$*, where* r *is the number of blocks in which each variety appears. [5]*

We are particularly interested in the GD designs where $\lambda_1 = 0$ and $\lambda_2 = 1$, and we will denote such designs as GD($v; k, m; 0, 1$) and as SRGD($v; k, m; 0, 1$) if the design is Semi-regular.

The $\lambda_1 = 0$ and $\lambda_2 = 1$ case is helpful to us because we can augment the design to create a pairwise balanced design of index unity. To do this, start with a GD($v; k, m; 0, 1$) design and add l new blocks that correspond to the l groups from that partition beforehand. The result is a pairwise balanced design of index unity and type $(v; k, m)$ [5]. Carrying out this procedure on our second example from earlier, we get

$$\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2\}, \{3, 4\},$$

a pairwise balanced design of index unity and type (4,2,2), which is the same as having type (4,2).

To demonstrate the utility of this construction, we need to define equiblock components and clear sets of equiblock components.

Definition 5.5. *Given a pairwise balanced design of index unity (D) and type* $(v; k_1, k_2, ..., k_m)$*, the subdesign* (D_i) *is the collection of blocks of size* k_i *for all* $i = 1, 2, ..., m$*. This collection is referred to as the ith equiblock component of* (D) *[4].*

Definition 5.6. *Given a pairwise balanced design of index unity and type* $(v; k_1, k_2, ..., k_t)$*, the set of equiblock components* (D_1)*,* (D_2)*,...,*(D_s)*,* $s < t$*, is a clear set if all blocks contained in the* (D_i) *are disjoint. [5]*

Looking at our construction before, the l blocks of size m we added are the only blocks of that size so they make up an equiblock component. They are also disjoint by design, so that equiblock component is by itself a clear set [5].

Next we'll look at a theorem we will need to prove the theorem that comes after it. The proof of this theorem is pretty long and complicated so I won't include it here.

Theorem 5.5. *Suppose there exists a pairwise balanced design of index unity and type* $(v; k_1, k_2, ..., k_t)$ *with a clear set of equiblock components* (D_1)*,* (D_2)*,...,*(D_s)*. Then*

$$N(v) \geq \min\{N(k_1) + 1, N(k_2) + 1, ..., N(k_s) + 1, N(k_{s+1}), N(k_{s+2}), ..., N(k_t)\}. \qquad [5]$$

For example, when we apply this to a pairwise balanced design of index unity that is an augmented GD($v; k, m; 0, 1$) (using the procedure from earlier), we find that

$$N(v) \geq \min\{N(m) + 1, N(k)\} - 1.$$

Now we have the materials we need to state and prove the theorem below!

Theorem 5.6. *If there exists a resolvable GD*$(v; k, m; 0, 1)$ *with each variety occurring in* r *blocks and* $1 < x < r$*, then*

$$N(v + x) \geq \min\{N(k), N(k + 1), N(m) + 1, N(x) + 1\} - 1. \qquad [5]$$

Proof. By the definition of resolvable designs, we can partition the blocks of GD$(v; k, m; 0, 1)$ into r sets such that each variety occurs exactly once in each set. We'll arbitrarily number these sets with the numbers between 1 and r. Next we'll perform the same augmentation that we did before, adding the l sets of size m, so GD$(v; k, m; 0, 1)$ is a pairwise balanced design of index unity and type $(v; k, m)$. Now we introduce x new varieties $\{\theta_1, \theta_2, ..., \theta_x\}$ to the design by adding θ_i to each block in the ith set (from the r sets we numbered earlier) for $1 \leq i \leq x$ and add a new block $\{\theta_1, \theta_2, ..., \theta_x\}$. This action produces a pairwise balanced design of index unity and type $(v + x; k_1, k_1 + 1, m, x)$. In addition, we know that the blocks of size m and size x together form a clear equiblock set. Lastly we apply Theorem 5.5 to this design to derive the lower bound for $N(v + x)$. [5] □

Now we're only one theorem away from the part where we prove $N(v) \geq 2$ for the rest of the v values! This next theorem is the combination of Theorem 5.6 and Lemma 5.2, which is stated below (without proof since that would be opening up another huge can of worms).

Lemma 5.2. *There exists a resolvable SRGD*$(km; k, m; 0, 1)$ *if* $k \geq N(m) + 1$*. Note that for this design,* $m = r$*. [5]*

Theorem 5.7. *If* $k \geq N(m) + 1$ *and* $1 < x < m$*, then*

$$N(km + x) \geq \min\{N(k), N(k + 1), 1 + N(m), 1 + N(x)\} - 1. \qquad [5]$$

As we can see, this theorem only relies on the properties of k and m, while group divisible designs and pairwise designs of index unity are no longer needed to calculate $N(v)$ (this isn't the first theorem like this, but the previous ones aren't as powerful as you're about to find out).

5.9 For v values below 730 and above 154

As the section title suggests, the next v values we look at are $154 < v < 730$. To do this, we've partitioned the v values (still only the ones that are $\equiv 2 \pmod 4$) into nine sets of closed intervals $I_i = [a_i, b_i]$ listed in the second column of Table 5.1. Note that any v in I_i can be written as

$$v = 4m_i + x_i, \qquad 10 \leq x_i \leq c_i,$$

using the m_i and c_i values from Table 5.1, as $a_i = 4m_i + 10$ and $b_i = 4m_i + c_i$. [5]

Using the prime factorization of each m_i, we know that $N(m_i) \geq n(m_i) \geq 3$. And since $10 \leq x_i \leq c_i \leq 154$, we know that $N(x_i) \geq 2$ for all x_i. Now we can apply Theorem 5.7 for $k = 4$, as the conditions $k \geq N(m_i) + 1$ and $1 \leq x_i \leq c_i < m_i$ are met. Therefore

i	$I_i = [a_i, b_i]$	m_i	c_i
1	[158,182]	37	34
2	[186,218]	44	42
3	[222,262]	53	50
4	[266,310]	64	54
5	[314,374]	76	70
6	[378,454]	92	86
7	[458,550]	112	102
8	[554,662]	136	118
9	[666,726]	164	70

Table 5.1: Values for I_i, m_i, and c_i [5]

$$\begin{aligned} N(v) = N(4m_i + x_i) &\geq \{\min\{N(k), N(k+1), N(m)+1, N(x)+1\} - 1 \\ &\geq \min\{3, 4, 4, 3\} - 1 \\ &\geq 2 \end{aligned}$$

for all v that are in an interval I_i, which means $N(v) \geq 2$ for all $6 < v \leq 726$. [5]

5.10 For v larger than 726

Now we're onto the final stretch! To prove $N(v) \geq 2$ for the remaining v values, we take a similar approach as the previous section, rewriting v as a function of variables and applying Theorem 5.7. However, this time we are able to deal with all of the v values at once instead of dividing them into intervals, as we will show below.

Theorem 5.8. *There exists a Graeco-Latin Square of all orders $v > 726$. [5]*

Proof. We'll start by noting that for every $v \equiv 2 \pmod 4$ and $v \geq 730$, the value $v - 10$ can be expressed as

$$v - 10 = 144g + 4u, \qquad g \geq 5, \qquad 0 \leq u \leq 35,$$

which simplifies to

$$v = 4(36g) + 4u + 10. \qquad [5]$$

Looking at the prime factorization of $36g = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_y^{\alpha_y}$, we can see that the smallest $p_i^{\alpha_i}$ is at least 4, so $N(36g) \geq n(36g) \geq 3$. Once again we will let $k = 4$, and we will let $m = 36g$ and $x = 4u + 10$. This makes $k = 4 \leq 1 + N(m) \leq 4$, $10 \leq x \leq 150$, and $m \geq 180$. This means that $1 < x < m$ and $N(x) \geq 2$. Therefore under Theorem 5.7, $N(v) \geq 2$ for all $v > 6$. [5] □

Now that we have covered all positive integer v values in this paper, we can combine all of the results to conclude that $N(v) \geq 2$ for any positive integer $v \neq 1, 2, 6$.

6 Sunset

Congratulations, we've made it to the end! It's been a long journey to get here, but we've finally proved (skipping a few details) that there is a Graeco-Latin Square of every order $v \neq 1, 2, 6$. Thanks for sticking with us to the end.

We have found Euler's Conjecture to be almost entirely incorrect despite Euler's fame for his numerous contributions to the field of mathematics. However, famous mathematicians or scientists being wrong is nothing new. In the mid 1600s Pierre de Fermat incorrectly conjectures that all numbers of the form $2^{2^n} + 1$ (with n being a positive integer) are prime. In fact, it is Euler who proves this to be incorrect by finding a contradiction when $n = 5$! [14]

Even though it is not correct, Euler's Conjecture has value in that it inspires many mathematicians after him to continue his research. In a way we can consider Euler's initial conjecture and the journey to its resolution all part of the discovery process that is sometimes necessary to achieve significant results.

But where do we go from here now that Euler's Conjecture is resolved? There is still much about Latin Squares and Graeco-Latin Squares that we have yet to discover. In particular, we've shown that $N(v) \geq 2$ for $v \neq 1, 2, 6$, but what about actually calculating $N(v)$? Is there a way for us to know when we've hit the maximum number of MOLS of order v (besides for prime powers) and not just improved the lower bound? Is there a formula for $N(v)$? Seeing as it took 200 years for Euler's Conjecture to be resolved and the last parts of the proof only came out a few decades ago, it might be quite a while before we see some of the answers to these questions. Then again, maybe the solutions will come more readily with the constantly improving technology at our fingertips. I guess we'll just have to wait and see.

Alphabetical Index

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