Goodness of Fit Test: Ornstein-Uhlenbeck Process

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Abstract

In literature, the Ornstein-Uhlenbeck process, a CAR(1) process, has been used extensively for data molding. We expand the classical OU process to be driven by a general Brownian motion. When such a process is observed at discrete times $0, h, 2h, ..., [N/h]h$, the sampled process $Y_n^{(h)}, n = 1, 2, \ldots, N$ and the approximation for the unobserved driving process (noise) are used to estimate the unknown CAR(1) parameters. These estimators are tested through simulations.

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2 Introduction

A Time Series is a sequence of data points indexed by time. Each data point is a discrete observation taken from an underlying process. Time series analysis looks at the methods used to create the models from the sampled data in order to study the continuous process. For example, one can use the recorded average monthly temperatures to create a model to study climate change by looking for significant trends or patterns.

This paper will focus on the Ornstein-Uhlenbeck (OU) process, a continuous-time autorerogressive process. This process is the continuous-time analogue of the well-known discrete-time autoregressive process and therefore has many applications. The OU process is an adaptation of Brownian Motion, which models the movement of a free particle through a liquid and was first developed by Albert Einstein [5]. The altered OU model introduces the concept of friction in that it describes a process with mean-reverting tendencies. It was first introduced as the following differential equation for the velocity $V_t, t \geq 0$ of a free particle in a fluid

$$m\ dV_t = -\lambda V_t dt + dB_t \tag{1}$$

where $B_t \geq 0$ is a Brownian motion (multiplied by a constant), m is the mass of the given particle and $\lambda > 0$ is a friction coefficient. For $\lambda = 0$, equation (1) reduces to the simple Brownian motion model of Einstein. From its emergence in the original paper, the model has been sourced in a diverse range of application areas. In Finance, it is best recognized in association with the Vasicek (1977) interest rate model.

Many applications assume that randomness is derived from a continuous-time stochastic process $(Y(t), t > 0)$. However, in practice this process is observed at discrete-time points $(Y_j^{(h)} = Y(jh),\ h > 0,\ j = 1, \ldots, n)$. Consequently, we would like to use the discretely observed process $Y_j^{(h)}$ to fit the continuous-time process $(Y(t), t > 0)$. In this paper, we attempt to study the statistical equivalence between the continuous-time OU process driven by Brownian Motion and its discrete-time analgoue, the autoregressive model of order one, AR(1). In particular, we will find the relationship between the analogous parameters of the processes and use the AR(1) estimators to recover OU parameters.

In particular, we will consider a CAR(1) model driven by a general Brownian Motion with mean μ and variance η^2 as opposed to a standard Brownian Motion with mean 0 and variance 1. This allows for easier application to real data sets. As in [1], we will use the inversion formula of Pham [8] (see Theorem 7.1) to find the increments of the continuous driving process using the samples of the process Y. This same strategy was explored in [3]. However, since the CAR(1) process can only be observed at discrete points, we must estimate the unobservable integral in Theorem 7.1 using Riemann Sum.

3 Preliminaries

3.1 Time Series and Stochastic Process

Definition 3.1 (Stochastic Process). *A stochastic process is a sequence of random variables $\{X(t)\}$ where t is an element of a set T. If T is a subset of the positive integers, then the process is a discrete time stochastic process. If T is a subset of the positive real numbers, then the process is a continuous time stochastic process.*

Definition 3.2 (Time Series). *A time series is a set of data realizations. Each random variable $x_t \in X_t$ where $t \in T$ may take on a value in the reals or be restricted to the integers.*

3.2 Autocorrelation and Autocovariance

Definition 3.3 (The Autocovariance Function). *If X is a process with a finite variance, we define the autocovariance function $\gamma_X(\cdot, \cdot)$ as*

$$\gamma_X(r, s) = \mathrm{Cov}(X(r), X(s)) = E\left[\left(X(r) - E\left[X(r)\right]\right)\left(X(s) - E\left[X(s)\right]\right)\right] \quad r, s \in T.$$

Remark 3.4. Covariance is a measure of the strength of the correlation between sets of random variables. In time series, the auto-covariance function measures how strongly observations are correlated as a function of the time lag between them.

Definition 3.5 (**Autocorrelation Function**). *The autocorrelation of a random process refers to the correlation between observations as a function of the time lag between them. The autocorrelation between times s and t of a random variable X with expected values μ_t and μ_s and standard deviation σ_t and σ_s at s and t respectively, is given by:*

$$\rho_X(s,t) = \frac{\gamma_X(r,s)}{\gamma(0,0)} = \frac{\text{Cov}(X(r), X(s))}{\sigma_t\ \sigma_s}$$

Remark 3.6. The correlation function gives the correlation coefficient, a measure of the statistical correlation, or dependence, between random variables taken at two different points. This coefficient will be between -1 and 1. A coefficient of 1 is perfectly positively correlated, while a 0 indicates no correlation.

3.3 Second-Order and Strict Stationarity

We begin with an arbitrary T-indexed stochastic process $X \equiv \{X(t), t \in T\}$, where $T \equiv [0, \infty)$.

Definition 3.7 (**Second-Order Stationarity**). *The process X is said to be second-order stationary if*

(i) $E[X^2(t)] < \infty$ *for all* $t \in T$,

(ii) $E[X(t)] = m$ *for all* $t \in T$,

(iii) $\gamma_X(r,s) = \gamma(r+t, s+t)$ *for all* $r, s, t \in T$.

Remark 3.8. If X is second-order stationary then $\gamma_X(r,s) = \gamma_X(r-s, 0)$ for all $r, s \in T$. Therefore, it is convenient to redefine the autocovariance function of a second-order stationary process as a function of only one variable:

$$\gamma(h) \equiv \gamma_X(h,0) = \text{Cov}\left(X(t+h), X(t)\right) \quad \text{for all} \quad t, h \in T.$$

Definition 3.9 (**Strict Stationarity**). *The process X is said to be strictly stationary if $(X(t_1), \cdots, X(t_k))$ and $(X(t_1+h), \cdots, X(t_k+h))$ have the same joint distribution for all integers $t_1, \cdots, t_k \in T$, $k \geq 1$ and $h > 0$.*

Remark 3.10. Note that any strictly stationary sequence with a finite second moment is second-order stationary.

3.4 First Order Autoregressive Process

The Autoregressive (AR) process is a strictly stationary process that models the conditional mean of a time series as a function of past observations. An AR process that depends on one past observation is called an AR model of degree 1, and is denoted by AR(1).

Definition 3.11 (**AR(1) Process**). *A First-Order Autoregressive, or AR(1) process, is a strictly stationary process $\{Y_n, n \in \mathbb{Z}\}$, satisfying the following equation*

$$Y_n = \rho Y_{n-1} + Z_n \tag{2}$$

where $\{Z_n\} \sim iid(m, \sigma_d^2)$, $|\rho| < 1$, and Z_n is uncorrelated with Y_s for each $s < n$.

Lemma 3.12. *If Y is the AR(1) process given in 3.11, then*

$$\gamma(h) = Cov[Y_0, Y_h] = \frac{\rho^h \sigma^2}{(1-\rho^2)}.$$

Proof Recall our definition of the autocovariance between two random variables, $\gamma(h)$, where h is the time lag between the two observations. Given our AR(1) model, we are interested in finding the autocovariance between observations at lag h.

We will find $\gamma(h)$ by first taking the expected values for each side of equation (2). Using $E[Z_t] = m$ we have

$$E[Y_n] = E\left[\rho Y_{n-1} + Z_n\right] = \rho E[Y_{n-1}] + E[Z_n] = \frac{m}{1-\rho} \tag{3}$$

since $E[Y_n] = E[Y_{n-1}]$ and $|\rho| < 1$.

Next, we will find $Var[Y_n]$,

$$\begin{aligned}
Var[Y_n] &= Var[\rho Y_{n-1} + Z_n] \\
&= Var[\rho Y_{n-1}] + Var[Z_n] + 2Cov[\rho Y_{n-1}, Z_n] \\
&= \rho^2 Var[Y_{n-1}] + \sigma^2 + 2 * 0 \\
Var[Y_n] &= \rho^2 Var[Y_n] + \sigma^2 \\
(1-\rho^2) Var[Y_n] &= \sigma^2 \\
Var[Y_n] &= \frac{\sigma^2}{(1-\rho^2)},
\end{aligned}$$

since $Var[Y_n] = Var[Y_{n-1}]$ by strict stationarity and $Cov[\rho Y_{n-1}, Z_n] = 0$ by definition. It follows from the definition of variance that

$$E[Y_n^2] = \frac{\sigma^2}{(1-\rho^2)} + \frac{m^2}{(1-\rho)^2}.$$

Lastly, we will find $E[Y_0 Y_h]$ by noting that

$$\begin{aligned}
E[Y_0 Y_h] &= E[Y_0(\rho Y_{h-1} + Z_h)] \\
&= E[Y_0(\rho^2 Y_{h-2} + \rho Z_{h-1} + Z_h)] \\
&\vdots \\
&= E[Y_0(\rho^h Y_0 + \sum_{i=0}^{h-1} \rho^i Z_{h-i})] \\
&= \rho^h E[Y_0^2] + \sum_{i=0}^{h-1} \rho^i E[Y_0] E[Z_{h-i}] \\
&= \frac{\rho^h \sigma^2}{(1-\rho^2)} + \frac{\rho^h m^2}{(1-\rho)^2} + \sum_{i=0}^{h-1} \rho^i \frac{m^2}{1-\rho}.
\end{aligned}$$

Using the formula for the sum of a geometric series,

$$\begin{aligned} E[Y_0Y_h] &= \frac{\rho^h\sigma^2}{(1-\rho^2)} + \frac{\rho^h m^2}{(1-\rho)^2} + \frac{m^2(1-\rho^h)}{(1-p)^2} \\ &= \frac{\rho^h\sigma^2}{(1-\rho^2)} + \frac{\rho^h m^2}{(1-\rho)^2} + \frac{m^2}{(1-p)^2} - \frac{m^2\rho^h}{(1-p)^2} \\ &= \frac{\rho^h\sigma^2}{(1-\rho^2)} + \frac{m^2}{(1-p)^2}. \end{aligned}$$

We can now clearly see that

$$\begin{aligned} \gamma(h) &= E[Y_0Y_h] - E[Y_0]E[Y_h] \\ &= \frac{\rho^h\sigma^2}{(1-\rho^2)} + \frac{m^2}{(1-p)^2} - \frac{m^2}{(1-\rho)^2} \\ &= \frac{\rho^h\sigma^2}{(1-\rho^2)}. \end{aligned}$$

Thus,

$$Cov[Y_0, Y_h] = \frac{\rho^h\sigma^2}{(1-\rho^2)}. \tag{4}$$

Plugging in $0 = h$, yields

$$\gamma(0) = \frac{\sigma^2}{(1-\rho^2)}.$$

□

Note that $Var[Y_n] = \frac{\sigma^2}{(1-\rho^2)} = \gamma(0)$.

3.5 Asymptotic Theorem

Definition 3.13 (Convergence in Probability). *Let $\{Y_n, n = 1, \ldots, \}$ be a discrete-time stochastic process, we say that Y_n converge in probability to Y (notation: $Y_n \xrightarrow{p} Y$ if and only if*

$$P\left(|Y_n - Y| > \epsilon\right) \to 0 \quad as \quad n \to \infty.$$

Theorem 3.14 (WWLO of AR(1)). *Let Y be an AR(1) process and then*

- $\frac{1}{N}\sum_{n=0}^{N} Y_n \xrightarrow{p} E[Y_n]$
- $\frac{1}{N}\sum_{n=0}^{N} Y_nY_{n-1} \xrightarrow{p} E[Y_nY_{n-1}]$
- $\frac{1}{N}\sum_{n=0}^{N} Y_n^2 \xrightarrow{p} E[Y_n^2]$

For Proof c.f. [2]

3.6 AR(1) Yule-Walker estimators

The Yule-Walker equations relate the AR(1) parameters to the autocovariance of the stochastic process. They are important because they allow us to estimate our model parameters using data. To utilize the Yule-Walker equations, we must first account for the fact that $E[Z_n] = m$, rather than 0. To do so, we will define a new random variable, Y^* as the following,

$$Y^* = Y - E[Y] = Y - \frac{m}{1-\rho} \tag{5}$$

Note that

$$E[Y^*] = E[Y - E[Y]] = 0.$$

To find our new noise, Z_n^*, we will do the following calculations,

$$\begin{aligned} Y_n^* &= Y_n - \frac{m}{1-\rho} \\ &= \rho Y_{n-1} + Z_n - \frac{m}{1-\rho} + \frac{\rho m}{1-\rho} - \frac{\rho m}{1-\rho} \\ &= \rho(Y_{n-1} + \frac{m}{1-\rho}) + Z_n + \frac{m(\rho-1)}{1-\rho} \\ &= \rho(Y_{n-1}^*) + Z_n - m. \end{aligned}$$

This yields the following equation for our new random variable,

$$Y_n^* = \rho(Y_{n-1}^*) + Z_n^*. \tag{6}$$

where $Z_n^* = Z_n - m$ is our new noise and $E[Z_n^*] = E[Z_n - m] = 0$.
Multiply 6 by Y_n* and also by Y_{n-1}^*,

$$Y_n^{*2} = \rho(Y_{n-1}^* Y_n^*) + Z_n^* Y_n^*$$

$$Y_n^* Y_{n-1}^* = \rho(Y_{n-1}^{*2}) + Z_n^* Y_{n-1}^*.$$

Take the expected value of both equations,

$$E[Y_n^{*2}] = \rho E[(Y_{n-1}^* Y_n^*)] + E[Z_n^* Y_n^*]$$

$$E[Y_n^* Y_{n-1}^*] = \rho E[(Y_{n-1}^{*2})] + E[Z_n^* Y_{n-1}^*].$$

Note that

$$E[Y_{n-1}^* Z_n^*] = E[Y_{n-1}^*]E[Z_n^*] = 0$$

and

$$\begin{aligned} E[Y_n^* Z_n^*] &= E[(\rho Y_{n-1}^* + Z_n^*)Z_n^*] \\ &= \rho E[Y_{n-1}^* Z_n^*] + E[Z_n^{*2}] \\ &= \rho E[Y_{n-1}^*]E[Z_n^*] + Var[Z_n^*] + E[Z_n^*]^2 \\ &= \sigma_d^2 \end{aligned}$$

Thus, we have

$$E[Y_n^{*2}] = \rho E[(Y_{n-1}^* Y_n^*)] + \sigma_d^2$$
$$E[Y_n^* Y_{n-1}^*] = \rho E[(Y_{n-1}^{*2})].$$

Using the method of moments, identify

$$E[\widehat{Y_{n-1}^* Y_n^*}] = \frac{1}{n}\sum_{t=1}^{n} Y_{t-1}^* Y_t^* \quad \text{and} \quad E[\widehat{Y_{n-1}^{*2}}] = \frac{1}{n}\sum_{t=1}^{n} Y_{t-1}^{*2}$$

Where $E(\widehat{Y_{n-1}^* Y_n^*})$ is the expected value of the sample mean of the product $(Y_{n-1}^* Y_n^*)$ and $\widehat{E(Y_{n-1}^{*2}})$ is the expected value of the square of the Y_{n-1}^*.

Note that : $E(Y_{n-1}^{*2}) = E(Y_n^{*2})$ by strict stationarity.
Using these estimators in equation we arrive at moment estimators $\hat{\rho}$ of ρ and $\hat{\sigma}_d^2$ of σ_d^2

$$\frac{1}{n}\sum_{t=1}^{n} Y_{t-1}^* Y_t^* = \hat{\rho}\frac{1}{n}\sum_{t=1}^{n} Y_{t-1}^{*2},$$
$$\frac{1}{n}\sum_{t=1}^{n} Y_{t-1}^{*2} = \hat{\rho}\frac{1}{n}\sum_{t=1}^{n} Y_{t-1}^{*2} + \hat{\sigma_d}^2.$$

Simplifying yields the following equations for the estimators of our desired parameters

$$\hat{\rho} = \frac{\sum_{t=1}^{n} Y_{t-1}^* Y_t^*}{\sum_{t=1}^{n} Y_{t-1}^{*2}}$$
$$\hat{\sigma}_d^2 = \frac{1}{n}\sum_{t=1}^{n} Y_{t-1}^{*2} - \hat{\rho}\frac{1}{n}\sum_{t=1}^{n} Y_{t-1}^* Y_t^*.$$

Theorem 3.15. *Let $\{Y_n\}$ be an $AR(1)$ process, and $\hat{\rho}, \hat{\sigma_d}^2$ defined as above. As $n \to \infty$ we have:*

$$\hat{\rho} \xrightarrow{p} \rho, \text{ and } \hat{\sigma}_d^2 \xrightarrow{p} \sigma_d^2.$$

Proof Recall our equation for $\hat{\rho}$. Note that,

$$\hat{\rho} = \frac{\frac{\sum_{t=1}^{n} Y_{t-1}^* Y_t^*}{n}}{\frac{\sum_{t=1}^{n} Y_{t-1}^{*2}}{n}}.$$

By Theorem 3.14 we have that,

$$\begin{aligned} \frac{\frac{\sum_{t=1}^{n} Y_{t-1}^* Y_t^*}{n}}{\frac{\sum_{t=1}^{n} Y_{t-1}^{*2}}{n}} &\xrightarrow{p} \frac{E\left[Y_{n-1}^* Y_n^*\right]}{E\left[Y_{n-1}^{*2}\right]} \\ &\xrightarrow{p} \frac{\rho E[(Y_{n-1} - E[Y_{n-1}])^2]}{E[(Y_n - E[Y_n])^2]} \\ &\xrightarrow{p} \rho. \end{aligned}$$

Now recall our equation for $\hat{\sigma_d^2}$. Again, by 3.14, we see that

$$\begin{aligned}
\hat{\sigma_d}^2 &= \frac{1}{n}\sum_{t=1}^{n} Y_{t-1}^{*2} - \hat{\rho}\frac{1}{n}\sum_{t=1}^{n} Y_{t-1}^{*}Y_t^{*} \\
&\xrightarrow{p} E[Y_{n-1}^{*2}] - \rho E[Y_{n-1}^{*}Y_n^{*}] \\
&= E[Y_{n-1}^{*2}] - \rho E[Y_{n-1}^{*}\rho Y_{n-1}^{*} + Y_{n-1}^{*}Z_n^{*}] \\
&= E[Y_{n-1}^{*2}] - \rho^2 E[Y_{n-1}^{*2}] \\
&= E[Y_{n-1}^{*2}](1-\rho^2) \\
&= \sigma_d^2.
\end{aligned}$$

Thus, as $n \to \infty$, $\hat{\rho} \to \rho$ and $\hat{\sigma_d^2} \to \sigma_d^2$. $\square$

4 Brownian Motion

The stochastic process of a standard Brownian Motion is motivated by the idea of the simple symmetric random walk, which tracks the movement of a particle along the real number line. In the random walk, the particle is equally likely to take one step to the right or the left, towards the positive or negative numbers respectively, every minute over some time interval. That is, $B(t) = S_0 + \sum_{i=1}^{t} X_i$, where $B(t)$ is the location of the particle at time t, X_i is the displacement of the particle at time i, and $P(X_i = 1) = .5 = P(X_i = -1)$. Note that the expected value of the change in location is 0 for the standard Brownian Motion. In the generalized Brownian Motion, the random particle is weighted. That is, the particle could be more likely to move in a positive direction than a negative one. In the general case, $E[X_i] = \mu$ for some constant μ. It folows that $E[B(t)] = \mu * t$.

To make this discrete process the continuous Brownian Motion, we let $k > 0$ and make the particle take a step every $\frac{1}{k}$ minutes and scale the size of the step by $\frac{1}{\sqrt{k}}$. Now our particle is take many small steps very frequently. Letting $k \longrightarrow \infty$, our discrete process converges to a continuous one:

$$B(t) = \lim_{k\to\infty} B_k(t) = \lim_{k\to\infty} \frac{1}{\sqrt{k}} \sum_{i=1}^{tk} X_{\frac{i}{k}}.$$

Note that for a constant k and times $s, t \in$ our time interval such that $t > s \geq 0$,

$$\begin{aligned}
B_k(t) - B_k(s) &= \frac{1}{\sqrt{k}} \sum_{i=1}^{tk} X_{\frac{i}{k}} - \frac{1}{\sqrt{k}} \sum_{i=1}^{sk} X_{\frac{i}{k}} \\
&= \frac{1}{\sqrt{k}} \left[\sum_{i=1}^{tk} X_{\frac{i}{k}} - \sum_{i=1}^{sk} X_{\frac{i}{k}} \right] \\
&= \frac{1}{\sqrt{k}} \sum_{i=sk}^{tk} X_{\frac{i}{k}}.
\end{aligned}$$

It follows that the distribution of the increment $B_k(t) - B_k(s)$ only depends on the length of the time interval, namely $(t - s)$. From this, we can deduce that the continuous Brownian motion will have stationary increments. This leads us to our formal definition of the general Brownian Motion.

Definition 4.1 (**General Brownian motion**). *A real-valued random process $B = \{B(t) : t \geq 0\}$ is called a General Brownian motion if the following is true:*

1. $B(0) = 0$.
2. *B has both stationary and independent increments.*
3. *$B(t) - B(s)$ has a normal distribution with mean $\mu(t - s)$ and variance $(t - s)$ for $0 \leq s < t$, $\mu \in \mathbb{R}$.*

Note that: $E[B_t] = \mu t$ and $\mathrm{Var}(B_t) = t$. Hence, if $s = t - \frac{1}{K}$ then

$$B(t) - B\left(t - \frac{1}{K}\right) \sim N\left(\frac{\mu}{K}, \frac{1}{K}\right).$$

5 Ornstein-Uhlenbeck Process

Definition 5.1 (**Ornstein-Uhlenbeck or CAR(1) Process**). *An Ornstein-Uhlenbeck process is the second-order stationary process $\{X_t\}$, that satisfies the following stochastic differential equation:*

$$dY(t) = -aY(t)\ dt + \sigma dB(t) \qquad t \geq 0.$$

where $\{B_t\}$ is standard Brownian motion, and a and $\sigma > 0$ are parameters and X_0 is a random variable that is independent of $\{B_t\}$.

In order to simulate Y_t, Brownian motion-driven CAR(1) process, we look at its definition through its stochastic differential equation

$$dY(t) = -aY(t)\ dt + \sigma dB(t),$$

which by Euler's scheme can be approximated by the difference equation:

$$Y_t - Y_{t-\frac{1}{K}} = -aY_{t-\frac{1}{K}}\frac{1}{K} + \sigma\left(B_t - B_{t-\frac{1}{K}}\right), \tag{7}$$

where t is of the form $t = i/K$, $i = 1, \ldots, KN$.
Consider the stochastic differential equation $dY(t) = -aY(t)dt + \sigma dB(t)$. We want to find a $Y(t)$ for which this equation is true. Using the following two lemmas, we will prove that $Y(t) = e^{-at}Y(0) + \sigma B(t) - a\sigma\int_0^t e^{a(s-t)}B(s)ds$ is a solution to this equation.

Lemma 5.2. *The following two equations are equivalent:*

$$Y(t) = e^{-at}Y(0) + \sigma B(t) - a\sigma\int_0^t e^{a(s-t)}B(s)ds \tag{8}$$

$$Y(t) = e^{-at}Y(0) + \sigma\int_0^t e^{a(s-t)}dB(s). \tag{9}$$

Proof 3.2 We will prove that equations 8 and 9 are equivalent equations by using integration by parts. Recall that since $B(t)$ is a Brownian motion, $B(0) = 0$.
Let $w = B(s)$ and $dv = e^{a(s-t)}ds$. Thus $dw = dB(s)$ and $v = \frac{1}{a}e^{a(s-t)}$.
Note that we can rewrite part of equation (3) as the following:

$$\int_0^t e^{a(s-t)}B(s)ds = B(s)\frac{1}{a}e^{a(s-t)}\bigg|_0^t - \int_0^t \frac{1}{a}e^{a(s-t)}dB.$$

Simplifying the right hand side of the equation, we get:

$$\int_0^t e^{a(s-t)}B(s)ds = \frac{1}{a}B(t) - \int_0^t \frac{1}{a}e^{a(s-t)}dB$$

$$\int_0^t e^{a(s-t)}B(s)ds = \frac{1}{a}\bigg(B(t) - \int_0^t e^{a(s-t)}dB\bigg). \tag{10}$$

Substituting our new equation into 8, we yield:

$$Y(t) = e^{-at}Y(0) + \sigma B(t) - a\sigma B(t)\frac{1}{a}\bigg(1 - e^{a(s-t)}\bigg).$$

Upon further simplification, we notice that:

$$Y(t) = e^{-at}Y(0) + \sigma\bigg(B(t) - a\bigg[\frac{1}{a}\bigg(B(t) - \int_0^t e^{a(s-t)}dB\bigg)\bigg]\bigg)$$

$$Y(t) = e^{-at}Y(0) + \sigma\int_0^t e^{a(s-t)}dB.$$

Thus we have shown that equation 8 is equivalent to 9.

Lemma 5.3. *Let $dY(t)$ be a stochastic differential equation given by*

$$dY(t) = -aY(t)dt + \sigma dB(t). \tag{11}$$

Then $Y(t) = e^{-at}Y(0) + \sigma\int_0^t e^{a(s-t)}dB(s)$ is a solution to this equation.

Proof 3.3 We will prove that 9 is a solution to 11. Note that by the previous lemma, 8 is equivalent to 9. Thus it is sufficient to prove that 8 is a solution to 11.
First, we integrate both sides of our stochastic differential equation to obtain integral form:

$$\begin{aligned} dY(t) &= -aY(t)dt + \sigma dB(t) \\ \int_0^s dY(t) &= \int_0^s -aY(t)dt + \int_0^s \sigma dB(t) \\ Y(s) - Y(0) &= -a\int_0^s Y(t)dt + \sigma B(s). \end{aligned}$$

Then, we substitute (8) in the right hand side of the equation. This yields,

$$Y(s) - Y(0) = -a\int_0^s \left[e^{-at}Y(0) + \sigma B(t) - a\sigma\int_0^t e^{a(v-t)}B(v)dv\right]dt + \sigma B(s).$$

Recalling 10, we can solve the integrals and simplify to get

$$Y(s) = Y(0)e^{-as} - a\int_0^s \sigma B(t)dt + a^2\sigma\int_0^s\int_0^t e^{-a(t-v)}B(v)dvdt + \sigma B(s).$$

Switching the limits of integration on our double integral and solving yields

$$\begin{aligned}
Y(s) &= Y(0)e^{-as} - a\int_0^s \sigma B(t)dt + a^2\sigma\int_0^s\int_v^s e^{a(v-t)}B(v)dtdv + \sigma B(s)\\
&= Y(0)e^{-as} - a\int_0^s \sigma B(t)dt + a\sigma\int_0^s (e^{a(v-s)} - e^0)dv + \sigma B(s)\\
&= Y(0)e^{-as} - a\int_0^s \sigma B(t)dt + a\sigma\int_0^s (e^{a(v-s)} - e^0)B(v)dv + \sigma B(s)\\
&= Y(0)e^{-as} - a\sigma\int_0^s B(t)dt + a\sigma\int_0^s e^{a(v-s)}B(v) - a\sigma\int_0^s B(v)dv + \sigma B(s).
\end{aligned}$$

Noting that $a\sigma\int_0^s B(t)dt = a\sigma\int_0^s B(v)dv$ and back substituting in 8, we can see that this becomes

$$\begin{aligned}
Y(s) &= Y(0)e^{-as} - a\sigma\int_0^s B(t)dt + a\sigma\int_0^s e^{a(v-s)}B(v) - a\sigma\int_0^s B(v)dv + \sigma B(s)\\
&= Y(s).
\end{aligned}$$

□

Remark 5.4. Recall the following,

$$E[B(t)] = \mu t \quad \text{and} \quad \mathrm{Var}(B(t)) = \eta^2 t, \quad t \geq 0,$$

and the Definition 5.3 of Y, a CAR(1) process:

$$\begin{aligned}
Y(t) &= e^{-at}Y(0) + \sigma\int_0^t e^{-a(t-u)}dB(u), \quad t \geq 0,\\
&= e^{-at}Y(0) + \int_0^t e^{-a(t-u)}d(\sigma B(u)), \quad t \geq 0,
\end{aligned}$$

where $a \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$.

We notice that σ and η are not identifiable, and so without loss of generality we can assume that $\eta^2 = 1$ since if Y is a CAR(1) process driven by B with $\eta^2 \neq 1$ then Y_t can be also seen as CAR(1) processes driven by the Brownian motion $B'(t) = \frac{B(t)}{\eta}$ with $\eta'^2 = 1$.

Now we have seen that $Y(t) = e^{-at}Y(0) + \sigma B(t) - a\sigma \int_0^t e^{a(s-t)}B(s)ds$ is a solution to the stochastic differential equation $dY(t) = -aY(t)dt + \sigma dB(t)$ and is therefore a CAR(1) process. We want to show that $Y(t)$ can be written in the form of an AR(1) model. To do so, we will define a relationship between the parameters of the AR(1) and CAR(1) models. Since we can use the discrete sampled data to estimate the AR(1) parameters, we can utilize these new relationships to then estimate the CAR(1) parameters.

The following Lemma shows that we can write $Y(t)$ as a function of past observations. We will then note that this equation is similar to that of an AR(1) process.

Lemma 5.5. *Let* $Y(t) = Y_0 e^{-at} + \sigma \int_0^t e^{-a(t-u)}dB(u)$ *be a true statement. Then for* $0 \le s \le t$,

$$Y(t) = e^{-a(t-s)}Y(s) + \sigma \int_s^t e^{-a(t-u)}dB(u). \tag{12}$$

Proof 3.4
Suppose that $Y(t) = Y_0 e^{-at} + \sigma \int_0^t e^{-a(t-u)}dB(u)$.
It follows that

$$\begin{aligned} Y(t) &= Y_0 e^{-at} + \sigma \int_0^s e^{-a(t-u)}dB(u) + \int_s^t e^{-a(t-u)}dB(u) \\ &= e^{-a(t-s)}\left[Y_0 e^{-as} + \sigma \int_0^s e^{-a(s-u)}dB(u)\right] + \int_s^t e^{-a(t-u)}dB(u). \end{aligned}$$

Recall that $Y(s) = e^{-as}Y(0) + \sigma \int_0^s e^{a(u-s)}dB(u)$. Substituting this into the left hand side of this equation, we get

$$Y(t) = e^{-a(t-s)}Y(s) + \int_s^t e^{-a(t-u)}dB(u).$$

□

6 The Sampled CAR(1) Process

In practice, continuous time processes are usually sampled at discrete times. Here we assume that the CAR(1) process is observed at equally spaced intervals of length h. To be precise, let Y be a strictly stationary CAR(1) process

$$Y(t) = e^{-at}Y(0) + \sigma \int_0^t e^{-a(t-u)}dB(u).$$

For $0 \le s < t$ we have:

$$Y(t) = e^{-a(t-s)}Y(s) + \sigma \int_s^t e^{-a(t-u)}dB(u).$$

For $h > 0$ and $n \in \mathbb{Z}_+$ choose $t = nh$ and $s = (n-1)h$. Define $Y_n^{(h)} \equiv Y(nh)$. Then

$$Y_n^{(h)} = e^{-ah}Y_{n-1}^{(h)} + \sigma \int_{(n-1)h}^{nh} e^{-a(nh-u)}dB(u).$$

The sampled process $\{Y_n^{(h)}, n = 0, 1, 2, \cdots\}$ can be written as:

$$Y_n^{(h)} = \phi Y_{n-1}^{(h)} + Z_n^{(h)}, \quad n = 0, 1, 2, \cdots \quad , \tag{13}$$

where

$$\phi = e^{-ah}, \quad \text{and} \quad Z_n^{(h)} = \sigma \int_{(n-1)h}^{nh} e^{-a(nh-u)} dB(u). \tag{14}$$

Now because that B is a Brownian motion, B has stationary and independent increments, $\{Z_n^{(h)}, n \geq 1\}$ is an i.i.d. sequence. Hence, the sampled process $\{Y_n^{(h)}, n = 0, 1, 2, \cdots\}$ is a discrete-time AR(1) process.

Lemma 6.1. *Let $dY(t)$ be a stochastic differential equation given by*

$$dY(t) = -aY(t)dt + \sigma dB(t).$$

Suppose

$$Z_n^{(h)} = \sigma \int_{(n-1)h}^{nh} e^{-a(nh-u)} dB(u), \tag{15}$$

where $\{Z_n^{(h)}, n \geq 1\}$ is an i.i.d. sequence, $B(u)$ is Brownian Motion with mean μu and variance u, and $\phi = e^{-ah}$.

Then

$$E[Z_n^{(h)}] = \frac{\sigma\mu(1-\phi)}{a}. \tag{16}$$

Proof Consider 15. We will use integration by parts to yield

$$\begin{aligned} Z_n^{(h)} &= \sigma\left(e^{-a(nh-u)} B(u) \Big|_{(n-1)h}^{nh} - \int_{(n-1)h}^{nh} B(u) a e^{-a(nh-u)} du \right) \\ &= \sigma\left(B(nh) - e^{-ah} B((n-1)h) - a \int_{(n-1)h}^{nh} B(u) e^{-a(nh-u)} du \right). \end{aligned}$$

Note that since B is Brownian motion, and therefore has stationary and independent increments, we know that $Z_n^{(h)}$ is *iid* for $n = 1, 2, \ldots$ hence $E[Z_n^{(h)}] = E[Z_1^{(h)}]$ for all n and the $\text{Var}(Z_n^{(h)}) = \text{Var}(Z_1^{(h)})$ for all n. Furthermore, recall $E[B(t)] = \mu t$ by property of Brownian Motion. Taking the expected value of $Z_n^{(h)}$,

$$\begin{aligned} E[Z_n^{(h)}] &= \sigma\left(E[B(nh)] - E[e^{-ah} B((n-1)h)] - E\left[a \int_{(n-1)h}^{nh} B(u) e^{-a(nh-u)} du \right] \right) \\ &= \sigma\left(\mu nh - \phi\mu(n-1)h)] - a \int_{(n-1)h}^{nh} E[B(u)] e^{-a(nh-u)} du \right) \\ &= \sigma\left(\mu nh - \phi\mu(n-1)h)] - a \int_{(n-1)h}^{nh} \mu u e^{-a(nh-u)} du \right). \end{aligned}$$

Again, we use integration by parts to simplify our equation,

$$
\begin{aligned}
E[Z_n^{(h)}] &= \sigma\Big(\mu nh - \phi\mu(n-1)h)] - a\Big(\frac{u}{a}e^{-a(nh-u)}\Big|_{(n-1)h}^{nh} - \int_{(n-1)h}^{nh}\frac{1}{a}e^{-a(nh-u)}du\Big) \\
&= \sigma\mu\Big(nh - \phi(n-1)h)] - \Big(nh - (n-1)he^{-ah} - \frac{1}{a}e^{-a(nh-u)}\Big|_{(n-1)h}^{nh}\Big)\Big) \\
&= \sigma\mu\Big(nh - \phi(n-1)h)] - nh + \phi(n-1)h + \frac{1}{a}(1-\phi)\Big) \\
&= \frac{\sigma\mu(1-\phi)}{a}.
\end{aligned}
$$

Thus, we have proven that $E[Z_n^{(h)}] = \frac{\sigma\mu(1-\phi)}{a}$. □

Lemma 6.2. *Suppose the assumptions in Lemma 6.1 are true, then*

$$\mathrm{Var}[Z_n^{(h)}] = \frac{\sigma^2(1-\phi^2)}{2a}.$$

Proof Recall that 13 is an AR(1) process. Thus we can find $E[Y_n]$ by substituting in e^{-ah} for ρ and 16 for m into 3 as shown below,

$$
\begin{aligned}
E[Y_1] &= \frac{m}{1-\rho} \\
&= \frac{\frac{\sigma\mu(1-e^{-ah}}{a})}{1-e^{-ah}} \\
&= \frac{\sigma\mu}{a}.
\end{aligned}
$$

Similarly, we can use 4 to see that

$$Cov[Y_0, Y_1] = \frac{\sigma^2}{2a}.$$

Now, note that

$$
\begin{aligned}
E[Z_1^2] &= E[(Y_1 - \rho Y_0)^2] \\
&= E[Y_1^2] - 2\rho E[Y_1 Y_0] + \rho^2 E[Y_0^2] \\
&= (1-\rho^2)\frac{\sigma^2}{2a} + (1-\rho)^2\frac{\mu\sigma^2}{a^2}.
\end{aligned}
$$

Thus,

$$Var[Z_n] = E[Z_n^2] - E[Z_n]^2 = (1-\rho^2)\frac{\sigma^2}{2a}. \tag{17}$$

□

7 Estimated Increments

Note that although we can observe the CAR(1) process Y on some interval $[0, T]$, we can not observe $dB(u)$ directly. Thus, we must derive the following inversion formula to extract the Brownian Motion increments from the observations. Suppose the we have continuously observed the CAR(1) process.

Theorem 7.1 (Inversion Formula). *Let Y be a CAR(1) process satisfying*

$$Y(t) = e^{-at}Y(0) + \sigma \int_0^t e^{-a(t-u)} dB(u).$$

Then

$$B(t) = \sigma^{-1} \left[Y(t) - Y(0) + a \int_0^t Y(s) ds \right]. \tag{18}$$

Proof:
Consider (18) Recall 5.3

$$\begin{aligned} B(t) &= \sigma^{-1} \left[Y(t) - Y(0) + a \int_0^t Y(s) ds \right] \\ &= \sigma^{-1} \left[-a \int_0^t Y(u) du + \sigma B(t) + a \int_0^t Y(s) ds \right] \\ &= \sigma^{-1} \left[-a \int_0^t Y(u) du + a \int_0^t Y(s) ds \right] + B(t). \end{aligned}$$

Since $\int_0^s Y(u)du$ and $\int_0^t Y(s)ds$ are equivalent, they cancel each other out and we get

$$B(t) = B(t).$$

□

Recall that the CAR(1) process is sampled discretely, and thus we cannot directly use the Inversion Formula to recover driving process B. Instead, we must approximate the increments of B over our interval.

Using 18 and assuming $h = 1$ without loss of generality, the increment of B over the interval $((n-1), n)$, where and $n = 1, ..., N$, is

$$\Delta B_n = B(n) - B((n-1)) = \sigma^{-1} \left[Y_n - Y_{n-1} + a \int_{(n-1)}^n Y(s) ds \right]. \tag{19}$$

Note that the above increment depends on a continuously observed process. In practice however, the process is often only able to be observed discretely at times nh. Thus we will use Riemann's sum to approximate the integral as follows,

$$\widehat{\Delta B_n} = \sigma^{-1} \left[Y_n - Y_{n-1} + a \frac{1}{k} \sum_{i=0}^{k} Y_{(n-1)+\frac{i}{k}} \right]. \tag{20}$$

We will refer to (20) as the *estimated increments*.

8 Model Construction

Recall that we are trying to construct a model,

$$Y_n^{(h)} = \phi Y_{n-1}^{(h)} + Z_n^{(h)}, \quad n = 0, 1, 2, ..., \tag{21}$$

where

$$\phi = e^{-ah}, \quad \text{and} \quad Z_n^{(h)} = \sigma \int_{(n-1)h}^{nh} e^{-a(nh-u)} dB(u),$$

and $B \sim N(\mu, 1)$ of a real, stochastic CAR(1) process,

$$Y \sim N\left(\frac{\mu\sigma}{a}, \frac{\mu^2\sigma^2}{2a}\right),$$

based on a set of discrete observations,

$$\hat{Y} = \{Y_0, Y_{\frac{1}{k}}, ..., Y_{\frac{Nk}{k}}\},$$

where $[0, N]$ is a time interval and k is the length of each time increment.
You will note that we have three unknowns we must extract from the data, namely σ, μ and a. Recall from 14 that $\rho = e^{-ah}$. Since we have assumed that $h = 1$, it follows that $a = -\ln \rho$. Using our estimator for ρ from Theorem 3.15, we obtain the following estimator for a,

$$\hat{a} = -\ln \hat{\rho} \xrightarrow{p} a. \tag{22}$$

Now we need to find an estimator for σ. Recall from 20 that,

$$\sigma\widehat{\Delta B_n} = \left(Y_n - Y_{n-1} + a\frac{1}{k}\sum_{i=0}^{k} Y_{(n-1)+\frac{i}{k}}\right).$$

By Theorem 20, we have estimators for the mean and standard deviation of our increments. Taking the mean of the increments multiplied by σ, we can find an estimator for $\sigma\mu$ as follows,

$$\frac{1}{N}\sum_{n=1}^{N}\left(Y_n - Y_{n-1} + a\frac{1}{k}\sum_{i=0}^{k} Y_{(n-1)+\frac{i}{k}}\right) = \sigma\frac{1}{N}\sum_{n=1}^{N}\widehat{\Delta B_n} = E[\sigma\widehat{\Delta B_n}] \xrightarrow{p} \sigma\mu.$$

Similarly, we can find an estimator for σ^2 by taking the variance of the increments multiplied by σ,

$$\frac{1}{N}\sum_{n=1}^{N}\left(\sigma\Delta B_n) - (\sigma\frac{1}{N}\sum_{n=1}^{N}\Delta B_n)\right) = Var[\sigma\Delta B_n] \xrightarrow{p} \sigma^2\eta^2 = \sigma^2.$$

We now have all the tools needed to model a continuous-time autoregressive process using discrete samples. The R code used to generate these estimators can be found in the appendix at the end of the paper. From our observations, we can use the above estimators to recover the following,

$$e^{-\hat{a}h} \xrightarrow{p} \phi$$

$$\frac{E[\widehat{\sigma \Delta B_n}](1-e^{-\hat{a}h})}{\hat{a}} \xrightarrow{p} m = E[Z_n]$$

$$\frac{Var[\widehat{\sigma \Delta B_n}](1-e^{-\hat{a}h})}{2a} \xrightarrow{p} \sigma_d^2 = Var[Z_n]$$

$$\frac{E[\widehat{\sigma \Delta B_n}]}{\hat{a}} \xrightarrow{p} E[Y_n].$$

8.1 Simulation

Now that we have constructed our model, we must investigate how well our estimators actually perform. To do so, we will simulate a CAR(1) process using R and compare our recovered values to the true values of our parameters. The R code used for this simulation may be found in the appendix below.

We will simulate a Brownian Motion driven CAR(1) process with the following parameters

$$N = 100$$
$$K = 1000$$
$$\mu = \frac{1}{2}$$
$$a = 0.9$$
$$\sigma = 2.$$

1. Divide time interval into K small chunks. Scale the size of each step by $\frac{1}{\sqrt{k}}$. For our model, we let $K = 1000$.

2. Using rnorm, generate $N * K + 1$ random numbers Z_t to represent the changes in location at times $t = 1, 2, ..., N * K + 1$.

3. Let $B(t) = B(t-1) + \frac{1}{\sqrt{k}} Z_i$ be the location of the Brownian Motion particle at time t.

4. Plot $B(t)$ with time t on the x-axis. Note that as this is a simulation of a random process, each iteration and subsequent plot may be different.

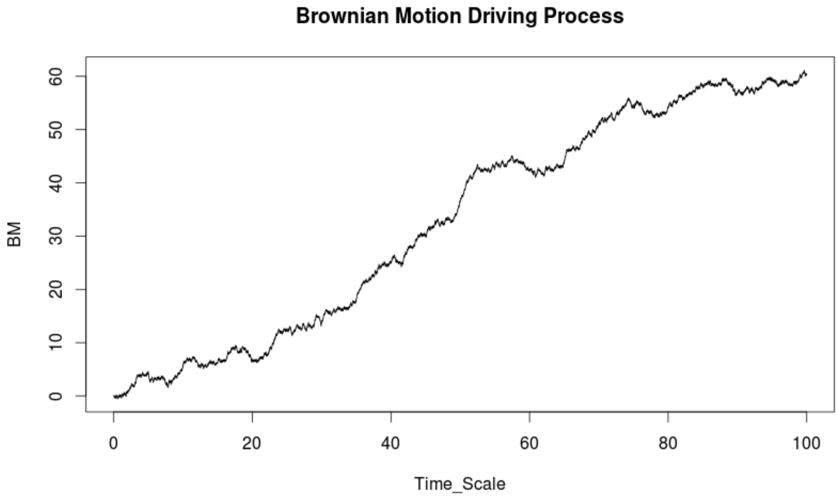

Figure 1: Brownian Motion with $\mu = \frac{1}{2}$, $N = 100$, $K = 1000$, $\sigma = 2$, $a = 0.9$.

Using difference equation (7, we will simulate the CAR(1) process using R. The R code used in the following simulation may be found in the appendix.

1. Since our equation is recursive, we must also generate our first iteration $Y(0)$. Using rnorm, generate 1 random number $Y0$ with $mean = \frac{\mu\sigma}{a}$ and standard deviation $= \frac{\sigma\eta}{\sqrt{2a}}$.

2. Simulate the Brownian motion $B(t)$ as detailed in the prior simulation section. Note that noise[i] is the increment of the Brownian Motion from time i to $i + 1$.

3. Define the CAR(1) function

$$Y_t = Y_{t-1} - aY_{t-1}\frac{1}{K} + \sigma Z_t,$$

   for $t = 1, ..., N * K + 1$. Simulate this process by generating $Y(t)$ for each $t = 1, ..., (NK + 1)$.

4. Plot the above process with time $1 < t \leq (NK + 1)$ on the x-axis and the corresponding values $Y(t)$ on the y-axis.

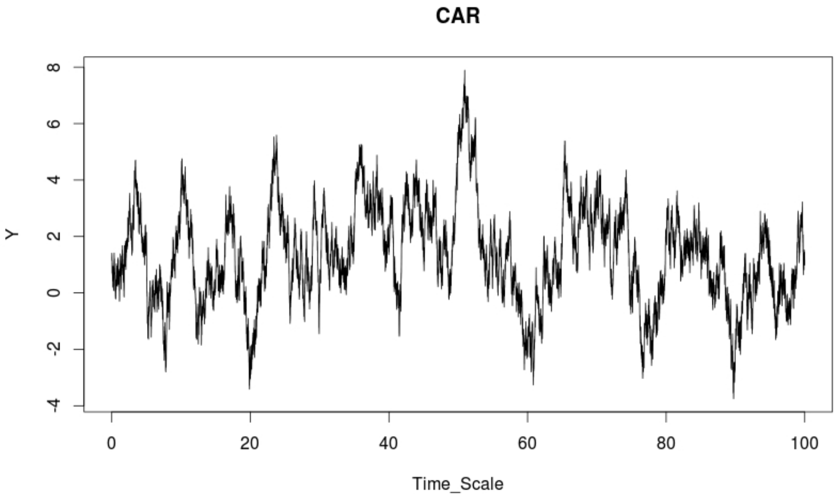

Figure 2: Continuous Brownian Motion Driven Process with $\mu = \frac{1}{2}$, $N = 100$, $K = 1000$, $\sigma = 2$, $a = 0.9$.

Now, we compare the recovered values to the true values,

Parameter	True	Recovered
a	0.9	0.99
$\mu\sigma$	1	1.34
σ^2	4	4.9698

Figure 3: Comparison of Recovered and True Values.

Note that our recovered values are significantly different than our true values. One method of improving the fit of the recovered parameters is to increase the number of increments taken. We will now simulate a CAR(1) process using $K = 5000$.

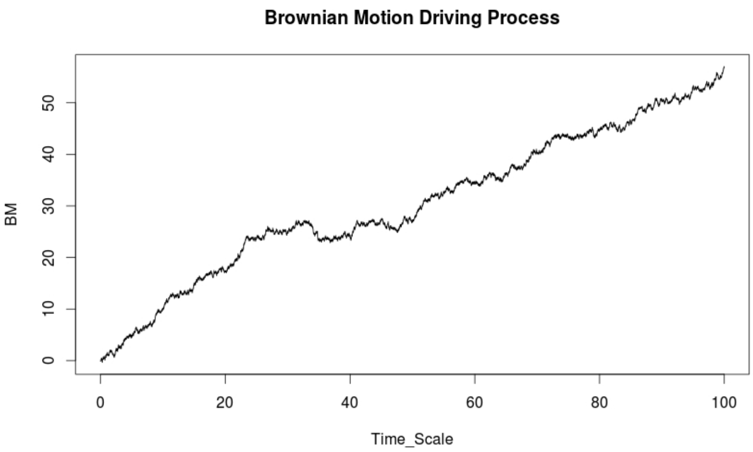

Figure 4: Brownian Motion with $\mu = \frac{1}{2}$, $N = 100$, $K = 1000$, $\sigma = 2$, $a = 0.9$.

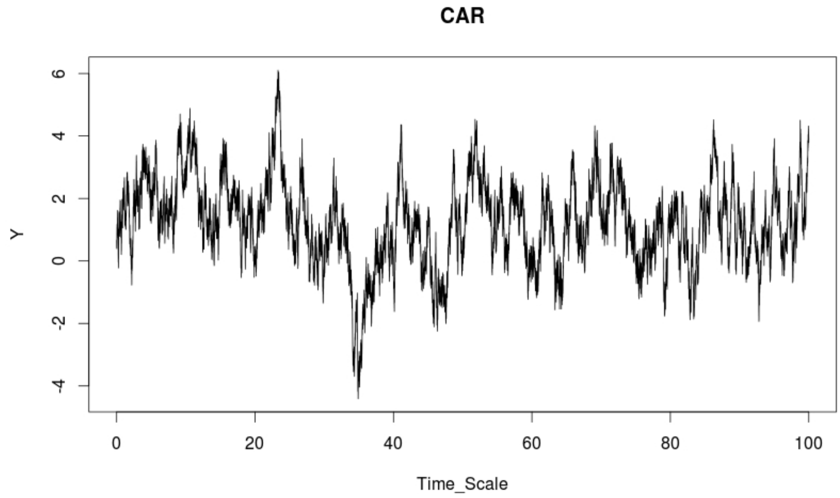

Figure 5: Continuous Brownian Motion Driven Process with $\mu = \frac{1}{2}$, $N = 100$, $K = 5000$, $\sigma = 2$, $a = 0.9$.

Parameter	True	Recovered
a	0.9	0.99
$\mu\sigma$	1	1.26
σ^2	4	3.54

Figure 6: Comparison of Recovered and True Values.

As shown in Figure 6, increasing our number of increments improved the fit of our recovered values. However, our estimators appear to still significantly differ from our true parameters. Performing the simulation with a variety of different true values generates similar discrepancies.

9 Appendix

9.1 R-codes

R Code for the simulation of the Brownian motion process and of the Brownian motion-driven CAR(1) process:

```
Driving_Process_BM<-function(N,K,noise)
{
  # Requires Noise as an input, must be of length N*K+1
  Dt <- 1/K;
  BM <- numeric(N*K+1);
  for(i in 2:(N*K+1)){
```

```
    BM[1]<- 0;
    BM[i] <- BM[i-1] + noise[i];
  }
  out<-BM;
}


CAR1<-function(a,sigma,N,K,noise,Y0)
{
  # Noise Z must be simulated first; the same noise is used for
  Driving_Process
  # Y0 must be simulated first
  Y <-numeric(N*K+1);
  Y[1] <- Y0;
  Dt=1/K;
  # Euler approximation scheme to CAR1 using the frequency K
  for(i in 2:(N*K+1)){
    Y[i] <- Y[i-1]- a*Y[i-1]*Dt + sigma*noise[i];
  }
  out<-Y
}


Estimated_increments=function(N=N, K=K,Y0=Y0, Y=Y, a = a_hat)
{
sigma_DeltaB=0
sigma_DeltaB[1]=Y[K]-Y0+ a/K* sum(Y[1:K])
for (i in 2:N){
upper=K*i;
lower=K*(i-1);
sigma_DeltaB[i]=Y[upper]-Y[lower]+ (a/K)* sum(Y[lower:upper])
}
out=sigma_DeltaB
}


CAR_MC<-function(N=100,K=1000,M=100,mu=1/2,a=0.9,eta=1,sigma=1)
{
  Z <- rnorm(N*K+1,mu/K,eta/sqrt(K));
  Y0 <- rnorm(1,mean=mu*sigma/a, sd=sigma*eta/sqrt(2*a));
  BM=Driving_Process_BM(N=N,K=K,noise=Z);
  Y<-CAR1(a=a,sigma=sigma,N=N,K=K,noise=Z,Y0=Y0);
  Time_Scale=seq(0,N,by=1/K);
  par(mfrow=c(1,1));
  plot(Time_Scale,BM,type="l",main="Brownian Motion Driving Process");
  readline(prompt = "Press to continue...");
```

```
 plot(Time_Scale,Y,type="l",main="CAR");
     Y_star=Y-mean(Y);
     upper=N*K;
     lower=(N*K)+1
     a_hat=sum(Y_star[1:upper]*Y_star[2:lower])/(sum(Y_star^2));
     sigma_DeltaB=Estimated_increments(N=N,K=K,Y0=Y0,Y=Y, a=a_hat);
     mu_sigma=mean(sigma_DeltaB);
     sigma_sq=var(sigma_DeltaB);
     AV=(Y_star[2:lower])^2-exp(-1*a_hat)*Y_star[1:upper]*Y_star[2:lower];
     sigmaD_sq= mean(AV);
     simga_sq_D=(2*a_hat*sigmaD_sq)/(1-exp(-2*a_hat));
     print(c(a_hat,mu_sigma,sigma_sq,simga_sq_D));
     out=Y
}
```

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