Sudoku Puzzles and Mathematical Expressions

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1 Background

A **Latin square** is a $n \times n$ array composed of n different symbols, each only occurring once in each row and once in each column. A completed Sudoku grid is a type of Latin square, having the additional property of no repeated values in any of the 9 blocks of contiguous 3×3 cells.

The number of 9×9 Sudoku solution grids is

$$6,670,903,752,021,072,936,960.$$

Though when symmetries are taken into account, this number is much smaller at 5,472,730,538.

The maximum number of given **cells** that does not render a unique solution is 77 (four short of a full **grid**). The fewest number of givens that produces a unique solution was proven to be 17 givens in January 2012.

In Sudoku a **minimal puzzle** is one in which no clue can be deleted without losing uniqueness of the solution. The number of minimal 9×9 puzzles is not precisely known. However, statisticians have determined that (with 0.065% relative error) there are 3.10×10^{37} minimal puzzles.

2 Introduction

I am sure most of you are familiar with the **Sudoku** puzzle. The idea of the puzzle is rather simple; you are faced with a 9×9 gird, which is divided into nine 3×3 blocks.

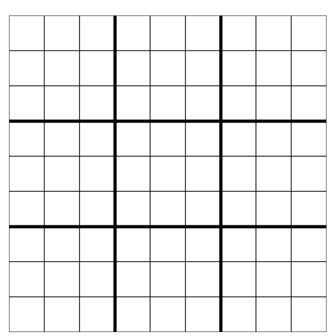

Figure 1: A blank Sudoku grid.

The puzzle will be given to the solver with some squares filled with digits $1 - 9$. The solver must complete the grid by placing a digit in every box, such that each row, column and 3×3 box contains each of the digits $1 - 9$ exactly once.

1	2	3	4	5	6	7	8	9
4	5	6						
7	8	9						
2								
3								
5								
6								
8								
9								

Figure 2: A solved Sudoku puzzle.

This puzzle is similar to the age old **Latin square**. A Latin square of order n is simply an $n \times n$ square containing each of the digits $1, \ldots, n$ in every row and column.

Latin squares are heavily related to set theory, more specifically particular subgroups and their cosets. For a Latin square of order n, there are n^2 elements of the square, i.e. $|\mathbb{Z}_n \times \mathbb{Z}_n|$ elements. Also, as every row and column contains every digit from 1 to n, we care about the subgroups $\{0\} \times \mathbb{Z}_n$ and $\mathbb{Z}_n \times \{0\}$. The subgroup $\{0\} \times \mathbb{Z}_n$ and its cosets (e.g. $\{n-1\} \times \mathbb{Z}_n$) indicate the columns of this Latin square and the subgroup $\mathbb{Z}_n \times \{0\}$ and its cosets indicate the rows of the square.

(0,0)	(0,3)	(0,6)	(0,1)	(0,4)	(0,7)	(0,2)	(0,5)	(0,8)
(3,0)	(3,3)	(3,6)						
(6,0)	(6,3)	(6,6)						
(1,0)								
(4,0)								
(7,0)								
(2,0)								
(5,0)								
(8,0)								

Figure 3: A Latin square of order 9.

When we are interested in a Sudoku puzzle we are concerned with one more subgroup $\langle 3 \rangle \times \langle 3 \rangle$, which corresponds to the first of the 3×3 blocks in any puzzle.

(0,0)	(0,3)	(0,6)
(3,0)	(3,3)	(3,6)
(6,0)	(6,3)	(6,6)

Figure 4: A Sudoku box with elements from $\langle 3 \rangle \times \langle 3 \rangle$.

3 Sudoku Puzzles and Group Theory

The main motivation for the parallels between Sudoku puzzles and group theory is labeling. Each individual square is labeled with an element from $\mathbb{Z}_9 \times \mathbb{Z}_9$ as shown below in Figure 5.

(0,0)	(0,3)	(0,6)	(0,1)	(0,4)	(0,7)	(0,2)	(0,5)	(0,8)
(3,0)	(3,3)	(3,6)						
(6,0)	(6,3)	(6,6)						
(1,0)								
(4,0)								
(7,0)								
(2,0)								
(5,0)								
(8,0)								

Figure 5: Sudoku grid with elements from $\mathbb{Z}_9 \times \mathbb{Z}_9$

Note that each element of the subgroup $\mathbb{Z}_9 \times \{0\}$ is in the first column. Also every element of $\{0\} \times \mathbb{Z}_9$ is in the first row and every element of $\langle 3 \rangle \times \langle 3 \rangle$ is in the first 3×3 box. Note that the coset of the subgroup $\mathbb{Z}_9 \times \{0\}$, $\mathbb{Z}_9 \times \{2\}$, is completely represented in the fourth row of the Sudoku grid. Furthermore, the coset of the subgroup $\langle 3 \rangle \times \langle 3 \rangle$, $\langle 3 \rangle \times \{1, 4, 7\}$, is completely represented in the topmost middle box of Figure 5.

This labeling simplifies the problem drastically. It is now obvious that any function that maps the row, column and box subgroups must be a bijection. For example, any function on the first row must contain all elements of $\mathbb{Z}_9 \times \{0\}$, meaning our function is surjective. However, each element needs to appear exactly once, meaning each image is unique, so our function is also injective.

Dealing with a grid that contains 81 elements can be tricky, so there is some further simplification that we can do. Note that each box is 3 by 3 cells and the entire grid is 3 boxes by 3 boxes. In other words, each box is 3^2 cells and each Sudoku grid is $(3^2)^2$ cells. Therefore we will construct a smaller similar puzzle which has 2^2 cells per box and $(2^2)^2$ cells per grid, this puzzle is usually referred to as a **Shidoku** puzzle, and is shown below.

Figure 6: A blank Shidoku grid.

4 The Total Number of Sudoku Puzzles

Now that we have finally simplified the Sudoku puzzle enough we can move on to actual results. The first of which is the total number of possible Sudoku grids. This is a popular problem and one that has been solved in many ways. Today I will be showing you what is believed to be the first method, which is possibly the most direct.

If you were curious, the total number of 9×9 Sudoku solution grids is 6,670,903,752,021,072,936,960. Though when symmetries are taken into account, this number is much smaller at 5,472,730,538. However, this is still quite a large number!

To begin tackling this problem, we will discover the total number of Shidoku 4×4 grids.

4.1 The 4×4 Case

We begin by constructing the first column of the 4×4 grid.

1			
2			
3			
4			

Figure 7: Step 1

As we said before, any remapping or reordering of these 4 numbers has to be a bijection, so the total number of ways we could fill out the first column is 4!. Now we need to place the two remaining digits in the first box: 3 and 4.

1	3		
2	4		
3			
4			

Figure 8: Step 2

As we have already placed the 1 and 2, it is obvious to see that there are only two ways to place the 3 and 4. To complete the first column we now need to place a 2 and 4. Again there are only two ways in which we can place these 2 numbers.

1	3	2	4
2	4		
3			
4			

Figure 9: Step 3

From here we can see that there are three 4's, so we can automatically place the last one.

1	3	2	4
2	4		
3		4	
4			

Figure 10: Step 4

At this point the more accomplished Sudoku players among you could tell that the bottom right cell could possibly be 1, 2 or 3. I ask that those others take this on faith. Once we place a 2 or 3 a unique solution to the grid is easily found.

1	3	2	4
2	4		
3		4	
4			2

1	3	2	4
2	4	1	3
3	2	4	1
4	1	3	2

Figure 11: Solution with a 2 placed.

When we place 1 the answer is less obvious, but still unique.

1	3	2	4
2	4		
3		4	
4			1

1	3	2	4
2	4	1	3
3	1	4	2
4	2	3	1

Figure 12: Solution with a 4 placed.

Now we're basically done! Throughout this process we tallied the number of total permutations to get all unique solutions, excluding symmetry: 4! ways of placing the first row, 2 ways of finishing the first box, 2 ways of finishing the first column, and finally 3 final ways of coming to a solution. Therefore the total number of 4×4 "Sudoku" grids is 288.

5 The Minimal Puzzle

In Sudoku a **minimal puzzle** is one in which no clue can be deleted without losing uniqueness of the solution. The number of minimal 9×9 puzzles is not precisely known. However, statisticians have determined that (with 0.065% relative error) there are 3.10×10^{37} minimal puzzles.

The fewest number of givens that produces a unique solution was proven to be 17 in January 2012. Minimal puzzles with 17 givens had been constructed for years before this, but the proof that disproved any minimal puzzle of 16 or fewer givens took a while to construct. In the opposite vein, the maximum number of given cells that does not render a unique solution is 77 (four short of a full grid). An example is given below.

3	4			7	6	5	9	8
7	8	9	3	4	5	1	2	6
5	6			8	9	3	4	7
4	1	3	7	6	2	8	5	9
6	2	5	8	9	1	7	3	4
8	9	7	4	5	3	6	1	2
9	7	8	5	3	4	2	6	1
2	5	6	9	1	8	4	7	3
1	3	4	6	2	7	9	8	5

Figure 13: A Sudoku puzzle with two possible solutions.

To discuss the construction of a minimal puzzle and the proof that determines the total number of minimal puzzles, we will again use the Shidoku puzzle.

5.1 Constructing a 4×4 Minimal Puzzle

To construct a minimal Shidoku puzzle, we want to affect as many possible cells as possible, with the fewest number of givens. To begin we will place a 1 in the top left corner and color those squares that can no longer contain a 1.

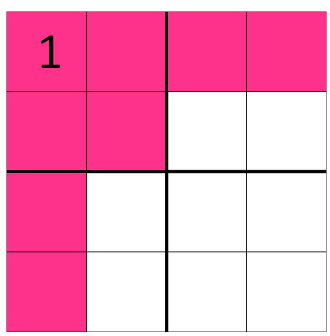

Figure 14: Step 1

We then place a 2 in the bottom right corner and again color all cells that can no longer contain a 2.

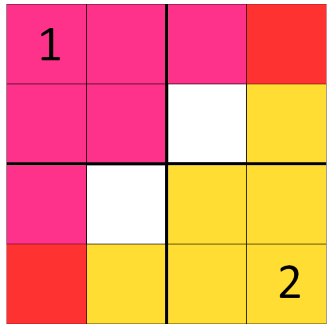

Figure 15: Step 2

Now there are only two unaffected cells. If we place a 3 into one of these cells we will uniquely determine some cells, but not all of them.

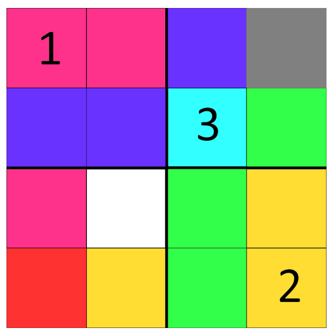

Figure 16: Step 3

Finally, when we place a 4 in the last unaffected cell, we get a unique solution.

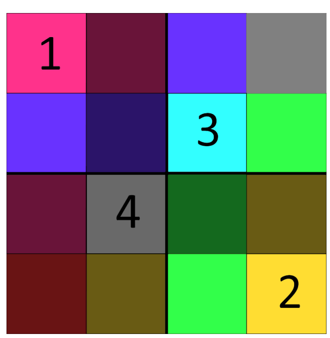

Figure 17: Step 4

1	3	2	4
4	2	3	1
2	4	1	3
3	1	4	2

Figure 18: The solution to the minimal puzzle.

It is quite easy to see that this puzzle has a unique solution, even without the provided solved grid. Once the 4 is placed, the 4 midmost cells are solved, along with the top-right and bottom-left cells. From there it is simple to fill in the rest of the grid.

6 Graph Theory

A Sudoku puzzle can very easily be represented as a graph. Graph theory can then be used to construct a lot of interesting results about Sudoku puzzles. One of the main results we are interested in is the total number of Sudoku puzzles and to compute this graph colorings are majorly important. All of the material in this chapter is standard and can be found in article **??** and textbook **??**.

6.1 Definitions

Definition 6.1. *A* ***simple graph*** *G is a set of elements called* ***vertices****, denoted $V(G)$, together with a collection of unordered pairs of vertices called edges, denoted by $E(G)$, that meets the following condition.*

$$E(G) \subseteq \{\{u, v\} \mid u, v \in V(G), u \neq v\}.$$

For simplicity we will refer to the unordered pair $\{u, v\}$ as uv or vu in the remainder of this article. Also we will use the term **graph** to abbreviate simple graph.

The order of a graph is equal to $V(G)$, the total number of vertices; the **size** is equal to $E(G)$, the number of edges in the graph. If u and v are two vertices of a graph and if the unordered pair uv is an edge, denoted by e, then we say that e **joins** u and v. This also means that the vertices u and v are **adjacent**.

A graph can be represented simply, using dots to represent vertices and lines (curved or straight), between unordered pairs of verteices, to represent edges. An example is below in Figure 19.

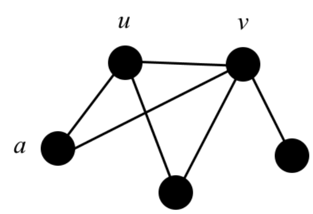

Figure 19: A visual illustration of a graph.

Definition 6.2. *The* ***neighborhood*** *of a vertex v, denoted by $N(v)$, is the collection of vertices which are adjacent to v. This is written as $N(v) = \{u \in V(G) \mid uv \in E(G)\}$.*

For example the neighborhood of a is $\{u, v\}$, or $N(a) = \{u, v\}$. If we want to talk about the size of $N(v)$, or its number of elements, we are referring to the **degree** of v. When all of the vertices have the same degree, the graph is considered **regular**.

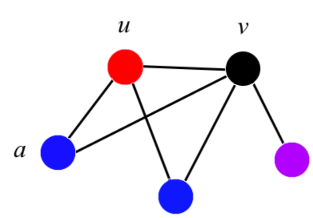

Figure 20: A depiction of graph coloring.

We now define what a graph coloring is. A λ **coloring** of a graph G is a function from G to the "colors" $\{1, 2, \ldots, \lambda\}$. We call this function a **proper coloring** if $f(u) \neq f(v)$ whenever u and v are adjacent in G. Figure 20 is an example of a proper coloring as whenever two vertices are adjacent they have a different color.

Definition 6.3. *The total number of ways one can properly color a graph G with λ colors is denoted $C_G(\lambda)$.*

This is where we begin to see the connections to Sudoku puzzles. Notice that if we look at a Sudoku grid as a graph then the following facts are true.

1. All cells in a box are adjacent.
2. All cells in a row are adjacent.
3. All cells in a column are adjacent.
4. The colors on a Sudoku puzzle are the numbers $\{1, 2, 3, \ldots, 9\}$.
5. A proper coloring of a Sudoku grid is a completed Sudoku puzzle.

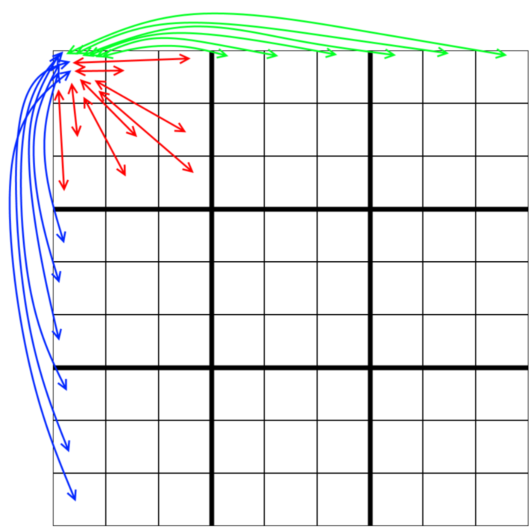

Figure 21: Any cell in a Sudoku grid is adjacent to 14 other cells.

6.2 Lemmata

We will now discuss some lemmata important to graph colorings. It is first important to realize that the total number of ways to color a graph can be equal to coloring combinations of the original graph, modified by an additional or subtracted edge, or by identifying two vertices.

Lemma 6.1. *Let G be a graph and let u and v be non-adjacent vertices in G. Then the number of proper λ-colorings of G that give u and v the same color is equal to $C_{G_{\cdot uv}}(\lambda)$.*

Proof. Let $w \in G_{\cdot uv}$ be the vertex that results in the identification of u and v. Let A be the set of proper λ-colorings of G that give u and v the same color. Let B be the set of proper λ-colorings of $G_{\cdot uv}$. Define $\alpha : A \to B$ by $\alpha(f) = f_\alpha$ where

$$f_\alpha = \begin{cases} f(x) & \text{if } x \in V(G_{\cdot uv}) \setminus \{w\}, \\ f(u) & \text{if } x = w. \end{cases}$$

Clearly, $f_\alpha : V(G_{\cdot uv}) \to \{1, 2, \ldots, \lambda\}$. Moreover, if x, y are adjacent vertices of $G_{\cdot uv}$ and $w \notin \{x, y\}$, then they are adjacent vertices of G, and so $f_\alpha(x) = f(x) \neq f(y) = f_\alpha(y)$. Also, if w is adjacent to a vertex x, then x is adjacent to either u or v in G, which implies that $f_\alpha(x) = f(x) \neq f(u) = f_\alpha(w)$. Thus, α is a well defined function from A to B, since each coloring of G will determine a unique coloring of $G_{\cdot uv}$. We show that α is one-to-one and onto.

To show that α is 1-1, let f_1 and f_2 be two different elements of A. Then for some $x \in V(G)$, $f_1(x) \neq f_2(x)$. There are two cases:

Case 1: $x \neq u$ and $x \neq v$. Then $x \in V(G_{\cdot uv}) \setminus \{w\}$. Then

$$(f_1)_\alpha(x) = f_1(x) \neq f_2(x) = (f_2)_\alpha(x).$$

Hence

$$(f_1)_\alpha(x) = (f_2)\alpha(x),$$

which implies that $(f_1)_\alpha \neq (f_2)_\alpha$.

Case 2: $x = u$ or $x = v$. Then $f_1(x) = f_1(u)$ and $f_2(x) = f_2(u)$. Now

$$(f_1)_\alpha(w) = f_1(u) = f_1(x) \neq f_2(x) = f_2(u) = (f_2)_\alpha(w).$$

Hence

$$(f_1)_\alpha \neq (f_2)_\alpha.$$

So α is 1-1 from A to B.

To show α is onto, let $g \in B$. Define f by

$$f(x) \begin{cases} g(x) & \text{if } x \in V(G) \setminus \{u, v\}, \\ g(w) & \text{if } x = u \text{ or } x = v. \end{cases}$$

First we show that $f \in A$. Clearly $f : V(G) \to \{1, 2, \ldots, \lambda\}$ and $f(u) = f(v)$, so we only need to show that f is a proper coloring.

Suppose $x, y \in V(G)$ and x, y are adjacent. Note that since u and v are non-adjacent, $\{u, v\} \neq \{x, y\}$. Again we have two cases:

Case 1: $x, y \in V(G) \setminus \{u, v\}$. Then $f(x) = g(x) \neq g(y) = f(y)$. So x and y are given different colors by the function f.

Case 2: $\{x, y\} \cap \{u, v\} \neq \emptyset$. Then by our previous remark, only one of x or y is an element of $\{u, v\}$. WOLOG, we let $x \in \{u, v\}$ and $y \in V(G) \setminus \{u, v\}$. Since x and y are adjacent in G, it must be that w and y are adjacent in $G_{\cdot uv}$. Hence $f(x) = g(w) \neq g(y) = f(y)$. So x and y are given different colors by the function f.

So if f is a function that, using λ colors, properly colors vertices in G with the stipulation that u and v are given the same color. Hence $f \in A$.

Now we show that $f_\alpha(x) = g(x)$. By definition,

$$f_\alpha(x) \begin{cases} f(x) = g(x) & \text{if } x \in V(G_{\cdot uv}) \setminus \{w\}, \\ f(u) = g(w) & \text{if } x = w. \end{cases}$$

So $f_\alpha(x) = g(x)$ for all $x \in V(G_{\cdot uv})$. Hence α maps A onto B. Since α is both 1-1 and onto, the order of A is equal to the order of B. □

Lemma 6.2. *Let G be a graph and let u and v be distinct vertices in G. Then $C_{G_{+uv}}(\lambda)$ is equal to the number of proper λ-colorings of G which give u and v different colors.*

Proof. Let A be the set of proper λ-colorings of G such that u and v receive different colors. Let B be the set of proper λ-colorings of G_{+uv}.

We define $\alpha : A \to B$ by $\alpha(f) = f_\alpha$, where for each $x \in V(G_{+uv})$ we have $f_\alpha(x) = f(x)$. Then α is a function from A to B, since each proper λ coloring of G that assigns different colors to u and v will determine a unique proper λ-coloring of G_{+uv}. We must show that α is 1-1 and onto.

For 1-1, let f_1 and f_2 be two separate elements of A. Then for some $x \in V(G)$, $f_1(x) \neq f_2(x)$. But then $(f_1)_\alpha(x) = f_1(x) \neq f_2(x) = (f_2)_\alpha(x)$. Hence α is 1-1 from A to B.

For onto, let $g \in B$. Define f by $f(x) = g(x)$. Then $f_\alpha(x) = f(x) = g(x)$. So $(f(x)) = g(x)$ for all $x \in V(G_{+uv})$. Therefore α is onto. Since α is both 1-1 and onto, the order of A is equal to the order of B. □

Lemma 6.3. *If u and v are non-adjacent vertices in a graph G, then*

$$C_G(\lambda) = C_{G_{+uv}}(\lambda) + C_{G \cdot uv}(\lambda).$$

Proof. In any proper coloring of the graph G that uses λ colors, there are two distinct possibilities. Either u and v will have the same color, or they will have different colors. By Lemma 6.1 the number of ways to color G giving u and v the same color is equal to $C_{G \cdot uv}(\lambda)$. By Lemma 6.2 the number of ways to color G giving u and v different colors is equal to $C_{G_{+uv}}(\lambda)$. Hence $C_G(\lambda) = C_{G_{+uv}}(\lambda) + C_{G \cdot uv}(\lambda)$. □

Lemma 6.4. *If u and v are adjacent vertices in a graph G, then*

$$C_G(\lambda) = C_{G_{-uv}}(\lambda) - C_{G \cdot uv}(\lambda).$$

Proof. Since u and v are adjacent, any proper coloring of G must assign different colors to u and v. Now, in any coloring of $C_{G_{-uv}}(\lambda)$, u and v may have different colors, or they may have the same color. But by Lemma 6.1, $C_{(G_{-uv}) \cdot uv}(\lambda)$ is equal to the number of ways to properly color $C_{G_{-uv}}$, with the stipulation that u and v be given the same color. We must subtract these possibilities so $C_G(\lambda) = C_{G_{-uv}}(\lambda) - C_{(G_{-uv}) \cdot uv}(\lambda)$. Since $C_{(G_{-uv}) \cdot uv}(\lambda) = C_{G \cdot uv}(\lambda)$ we have $C_G(\lambda) = C_{G_{-uv}}(\lambda) - C_{G \cdot uv}(\lambda)$. □

7 Polynomials

As we have mentioned, but have not yet shown, the number of ways one can fill out a Sudoku puzzle is the same as the number of proper colorings of a corresponding graph. Hence we are interested in how to determine the number of ways to properly color a graph with λ colors, and hence the number of ways to fill out a Sudoku puzzle is equal to a monic polynomial evaluated at λ. In this chapter we will develop and apply these ideas.

Definition 7.1. *A (complex or real)* ***polynomial*** *of x is a function of the form*

$$p(x) = \sum_{i=1}^{\infty} a_i x^i,$$

where only finitely many of the a_i are nonzero (and each a_i is complex or real, alternatively). The a_i are called the ***coefficients*** *of the polynomial.*

Note that the above definition implies that a polynomial $p(x)$ can be written in the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

which we will do from now on.

Definition 7.2. *A polynomial $p(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0$, is the* ***zero polynomial*** *if each of the a_i are zero; in other words $p(x) = 0$. If $p(x)$ is a nonzero polynomia, then its* ***degree*** *is n if $a_n \neq 0$ and $a_i = 0$ for all $i \geq n$.*

We will need some more definitions:

Definition 7.3. *A polynomial with degree n is* ***monic*** *if and only if $a_n = 1$.*

Definition 7.4. *Let $p(x)$ be a polynomial. The number x_0 is a* ***root*** *of $p(x)$ if $p(x_0) = 0$.*

Theorem 7.1 (Fundamental Theorem of Albegra)**.** *Let $p(x)$ be a non-zero polynomial of degree n with complex coefficients. Then $p(x)$ has n roots, when repeated roots are counted up to their multiplicity.* **??**

Corollary 7.1.1. *Let $P(x)$ and $Q(x)$ be two monic polynomials, and assume that there exists an integer m such that $P(\lambda) = Q(\lambda)$ for all integers λ with $\lambda \geq m$. Then $P(x) = Q(x)$.*

Proof. Assume there exits $Q(x)$ which equals $P(x)$ for all $\lambda \geq m$. Assume that the maximum of the degreees of $P(x)$ and $Q(x)$ is n. Then $(P - Q)(x)$ is a polynomial of degree $\leq n$ with an infinite number of zero roots. This contradicts the Fundamental Theorem of Algebra. □

Later we will make use of the following Lemma:

Lemma 7.2. *Let $p(x)$ be a nonzero polynomial of degree n with integer coefficients and a be an integer root of $p(x)$. Then $p(x) = (x - a)q(x)$, where $q(x)$ is a polynomial of degree $n - 1$ and has integer coefficients.*

Proof. Let □

8 Exploring Additional Problems

8.1 Intro

A Sudoku "puzzle" traditionally demands that there only exist a single unique solution to the given grid. This is why so much of the documentation surrounding Sudoku focuses on minimal puzzles. Those who desire to create an arbitrary Sudoku puzzle constrain themselves by only focusing on those partially colored Sudoku grids that have a unique proper coloring.

In this section we will explore Sudoku grids that do not have unique solutions.

8.2 How many solutions can a Sudoku grid have?

More specifically we will be determining for what values of k does a certain partially filled Sudoku grid have k possible solutions, which will be labeled the set K. However, in the interest of simplicity we will begin by answering this question for Shidoku grids, instead of Sudoku grids.

As was shown previously, there are 288 possible Shidoku grids. It follows easily that a completely blank Shidoku grid will have 288 possible solutions.

Now let us fill in just one cell of a Shidoku grid with a (see Figure 22).

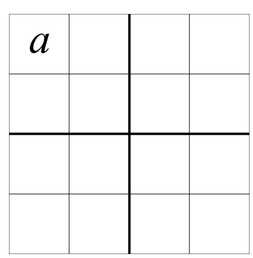

Figure 22: A Shidoku grid with one cell filled.

A proper coloring of a Shidoku grid only has four colors, $\{1, 2, 3, 4\}$. As it is possible for a to be any of these four colors, once we have assigned a value to a, three quarters of the possible solutions to the grid are no longer valid. This means that the second largest k value is $288 \div 4 = 72$.

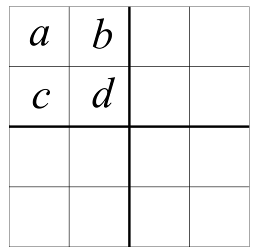

Figure 23: A Shidoku grid with one box filled.

If we were to fill in just a single box of a Shidoku grid, the number of solutions to that grid would be reduced to 12. Looking at Figure 23, it is obvious that, if we fill the cells in alphabetical order, cell a can possibly be one of four colors; cell b can then be one of three colors; cell c could be one of two remaining colors and cell d will then be determined by the other three cells. Therefore there are $4 \times 3 \times 2 \times 1 = 24$ possible Shidoku grids resembling Figure 23. All of these possible values of a, b, c and d create valid Shidoku grids, so to find the total number of possible solutions to Figure 23, we divide the total number of possible Shidoku grids (288) by the possible configurations of Figure 23 (24), to get 12 possible solutions.

One might imagine then that the only possible values of k are divisors of 288, so that K could possibly be

$$\{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 32, 36, 48, 72, 96, 144, 288\}.$$

We already know that these sets are not equal however, as there cannot be any puzzle with 96 or 144 solutions. We will now try to determine what other values of k are possible from the divisors of 288.

8.3 Values of k that divide 288

The puzzles found below were found through trial and error. The author will do his best to explain how each puzzle has the number of solutions claimed in the most simple terms possible.

The figure below, Figure 24, has two possible solutions. To see this look at the cell labeled a. The cell a can only be two possible colors to yield a proper coloring: $\{2, 4\}$. Once we have colored a, the only cell remaining in the column will be determined by a. For example, if a is colored 2, then the last cell in the column will be 4. Clearly, the two remaining cells will then be the last cells in a row and will be uniquely determined by the other three cells also in that row. Again, if cell a is colored 2, then the final cell in the row will be colored 4. This means that once a is determined, there is a unique solution to Figure 24, so Figure 24 has only as many solutions as a can be colored, namely 2.

1	2	4	3
3	4	2	1
$\underline{a}$	3	1	
	1	3	

Figure 24: A Shidoku grid with two solutions.

1	2		
3	4		
		1	
			$\underline{a}$

Figure 25: A Shidoku grid with three solutions.

Figure 25 has only three solutions. From our proof of the total number of Shidoku puzzles, we showed that once a has been colored, there exists a unique solution for Figure 25. Therefore, Figure 25 has 3 solutions, because a can only be colored in 3 ways: $\{2, 3, 4\}$.

1	2	4	3
3	4	2	1
$\underline{a}$	$\underline{b}$		

Figure 26: A Shidoku grid with four solutions.

Figure 26 is similar to Figure 24, however the two middle columns are not completely determined. Therefore, to find the total number of solutions to Figure 26, we will solve it to the point that it resembles Figure 24. To do this we will need to completely color the middle two columns. Clearly cell b can only possibly be colored $\{1, 3\}$. Once cell b is colored the other cell in its column is determined as well. Now we will try to determine the second column. Again it is clear that the remaining cells have to be colored with 1 and 3. However, since we have already colored cell b with either a 1 or 3, then the cell to the right of it will be uniquely determined by cell b. For example if b is colored 3, then the cell of the right of it could originally be either 1 or 3, but since b is in the same row, that cell can no longer be the same color as b, so it has to be a 1. We have now colored all cells in the middle two columns and Figure 26 is identical to Figure 24. This means that we have two possible solutions once Figure 26 has been filled out this way. Meaning in total Figure 26 has $2 \times 2 = 4$ solutions, as cells a, b each have two possible colorings.

1	$\underline{a}$		
2			
		1	
			$\underline{b}$

Figure 27: A Shidoku grid with six solutions.

Figure 27 Is similar to Figure 25. Note that if we color cell a 3 and then complete its box, Figure 27 becomes identical to Figure 25. This means that Figure 27 has at least three solutions. Note that if cell a is colored 3, this is another valid coloring and is not difficult to check that this also generates another three proper colorings for Figure 27. This means that Figure 27 has six possible solutions as each possible coloring of a has three valid colorings for b that produce a unique proper coloring of the graph.

1	<u>a</u>		3
3			1
<u>b</u>	<u>c</u>		

Figure 28: A Shidoku grid with eight solutions.

Note that coloring cell a, of Figure 28, with 3 will determine the rest of the cells in the top two boxes and make Figure 28 identical to Figure 26; meaning it has at least four solutions. If we were color cell $a_2 8$ with 4, this would also allow us to color the rest of the cells in the top two boxes. Then using the same reasoning as we did with Figure 26, this renders another four possible solutions for Figure 28. In total Figure 28 has eight solutions, four for each of the two possible colorings of cell a. In other words, cell a has two possible colorings, which yields two possible colorings for cell c, which in turn yields another two possibilities for cell c, so there are $2 \times 2 \times 2 = 8$ total possible proper colorings for Figure 28.

The partially colored grid that has nine solutions is more complicated and the explanation will be shortened for brevity.

	1	2	
	3		

Figure 29: A Shidoku grid with nine solutoins.

ghg

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.

.

Other possibilities??

.

.

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While all puzzles we constructed previously had k solutions where $k \mid 288$, it is possible to construct puzzles with k solutions, when $k \nmid 288$.

8.4 Values of k that do not divide 288

.... As of yet I don't have a proof/reason for why these exist. Wanted to explain how its possible but for now I'll just give the examples....

		1	
	1	2	$\underline{a}$
	3	4	
		3	

Figure 30: A Shidoku grid with five solutions.

In Figure 30 we have a partial coloring of a graph that yields five possible proper colorings.

To see this we will begin by coloring cell a_{30} with 3, to produce Figure 31a. We will then complete all the rows and columns and boxes that we can (See Figure 31b). Note that three cells have been colored 3, so we can determine the final cell that will be colored 3. Once that has been completed we can complete the graph until it resembles Figure 24 (see Figure 31c). This means that if we color cell a_{30} with 3, then Figure 30 has two possible proper colorings.

		1	
	1	2	$\underline{3}$
	3	4	
		3	

(a) Colored a 3

		1	4
4	1	2	$\underline{3}$
	3	4	
		3	

(b) Step 1

3	4	1	4
4	1	2	$\underline{3}$
	3	4	
	4	3	

(c) Step 2

		1	
	1	2	$\underline{4}$
	3	4	
		3	

(d) Colored a 4

		1	3
3	1	2	$\underline{4}$
	3	4	
		3	

(e) Step 1

Figure 31: Solutions to Figure 30.

Next we color cell a_{30} with 4 to produce Figure 31d. Once this has been done we can solve the puzzle up to Figure 31e. Note that this figure resembles Figure 25, and by the same logic we used earlier, Figure 31e has three possible proper colorings.

To sum up, if we color cell a_{30} with 3, then the graph has three solutions and if we color it with four, then the graph has three solutions. Therefore, the graph has five possible solutions.

			$\underline{a}$
	1	2	
	3	4	

Figure 32: A Shidoku grid with seven solutions.

It is also possible to partially color a Shidoku grid so that it has seven possible proper colorings (see Figure 32). The explanation of this fact will be separated into three parts. Three parts for each of the three possible colorings of cell a: $\{1,3,4\}$.

			1
	1	2	
	3	4	

(a) Color a with 1

			1
	1	2	
1	3	4	
		1	

(b) Step 1

		3	1
3	1	2	4
1	3	4	2
		1	

(c) Step 2

			3
	1	2	
	3	4	

(d) Color a with 3

			3
3	1	2	
	3	4	
		3	

(e) Step 1

		1	3
3	1	2	4
	3	4	
		3	

(f) Step 2

			a
	1	2	
	3	4	

(g) Color a with 4

			4
4	1	2	
	3	4	
	4		

(h) Step 1

3	2	1	4
4	1	2	3
	3	4	
	4	3	

(i) Step 2

Figure 33: Solutions to Figure 32

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