Exploring Fibonacci Numbers

Jessica Shatkin

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1 Abstract

This paper will illustrate a multitude of properties involving the Fibonacci and Lucas numbers. In an attempt to cover an array of different properties, this paper will include concepts from Calculus, Linear Algebra, and Number Theory. It will also include three distinct derivations of the Binet form for the nth Fibonacci number. To begin, a brief discussion of historical background is offered.

2 Historical Background

Fibonacci was born around 1170 in Pisa, Italy. He is known throughout history by a few different nomenclatures, Leonardo Bonacci, Leonardo Pisano, Leonardo of Pisa, and most commonly Fibonacci. Below is an image of Fibonacci.

At the time of his birth, Pisa was a hub of trade and commerce. Fibonacci's father, Guilielmo was a merchant and, as such, travelled throughout Mediterranean lands to conduct trade. Guilielmo brought his son with him to the northern coast of Africa in order to learn skills central to being a merchant. It was there that he was taught the Hindu-Arabic numbers, which are

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 0.$$

It is important to note that in Europe at the time, they solely used Roman numerals. As Fibonacci learned how to perform calculations with these foreign numbers, he integrated them into his business practices. Throughout many years of traveling as a merchant, Fibonacci grew to believe that the Hindu-Arabic numbers demonstrated superiority over Roman numerals in many ways. Motivated by this belief, he wrote *Liber Abaci* upon returning to Pisa. In it, Fibonacci introduced the Western world to the Hindu-Arabic numbers and elaborated on methods of calculations and example problems. Chapter twelve of *Liber Abaci* includes the famous "problem of the rabbits", from which arises the **Fibonacci numbers or sequence** that he is so well-known for today. It is worth acknowledging that Fibonacci's greatest contribution to mathematics was arguably influencing the switch in Europe from Roman numerals to Hindu-Arabic numbers. Interestingly, he is most commonly known today not for that reason but for the "Fibonacci numbers", which he did not actually study in depth.

Not until over 600 years after Fibonacci's death did French mathematician, Eduard Lucas (1842-1891), take a closer look at the numbers that came from the problem of the rabbits. Lucas deemed the sequence worthy of study and named them the "Fibonacci numbers". He also played around with changing the first numbers in the sequence, and gave us the "Lucas numbers", which will be defined later in this paper [3]. Below is an image of Lucas.

3 Worldly Examples

3.1 The Problem of the Rabbits

This problem revolves around breeding rabbits. It was first introduced in Fibonacci's *Liber Abaci*. If a breeder begins with a pair of rabbits, male and female, how many pairs of rabbits will they have at the end of one year? To eliminate randomness, the problem follows three assumptions:

1. Each newborn pair, female and male, matures in one month then starts to breed.
2. A mature pair breeds at the beginning of each month, resulting in a newborn pair, female and male.
3. No rabbits die in the first year.

Starting with one newborn pair, the distribution of rabbits for each month is as follows.

Month	Newborn Pairs	Mature Pairs	Total Pairs
Jan	1	0	1
Feb	0	1	1
March	1	1	2
April	1	2	3
May	2	3	5
June	3	5	8
July	5	8	13
Aug	8	13	21
Sep	13	21	34
Oct	21	34	55
Nov	34	55	89
Dec	55	89	144

In any given month, the number of newborn pairs is equal to the number of mature pairs in the previous month, while the number of mature pairs is the number of total pairs in the previous month. Since total pairs is the sum of the newborn and mature pairs, then total pairs in a given month is the sum of the previous two total pairs. This leads us to the Fibonacci numbers [2].

3.2 Honeybees

Honeybees differ from rabbits becuase, for one, they can have either one or two parents. Explicitly, the males, or drones, have just one female parent. The females, either workers or queens, have two parents, one male and one female [6]. The Fibonacci numbers arise from the family tree of a single drone. Let the parent of the drone be "generation 1", and let the grandparents be "generation 2". Note that we are numbering the generations in the opposite way one would normally do so. Furthermore, denote the total females of a given generation n as e_n and the total males of generation n as m_n. Then the number of males and females of the first six generations of a single drone's lineage is given in the table below.

n	e_n	m_n
1	1	0
2	1	1
3	2	1
4	3	2
5	5	3
6	8	5

Since females have one parent of each sex, and males have a only one female parent, then explicitly,

$$e_{n+1} = e_n + m_n \qquad \text{and} \qquad m_{n+1} = e_n.$$

Then $m_{n+2} = e_{n+1}$, and substitution yields

$$m_{n+2} = m_{n+1} + m_n.$$

This recursive relationship is exactly that of the Fibonacci numbers [5].

3.3 Climbing a Staircase

This problem also reveals the Fibonacci numbers. The premise is a staircase with n stairs. Using either one or two stairs at a time, how many different ways are there to ascend (or descend) the stairs? Let S_n be the number of ways to climb the n stairs. Note that there are two different options to begin climbing, either climb one stair or two. After one stair, the number of different ways to climb the remaining stairs is S_{n-1}. If instead we begin with two stairs, the number of different ways to climb the remaining stairs is S_{n-2} [5]. Therefore the total number of ways to climb n stairs is given by

$$S_n = S_{n-1} + S_{n-2}.$$

This leads us to the definition of Fibonacci numbers.

4 The Recursive Definition

To define the Fibonacci sequence, let f_n denote the nth Fibonacci number. For $n \geq 0$, we have

1. $f_0 = 0,\ f_1 = 1$
2. $f_n = f_{n-1} + f_{n-2},\ n \geq 2$

Below is a table of the first 24 Fibonacci numbers.

$f_0 = 0$	$f_6 = 8$	$f_{12} = 144$	$f_{18} = 2584$
$f_1 = 1$	$f_7 = 13$	$f_{13} = 233$	$f_{19} = 4181$
$f_2 = 1$	$f_8 = 21$	$f_{14} = 377$	$f_{20} = 6765$
$f_3 = 2$	$f_9 = 34$	$f_{15} = 610$	$f_{21} = 10946$
$f_4 = 3$	$f_{10} = 55$	$f_{16} = 987$	$f_{22} = 17711$
$f_5 = 5$	$f_{11} = 89$	$f_{17} = 1597$	$f_{23} = 28657$

5 Formula for f_n: The Binet Form

Any Fibonacci number can be found by adding the two numbers that preceded it in the sequence. However, say we want to find f_{43}. It is unlikely that we know f_{42} and f_{41}, and although finding them would be doable, it would also be cumbersome. It is therefore useful to define f_n without using a recurrence relation, among other reasons.

Binet Form: Let α and β be roots of the quadratic equation $x^2 = x + 1$ such that $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Since α and β are the roots of $x^2 = x + 1$, some useful properties of α and β are:

- $\alpha\beta = -1$
- $\alpha + \beta = 1$
- $\alpha^2 = \alpha + 1$
- $\beta^2 = \beta + 1$

Furthermore,

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \alpha,$$

and α is known as **the golden ratio**. The proof of this last result is as follows.

Proof. The Binet form for f_{n+1} and f_n yields

$$\frac{f_{n+1}}{f_n} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n}.$$

Then since $|\beta| < 1$, it follows that

$$\lim_{n\to\infty} \beta^{n+1} = \lim_{n\to\infty} \beta^n = 0.$$

Therefore,

$$\lim_{n\to\infty} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} = \lim_{n\to\infty} \frac{\alpha^{n+1}}{\alpha^n} = \alpha.$$

$\square$

To see how the values α and β arise, here are three different derivations of the Binet form.

5.1 Recurrence Relation

Suppose $\{a_n\}$ is a sequence that satisfies

$$a_n = a_{n-1} + a_{n-2}, \quad n \geq 2.$$

Assume $a_n = Ar^n$, where A and r are nonzero constants. Substituting $a_i = Ar^i$, the result is

$$Ar^n = Ar^{n-1} + Ar^{n-2}.$$

Dividing by A and r^{n-2} gives

$$r^2 - r - 1 = 0.$$

Applying the quadratic formula yields

$$\begin{aligned} r &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{1 \pm \sqrt{5}}{2}. \end{aligned}$$

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

In place of $\{a_n\}$, use $\{f_n\}$. Then for constants c_1 and c_2,

$$f_n = c_1\alpha^n + c_2\beta^n, \; n \geq 0.$$

Solving for c_1 and c_2 using initial values f_0 and f_1 yields

$$\begin{aligned} 0 = f_0 =& c_1 + c_2 \\ 1 = f_1 =& c_1\alpha + c_2\beta \\ =& c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right). \end{aligned}$$

Solving for c_1 and c_2 yields $c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = \frac{-1}{\sqrt{5}}$. Therefore, f_n can be expressed by

$$f_n = \frac{1}{\sqrt{5}}\alpha^n - \frac{1}{\sqrt{5}}\beta^n = \frac{\alpha^n - \beta^n}{\sqrt{5}}.$$

This concludes one proof of the Binet form.

5.2 Matrix Multiplication

If f_n represents the nth Fibonacci number, then

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}.$$

Letting

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

it will be shown that for $n \geq 0$

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

To see this, note that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_0 \end{bmatrix} = \begin{bmatrix} f_2 \\ f_1 \end{bmatrix}.$$

Then

$$\begin{bmatrix} f_3 \\ f_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_2 \\ f_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and so on. To help find A^n for any n requires eigenvalues and eigenvectors of matrix A. The determinant of the matrix

$$\begin{bmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{bmatrix}$$

is equal to $\lambda^2 - \lambda - 1$. Setting $\lambda^2 - \lambda - 1 = 0$, it follows that the eigenvalues are $\lambda_1 = \alpha$ and $\lambda_2 = \beta$ with corresponding eigenvectors $\begin{bmatrix} \alpha \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \beta \\ 1 \end{bmatrix}$, respectively. Let

$$P = \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}.$$

Then $A = PDP^{-1}$, and therefore $A^n = PDP^{-1}PDP^{-1} \dots PDP^{-1} = PD^nP^{-1}$. Now it is possible to solve for A^n:

$$A^n = PD^nP^{-1} = \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha-\beta} & \frac{-\beta}{\alpha-\beta} \\ \frac{-1}{\alpha-\beta} & \frac{\alpha}{\alpha-\beta} \end{bmatrix}$$

$$= \left(\tfrac{1}{\alpha-\beta}\right) \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} \begin{bmatrix} 1 & -\beta \\ -1 & \alpha \end{bmatrix}$$

$$= \left(\tfrac{1}{\alpha-\beta}\right) \begin{bmatrix} \alpha^{n+1} - \beta^{n+1} & \alpha^n - \beta^n \\ \alpha^n - \beta^n & \alpha^{n-1} + \beta^{n-1} \end{bmatrix}$$

Using the formula $\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we get:

$$f_n = \left(\frac{1}{\alpha - \beta}\right)\left(\alpha^n - \beta^n\right) = \frac{\alpha^n - \beta^n}{\sqrt{5}}.$$

Thus, we can derive the Binet form using matrix multiplication.

5.3 Power Series

Definition 5.1. *Given a real number c and a sequence $\{a_k\}_{k=0}^{\infty}$ of real numbers, the expression $\sum_{k=0}^{\infty} a_k(x-c)^k$ is called a* ***power series*** *centered at c. The numbers a_k are the* ***coefficients*** *of the power series and the number c is the* ***center*** *of the power series [1].*

Define a sequence of real numbers by $a_0 = 1,\ a_1 = 1$, and $a_{k+1} = a_k + a_{k-1}$ for all $k \geq 1$. Note that $a_k = f_{k+1}$. Define a function g by $g(x) = \sum_{k=0}^{\infty} a_k x^k$ for each value of x for which the series converges. We first use induction to show that $a_k \leq 2^k$ for all k.

Proof. For a base case, $a_0 = 1$, $2^0 = 1$ and $1 \leq 1$. Also for $k = 1$, we have $a_1 = 1$, $2^1 = 2$ and $1 \leq 2$. Now assume the inequality is true for $n = 0, 1, \ldots k$. Then for $k + 1$,

$$a_{k+1} = a_k + a_{k-1} \leq 2^k + 2^{k-1} = (2+1)2^{k-1} = 3 \cdot 2^{k-1} < 2^{k+1}.$$

This shows that $a_k \leq 2^k$ for all k. □

Next will be another proof of the Binet form for f_n. It will be shown that if r and s are roots of $x^2 + x - 1$, then

$$a_k = \frac{1}{\sqrt{5}}\left(\alpha^{k+1} - \beta^{k+1}\right)$$

for each integer $k \geq 0$.

First it will be shown that

$$xg(x) + x^2 g(x) = g(x) - 1$$

.

Using the fact that $a_0 = 1 = a_1$, we have

$$\begin{aligned}
xg(x) + x^2 g(x) &= x\sum_{k=0}^{\infty} a_k x^k + x^2 \sum_{k=0}^{\infty} a_k x^k \\
&= \sum_{k=0}^{\infty} a_k x^{k+1} + \sum_{k=0}^{\infty} a_k x^{k+2} \\
&= \sum_{k=1}^{\infty} a_{k-1} x^k + \sum_{k=2}^{\infty} a_{k-2} x^k \\
&= a_0 x + \sum_{k=2}^{\infty} \Big(a_{k-1} + a_{k-2}\Big) x^k \\
&= a_1 x + \sum_{k=2}^{\infty} a_k x^k \\
&= \sum_{k=1}^{\infty} a_k x^k \\
&= g(x) - 1.
\end{aligned}$$

Then factoring $xg(x) + x^2 g(x) = g(x) - 1$ yields

$$g(x)(x + x^2 - 1) = -1.$$

Thus

$$g(x) = \frac{-1}{x^2 + x - 1}.$$

Now let r and s be the roots of $x^2 + x - 1$. We will use the quadratic formula to calculate values for r and s. The resulting roots of $x^2 + x - 1$ are

$$\frac{-1-\sqrt{5}}{2} \qquad \text{and} \qquad \frac{-1+\sqrt{5}}{2}\ .$$

Without loss of generality, let

$$r = \frac{-1-\sqrt{5}}{2}\ , \qquad s = \frac{-1+\sqrt{5}}{2}\ .$$

Note that

$$\frac{1}{r} = \frac{2}{-1-\sqrt{5}} \cdot \frac{-1+\sqrt{5}}{-1+\sqrt{5}} = \frac{1-\sqrt{5}}{2} = \beta$$

and similarly

$$\frac{1}{s} = \frac{1+\sqrt{5}}{2} = \alpha \ .$$

Using partial fractions, it will be shown that

$$g(x) = \frac{-1}{(r-x)(s-x)} = \frac{A}{r-x} + \frac{B}{s-x},$$

where A and B are constants that depend on r and s. Multiplying to get a common denominator gives

$$\frac{(s-x)A + (r-x)B}{(r-x)(s-x)} = \frac{-1}{x^2+x-1}.$$

Working with the numerator yields the system of equations

$$\begin{aligned} sA + rB &= -1 \\ -A - B &= 0 \end{aligned}$$

It follows that $A = \frac{1}{r-s}$ and $B = \frac{-1}{r-s}$.

From here, it is straight forward to solve for A and B. Since $r - s = -\sqrt{5}$, it follows that

$$A = \frac{1}{-\sqrt{5}} \qquad \text{and} \qquad B = \frac{1}{\sqrt{5}} \ .$$

Recall that the formula for the sum of a geometric series of the form $\sum\limits_{k=0}^{\infty} a_0 r^k$ is

$$S = \frac{a_0}{1-r} \ ,$$

where a_0 is the first term and r is the common ratio. Therefore $\frac{A}{r-x}$ and $\frac{B}{s-x}$ can be rewritten as geometric sums by multiplying each by 1 in a helpful way. Explicitly, for $|x| < |r|$, and $|x| < |s|$,

$$\frac{1/r}{1/r} \cdot \frac{A}{r-x} = \frac{A/r}{1-x/r} = \sum_{k=0}^{\infty} \frac{A}{r} \cdot \left(\frac{x}{r}\right)^k.$$

Similarly,

$$\frac{B}{s-x} = \sum_{k=0}^{\infty} \frac{B}{s} \cdot \left(\frac{x}{s}\right)^k.$$

It follows that

$$g(x) = \sum_{k=0}^{\infty} \frac{A}{r} \cdot \left(\frac{x}{r}\right)^k + \sum_{k=0}^{\infty} \frac{B}{s} \cdot \left(\frac{x}{s}\right)^k = \sum_{k=0}^{\infty} \left(\frac{A}{r^{k+1}} + \frac{B}{s^{k+1}}\right) x^k.$$

Then this is the Maclaurin series representation for $g(x)$. Since the Maclaurin series representation is unique, then plugging in values for A, B, r and s yields

$$a_k = \frac{1}{\sqrt{5}} \left(-\beta^{k+1} + \alpha^{k+1} \right)$$

for each integer $k \geq 0$. Since $a_k = f_{k+1}$, we see that

$$f_k = \frac{\alpha^k - \beta^k}{\sqrt{5}}.$$

This concludes our final proof of the Binet form.

6 Properties

The Fibonacci numbers have a myriad of interesting algebraic properties. Many such properties can be proven in more than one way, as with the Binet form alone. As demonstrated in this section, the recursive definition of the Fibonacci numbers lends itself well to proofs by induction.

Property 6.1. *For each positive integer n, $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.*

Proof. This will be a proof by induction [4]. To establish a base case with $n = 1$, we have $f_1 = 1$ and $f_3 - 1 = 2 - 1 = 1$. Now, we assume the equation is true for $n = 1, \ldots, k$ and prove it true for $k + 1$. Using addition and our hypothesis, we get

$$f_1 + f_2 + \cdots + f_k + f_{k+1} = f_{k+2} - 1 + f_{k+1}.$$

By the recursive definition of the Fibonacci numbers, $f_{k+1} + f_{k+2} - 1 = f_{k+3} - 1$. Therefore, it is proven that the sum of the first n Fibonacci numbers is one less then f_{n+2}. □

Property 6.2. *For each positive integer n, $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$.*

Proof. This will be a proof by induction [4]. With $n = 1$ as our base case, we see that $f_1 = 1 = f_2$. Now assume the conjecture is true for $n = 1, \ldots, k$. If $n = k + 1$, then

$$f_1 + f_3 + f_5 + \cdots + f_{2k-1} + f_{2(k+1)-1} = f_{2k} + f_{2k+1} = f_{2k+2} = f_{2(k+1)}.$$

This proves that the sum of the first n odd numbered Fibonacci numbers is f_{2n}. □

Alternatively, we can prove this property using the result from ***Property 6.1****.*

Proof. By the recursive definition of the Fibonacci numbers, and since $f_1 = 1$, then

$$\begin{aligned}
f_1 + f_3 + f_5 + \cdots + f_{2n-1} &= f_1 + (f_1 + f_2) + (f_3 + f_4) + \cdots + (f_{2n-3} + f_{2n-2}) \\
&= f_1 + f_{(2n-2)+2} - 1 \\
&= f_{2n}.
\end{aligned}$$

□

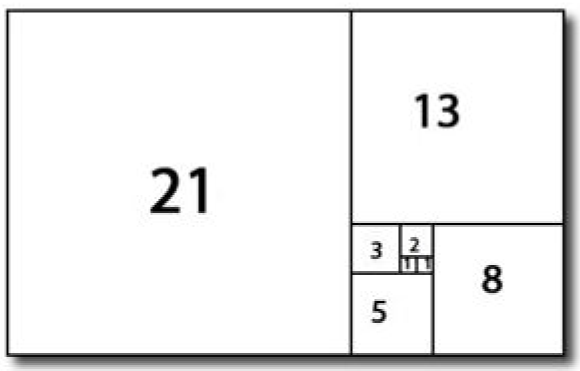
Property 6.3. *For each positive integer n, $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$.*

Proof. This will be a proof by induction [2]. When $n = 1$, $f_1^2 = 1^2 = 1 = f_1 f_2$. Now assume the equation is true for $n = 1, \ldots, k$. Note that

$$\begin{aligned} f_1^2 + \cdots + f_{k+1}^2 &= f_1^2 + \cdots + f_k^2 + f_{k+1}^2 \\ &= f_k f_{k+1} + f_{k+1}^2 \\ &= f_{k+1}(f_k + f_{k+1}) \\ &= f_{k+1} f_{k+2}. \end{aligned}$$

This proves that the sum of the squares of the first n Fibonacci numbers is equal to the product of the nth and the $(n+1)$st Fibonacci numbers. □

Below is a geometric representation of this formula, in which the numbers represent the side length of each square:



Property 6.4. *For each positive integer n, $f_1 f_2 + f_2 f_3 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$.*

Proof. This will be a proof by induction [5]. For $n = 1$, we have $f_1 f_2 = 1 = f_2^2$. Now we assume the equation to be true for $n = 1, \ldots, k$ and we shall prove it true for $k + 1$. Using the induction hypothesis,

$$\begin{aligned}
f_1f_2+\cdots+f_{2k+1}f_{2k+2} &= f_{2k}^2+f_{2k}f_{2k+1}+f_{2k+1}f_{2k+2}\\
&= f_{2k}^2+f_{2k}f_{2k+1}+f_{2k+1}(f_{2k}+f_{2k+1})\\
&= f_{2k}^2+2f_{2k}f_{2k+1}+f_{2k+1}^2\\
&= (f_{2k}+f_{2k+1})^2\\
&= f_{2k+2}^2.
\end{aligned}$$

This shows that the sum of all products of consecutive Fibonacci numbers from 1 to $2n$ is the square of the $2n$th Fibonacci number. ☐

Property 6.5. *For each positive integer n, $f_{n+1}f_{n-1}=f_n^2+(-1)^n$.*

Proof. This will be a proof by induction [5]. When $n=1$, $f_2f_0=0$ and $f_1^2+(-1)^1=1-1=0$. When $n=2$, $f_3f_1=2$ and $f_2^2+(-1)^2=1+1=2$. Assume the equation is true for $n=1,\dots,k$, where $k>2$. Now if $n=k+1$, we have

$$\begin{aligned}
f_{k+2}f_k &= (f_{k+1}+f_k)f_k\\
&= f_{k+1}f_k+f_k^2\\
&= f_{k+1}f_k+f_{k+1}f_{k-1}-(-1)^k\\
&= f_{k+1}(f_k+f_{k-1})-(-1)^k\\
&= f_{k+1}^2+(-1)^{k+1}.
\end{aligned}$$

This completes the proof. ☐

Property 6.6. *For each integer $n\geq 2$, $f_{n+2}f_{n-2}=f_n^2+(-1)^{n+1}$.*

Proof. This will be a direct proof [5]. For $n=2$,

$$f_4f_0=0=f_2^2-1.$$

Let us also look at $n=3$. We have

$$f_5f_1=5, \qquad \text{and} \qquad f_3^2+(-1)^4=2^2+1=5.$$

Now assume the equation is true for some $n \geq 3$. Using the result from **Property 6.5**,

$$\begin{aligned}
f_{(n+1)+2}f_{(n+1)-2} &= f_{n+3}f_{n-1} \\
&= (f_{n+2} + f_{n+1})(f_{n+1} - f_n) \\
&= f_{n+2}f_{n+1} - f_{n+1}f_n + f_{n+1}^2 - f_{n+2}f_n \\
&= f_{n+1}(f_{n+2} - f_n) + f_{n+1}^2 - (f_{n+1}^2 + (-1)^{n+1}) \\
&= f_{n+1}^2 + f_{n+1}^2 - f_{n+1}^2 - (-1)^{n+1} \\
&= f_{n+1}^2 + (-1)^{n+2}
\end{aligned}$$

This property will become useful later when looking at infinite series involving Fibonacci properties. □

Definition 6.1. *Greatest Common Divisor*
If a and b are integers, then the greatest common divisor of a and b is the largest positive integer that divides both a and b. It is denoted $\gcd(a, b)$.

Definition 6.2. *Relatively Prime*
If a and b are integers such that $\gcd(a, b) = 1$, then a and b are considered to be relatively prime.

Lemma 6.7. *If a and b are positive integers, and q and r are integers such that*

$$a = bq + r,$$

then

$$\gcd(a, b) = \gcd(b, r).$$

Proof. Using substitution, we have

$$\gcd(a, b) = \gcd(bq + r, b) = \gcd(r, b).$$

□

Property 6.8. *For each positive integer n,* $\gcd(f_{n-1}, f_n) = 1$*. In words, any two consecutive Fibonacci numbers are relatively prime [2].*

Proof. This will be a proof by induction on n. As a base case, note that

$$\gcd(f_0, f_1) = \gcd(0, 1) = 1.$$

Now assume the conjecture is true for k. Then since $f_{k+1} = f_k + f_{k-1}$ and using **Lemma 6.7** stated above, it follows that

$$\gcd(f_{k+1}, f_k) = \gcd(f_k, f_{k-1}) = 1.$$

This completes the proof. Therefore, every Fibonacci number is relatively prime to both its previous and subsequent Fibonacci number. This property can also be proven by contradiction. Such a proof will be left to the reader. $\square$

Property 6.9. *For each positive integer n,* $\gcd(f_n, f_{n+2}) = 1$*.*

Proof. This will be a proof by contradiction [2]. Suppose that $\gcd(f_n, f_{n+2}) = d$ such that $d > 1$. Then by the recursive definition, we have $f_{n+2} = f_{n+1} + f_n$, and since $d|f_{n+2}$ and $d|f_n$, then it follows that $d|f_{n+1}$. But this contradicts **Property 6.8**, which says that $\gcd(f_n, f_{n+1}) = 1$. Therefore, $\gcd(f_n, f_{n+2}) = 1$. $\square$

Property 6.10. *The sum of any six consecutive Fibonacci numbers is divisible by four [2]. Explicitly, for each positive integer n, $\sum\limits_{r=0}^{5} f_{n+r} = 4f_{n+4}$.*

Proof. Using our recursive definition, we see that,

$$\begin{aligned}
\sum_{r=0}^{5} f_{n+r} &= (f_n + f_{n+1}) + (f_{n+2} + f_{n+3}) + (f_{n+4} + f_{n+5}) \\
&= f_{n+2} + f_{n+4} + f_{n+4} + f_{n+5} \\
&= f_{n+2} + 2f_{n+4} + f_{n+3} + f_{n+4} \\
&= 4f_{n+4}.
\end{aligned}$$

This shows that the sum of any six consecutive Fibonacci numbers is a multiple of four. $\square$

Property 6.11. *The sum of any ten consecutive Fibonacci numbers is divisible by 11 [2]. Furthermore, for each positive integer n,* $\sum_{r=0}^{9} f_{n+r} = 11 f_{n+6}$ *.*

Proof. Similar to the previous property, we will rewrite the sum of an arbitrary ten consecutive Fibonacci numbers using the recursive definition.

$$
\begin{aligned}
\sum_{r=0}^{9} f_{n+r} &= (f_n + f_{n+1}) + (f_{n+2} + f_{n+3}) + (f_{n+4} + f_{n+5}) + f_{n+6} + f_{n+7} + f_{n+8} + f_{n+9} \\
&= f_{n+2} + f_{n+4} + f_{n+6} + f_{n+6} + (f_{n+5} + f_{n+6}) + (f_{n+6} + f_{n+7}) + (f_{n+7} + f_{n+8}) \\
&= f_{n+2} + f_{n+4} + f_{n+5} + 4f_{n+6} + 2f_{n+5} + 2f_{n+6} + f_{n+6} + f_{n+7} \\
&= f_{n+2} + f_{n+4} + 4f_{n+5} + 8f_{n+6} \\
&= f_{n+2} + 3f_{n+5} + 9f_{n+6} \\
&= f_{n+2} + 3f_{n+3} + 3f_{n+4} + 9f_{n+6} \\
&= 2f_{n+3} + 4f_{n+4} + 9f_{n+6} \\
&= 2f_{n+4} + 2f_{n+5} + 9f_{n+6} \\
&= 11f_{n+6}
\end{aligned}
$$

This shows that the sum of any ten consecutive Fibonacci numbers is divisible by 11. □

Property 6.12. *For each positive integer n,* $\sum_{i=1}^{n} (-1)^{i+1} f_i = (-1)^{n-1} f_{n-1} + 1.$

Proof. This will be a proof by induction [2]. When $n = 1$, we have $(-1)^2 f_1 = 1$ and $(-1)^0 f_0 + 1 = 1$. Now assume the equation is true for $n = 1, \ldots, k$. Then

$$
\begin{aligned}
\sum_{i=1}^{k+1} (-1)^{i+1} f_i &= \sum_{i=1}^{k} (-1)^{i+1} f_i + (-1)^{k+2} f_{k+1} \\
&= (-1)^{k-1} f_{k-1} + 1 + (-1)^k f_{k+1} \\
&= (-1)^k (f_{k+1} - f_{k-1}) + 1 \\
&= (-1)^k f_k + 1.
\end{aligned}
$$

This completes the proof. □

Property 6.13. *For $m \geq 0$ and $n > 0$, $f_{m+n} = f_{m+1}f_n + f_m f_{n-1}$.*

Proof. First we fix m, then do a proof by induction on n [4]. For $n = 1$, we have $f_{m+1}f_1 + f_m f_0 = f_{m+1}$. Now assume true for $n = 1, \ldots, k$ and prove true for $n = k + 1$. Then

$$\begin{aligned}
f_{m+1}f_{k+1} + f_m f_k &= f_{m+1}(f_k + f_{k-1}) + f_m(f_{k-1} + f_{k-2}) \\
&= (f_{m+1}f_k + f_m f_{k-1}) + (f_{m+1}f_{k-1} + f_m f_{k-2}) \\
&= f_{m+k} + f_{m+k-1} \\
&= f_{m+k+1}.
\end{aligned}$$

This completes the proof. □

Property 6.14. *For all $n \geq 0$, $\alpha^n = \alpha f_n + f_{n-1}$.*

Proof. To prove this we will use the Binet Form of f_n and the fact that $\alpha\beta = -1$ and $\alpha + \beta = 1$ [4].

$$\begin{aligned}
\alpha f_n + f_{n-1} &= \frac{\alpha(\alpha^n - \beta^n)}{\sqrt{5}} + \frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{5}} \\
&= \frac{\alpha^{n+1} + \beta^{n-1} + \alpha^{n-1} - \beta^{n-1}}{\sqrt{5}} \\
&= \frac{\alpha^{n+1} + \alpha^{n-1}}{\sqrt{5}} \\
&= \frac{\alpha^n(\alpha + \frac{1}{\alpha})}{\sqrt{5}} \\
&= \frac{\alpha^n(\alpha - \beta)}{\sqrt{5}} \\
&= \alpha^n.
\end{aligned}$$

□

Property 6.15. *For all $n \geq 0$, $\beta^n = \beta f_n + f_{n-1}$.*

Proof. This is very similar to the previous proof, and as such shall be left out [4]. □

7 Lucas Numbers

The Lucas numbers are very similar to the Fibonacci number. Like the Fibonacci numbers, any number in the sequence is the sum of the previous two. The only difference, in fact, is the starting point. The formal recursive definition follows.

Definition 7.1. *For $n \geq 0$, we have*

1. $\ell_0 = 2,\ \ell_2 = 1$
2. $\ell_n = \ell_{n-1} + \ell_{n-2},\ n \geq 2$

Below is a list of the first 24 Lucas numbers.

$\ell_0 = 2$	$\ell_6 = 18$	$\ell_{12} = 322$	$\ell_{18} = 5778$
$\ell_1 = 1$	$\ell_7 = 29$	$\ell_{13} = 521$	$\ell_{19} = 9349$
$\ell_2 = 3$	$\ell_8 = 47$	$\ell_{14} = 843$	$\ell_{20} = 15127$
$\ell_3 = 4$	$\ell_9 = 76$	$\ell_{15} = 1364$	$\ell_{21} = 24476$
$\ell_4 = 7$	$\ell_{10} = 123$	$\ell_{16} = 2207$	$\ell_{22} = 39603$
$\ell_5 = 11$	$\ell_{11} = 199$	$\ell_{17} = 3571$	$\ell_{23} = 64079$

The Lucas numbers also have a nice formula for the nth Lucas numbers. The Binet form for the Lucas numbers is given below.

Binet Form: For $n \geq 0$,

$$\ell_n = \alpha^n + \beta^n.$$

Proof. This will be a proof by induction. As a base case, let $n = 1$:

$$\ell_1 = 1 = \alpha + \beta.$$

Then assume the formula is true for $n = 1, \ldots, k$, where $k > 1$. Using the fact that

$\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$, the following holds true:

$$\begin{aligned}
\ell_{k+1} &= \ell_k + \ell_{k+1} \\
&= \alpha^k + \beta^k + \alpha^{k-1} + \beta^{k-1} \\
&= \alpha^{k-1}(\alpha + 1) + \beta^{k-1}(\beta + 1) \\
&= \alpha^{k-1}(\alpha^2) + \beta^{k-1}(\beta^2) \\
&= \alpha^{k+1} + \beta^{k+1}.
\end{aligned}$$

This shows that the nth Lucas number is the sum of α^n and β^n. □

7.1 Properties of Lucas Numbers

Property 7.1. *For each positive integer n, $\ell_1 + \ell_2 + \cdots + \ell_n = \ell_{n+2} - 3$.*

Proof. This proof will be left to the reader. It is smilar to that of **Property 6.1** [4]. □

Property 7.2. *For each positive integer n, $\ell_2 + \ell_4 + \ell_6 + \cdots + \ell_{2n} = \ell_{2n+1} - 1$ [4].*

Proof. Using **Property 7.1**, we have

$$\begin{aligned}
\ell_2 + \ell_4 + \ell_6 + \cdots + \ell_{2n} &= (\ell_0 + \ell_1) + (\ell_2 + \ell_3) + \cdots + (\ell_{2n-2} + \ell_{2n-1}) \\
&= 2 + \ell_{(2n-1)+2} - 3 \\
&= \ell_{2n+1} - 1.
\end{aligned}$$

This completes the proof. □

Property 7.3. *For each positive integer n, $\ell_1 + \ell_3 + \ell_5 + \cdots + \ell_{2n-1} = \ell_{2n-2} - 2$ [4].*

Proof. This can be proven by induction or by using **Property 7.1**. Both will be left to the reader. □

Property 7.4. *For each positive integer* n, $\ell_n^2 - 2(-1)^n = \ell_{2n}$ *[2].*

Proof. Using the Binet form of ℓ_n and the fact that $\alpha\beta = -1$, we get

$$\begin{aligned} \ell_n^2 - 2(-1)^n &= (\alpha^n + \beta^n)^2 - 2(-1)^n \\ &= \alpha^{2n} + 2\alpha^n\beta^n + \beta^{2n} - 2(-1)^n \\ &= \alpha^{2n} + 2(-1)^n + \beta^{2n} - 2(-1)^n \\ &= \alpha^{2n} + \beta^{2n} \\ &= \ell_{2n}. \end{aligned}$$

This completes the proof. □

8 Properties Involving both Fibonacci and Lucas Numbers

Below are various properties that encompass Fibonacci and Lucas numbers together.

Property 8.1. *For each positive integer* n, $f_{n+1} + f_{n-1} = \ell_n$.

Proof. This will be a proof by induction [2]. When $n = 1$, we have $f_2 + f_0 = 1 + 0 = 1$ and $\ell_1 = 1$. Now assume the equation is true for $n = 1, \ldots, k$ and prove it true for $n = k + 1$. Using the recursive definition, we get

$$\begin{aligned} f_{k+2} + f_k &= f_k + f_{k+1} + f_{k-2} + f_{k-1} \\ &= (f_k + f_{k-2}) + (f_{k+1} + f_{k-1}) \\ &= \ell_{k-1} + \ell_k \\ &= \ell_{k+1}. \end{aligned}$$

This completes the proof. Alternatively, this property can be proven using the Binet forms. Such a proof will be left to the reader. □

Property 8.2. *For each positive integer n, $\ell_{n+1} + \ell_{n-1} = 5f_n$ [2].*

Proof. Using the previous result, we find that

$$\begin{aligned}\ell_{n+1} + \ell_{n-1} &= (f_{n+2} + f_n) + (f_n + f_{n-2}) \\ &= f_n + f_{n+1} + 2f_n + f_{n-2} \\ &= 3f_n + f_n + f_{n-1} + f_{n-2} \\ &= 5f_n.\end{aligned}$$

This completes the proof. □

Property 8.3. *For each positive integer m and n, $\ell_{m+1}f_n + \ell_m f_{n-1} = \ell_{m+n}$.*

Proof. We fix m and prove this by induction on n [4]. When $n = 1$, $\ell_{m+1}f_1 + \ell_m f_0 = \ell_{m+1}$. Now assume true for $n = 1, \ldots, k$ and prove true for $n = k + 1$. Then

$$\begin{aligned}\ell_{m+1}f_{k+1} + \ell_m f_k &= \ell_{m+1}f_k + \ell_{m+1}f_{k-1} + \ell_m f_{k-1} + \ell_m f_{k-2} \\ &= \ell_{m+k} + \ell_{m+k-1} \\ &= \ell_{m+k+1}.\end{aligned}$$

This completes the proof. □

Property 8.4. *For each positive integer n, $\ell_{m+1}\ell_n + \ell_m \ell_{n-1} = 5f_{m+n}$.*

Proof. We fix m and prove by induction on n [4]. From **Property 8.2** we know that $\ell_{n+1} + \ell_{n-1} = 5f_n$. When $n = 1$, we have

$$\begin{aligned}\ell_{m+1}\ell_1 + \ell_m \ell_0 &= \ell_{m+1} + 2\ell_m \\ &= \ell_{m+2} + \ell_m \\ &= 5f_{m+1}.\end{aligned}$$

Now we assume the hypothesis is true for $n = 1, \ldots, k$ and prove it true for $n = k + 1$.

Then

$$\begin{aligned}\ell_{m+1}\ell_{k+1} + \ell_m\ell_k &= \ell_{m+1}\ell_{k-1} + \ell_{m+1}\ell_k + \ell_m\ell_{k-2} + \ell_m\ell_{k-1}\\ &= 5f_{m+k-1} + 5f_{m+k}\\ &= 5f_{m+k+1}.\end{aligned}$$

This completes the proof. □

Property 8.5. *For each positive integer n, $f_n\ell_n = f_{2n}$ [2].*

Proof. Using the Binet forms of f_n and ℓ_n, we find that

$$f_n\ell_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}(\alpha^n + \beta^n) = \frac{\alpha^{2n} - \beta^{2n}}{\sqrt{5}} = f_{2n}.$$

Thus, the product of the nth Fibonacci and n Lucas numbers is f_{2n}. □

9 Divisibility Properties and Consequences

This section will delve into interesting properties regarding divisibility. First, it will be shown that for nonnegative integers n and k,

$$f_n \text{ divides } f_{kn}.$$

Proof. It was shown in **Property 6.13** that for $m \geq 0$ and $n \geq 1$,

$$f_{m+n} = f_{m+1}f_n + f_mf_{n-1}.$$

To prove that f_n divides f_{kn} for $k \geq 1$, it is necessary to fix n and perform induction on k. First, if $m = n \geq 1$, then

$$\begin{aligned}f_{2n} &= f_{n+n}\\ &= f_{n+1}f_n + f_nf_{n-1}\\ &= f_n(f_{n+1} + f_{n-1}).\end{aligned}$$

Thus f_n divides f_{2n}, for $m \geq 1$.
Also

$$f_{3n} = f_{n+2n} = f_{n+1}f_{2n} + f_n f_{2n-1}.$$

Since f_n divides f_{2n}, then for $n \geq 1$, f_n divides f_{3n}. Now assume that $f_n | f_{kn}$ for $k = 1, \dots j-1$.
For $n \geq 1$, $j \geq 1$,

$$f_{jn} = f_{n+(j-1)n} = f_{n+1}f_{(j-1)n} + f_n f_{(j-1)n-1}.$$

Since f_n divides $f_{(j-1)n}$ and f_n divides f_n, then f_n must divide f_{jn}. This proves that for all $k > 0$

$$f_n \text{ divides } f_{kn}.$$

□

Next is in interesting property regarding greatest common divisors. It will be shown that

$$\gcd(f_m, f_n) = f_{\gcd(m,n)}.$$

For example, if $m = 14$ and $n = 21$, then

$$\gcd(f_{14}, f_{21}) = \gcd(377, 10946) = 13$$

and

$$f_{\gcd(14,21)} = f_7 = 13.$$

This proof requires four lemmas, two of which are **Property 6.8** and **Property 6.13**.

Lemma 9.1. *For integers n and m,* $\gcd(m, n) = \gcd(m, n \pm m)$*.*

Proof. If q is a divisor of both m and n, then there exist integers s and t such that $m = qt$ and $n = qs$. Then $m \pm n = qt \pm qs$, and therefore q is a factor of $m \pm n$. Since this is true for any divisor of both m and n, it is true for the greatest common divisor.

Now, if q is a divisor of both m and $m \pm n$, then there exist integers s and t such that $m = qt$ and $m \pm n = qs$. Then since q divides both m and qs, q must divide n. This completes the proof. $\square$

Theorem 9.2. *For nonnegative integers m and n,*

$$\gcd(f_m, f_n) = f_{\gcd(m,n)}.$$

Proof. Let $n = mk + r$, where $0 \leq r < m$. By **Property 6.13**,

$$f_n = f_{mk+1} f_r + f_{mk} f_{r-1}.$$

Therefore

$$\gcd(f_m, f_n) = \gcd(f_m, f_{mk+1} f_r + f_{mk} f_{r-1})$$

Since f_m divides f_{km}, then

$$\gcd(f_m, f_{mk+1} f_r + f_{mk} f_{r-1}) = \gcd(f_m, f_{mk+1} f_r).$$

Then since $\gcd(f_m, f_{mk+1}) = 1$, we have

$$\gcd(f_m, f_{mk+1} f_r) = \gcd(f_m, f_r).$$

Notice that the subscripts of the Fibonacci numbers form a step in the Euclidean algorithm [5]. This shows that the Euclidean algorithm for Fibonacci numbers f_m and f_n goes parallel with the Euclidean algorithm for integers m and n. Therefore, if the algorithm outputs the greatest common divisor of m and n to s, or $\gcd(m, n) = s$, then we also get $\gcd(f_m, f_n) = f_s = f_{\gcd(m,n)}$. $\square$

10 Infinite Series

An **infinite series** is an expression of the form

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \cdots .$$

The **partial sums** of a series are given by

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

Definition 10.1. *An infinite series* $\sum_{k=1}^{\infty} a_k$ ***converges*** *if its corresponding sequence* s_n *of partial sums converges. If* S *is the limit of the sequence* $\{s_n\}$*, then the series* $\sum_{k=1}^{\infty} a_k$ *converges to* S*. We write* $\sum_{k=1}^{\infty} a_k = S$ *and refer to* S *as the sum of the series [1].*

Property 10.1.

$$\sum_{k=1}^{\infty} \frac{1}{f_{2k}f_{2k+2}} = \frac{1}{\alpha^2}.$$

Proof. First, it will be shown that the partial sums s_n are equal to $\frac{f_{2n}}{f_{2n+2}}$. This will be a proof by induction on n. As a base case, when $n = 1$, then

$$\begin{aligned}\sum_{k=1}^{1} \frac{1}{f_{2k}f_{2k+2}} &= \frac{1}{f_2 f_4} \\ &= \frac{1}{3} \\ &= \frac{f_2}{f_4}.\end{aligned}$$

Now assume the conjecture is true for $n = 1, \ldots, j$. Then using **Property 6.6**,

$$\sum_{k=1}^{j+1} \frac{1}{f_{2k} f_{2k+2}} = \frac{f_{2j}}{f_{2j+2}} + \frac{1}{f_{2j+2} f_{2j+4}}$$

$$= \frac{f_{2j+4} f_{2j} + 1}{f_{2j+2} f_{2j+4}}$$

$$= \frac{f_{2(j+1)}^2 + (-1)^{2(j+1)+1} + 1}{f_{2j+2} f_{2j+4}}$$

$$= \frac{f_{2j+2}^2 - 1 + 1}{f_{2j+2} f_{2j+4}}$$

$$= \frac{f_{2j+2}}{f_{2j+4}}.$$

Therefore,

$$s_n = \frac{f_{2n}}{f_{2n+2}}.$$

Now to find the value of the infinite sum, it is necessary to take the limit of the partial sum s_n as n goes to infinity. Using the Binet form for f_n and the fact that $\alpha^2 = \alpha + 1$,

$\beta^2 = \beta + 1$ and $\alpha\beta = -1$, we get

$$s_n = \frac{f_{2n}}{f_{2n+2}}$$

$$= \frac{\alpha^{2n} - \beta^{2n}}{\alpha^{2n+2} - \beta^{2n+2}}$$

$$= \frac{\alpha^{2n} - \left(\frac{-1}{\alpha}\right)^{2n}}{\alpha^{2n+2} - \left(\frac{-1}{\alpha}\right)^{2n+2}}$$

$$= \frac{\alpha^{2n} - \frac{1}{\alpha^{2n}}}{\alpha^2\alpha^{2n} - \frac{1}{\alpha^{2n+2}}}.$$

As n approaches infinity, both $\frac{1}{\alpha^{2n}}$ and $\frac{1}{\alpha^{2n+2}}$ go to zero. Therefore,

$$\lim_{n\to\infty} s_n = \frac{1}{\alpha^2}.$$

□

.

This is just one example of Fibonacci numbers in infinite series. Furthermore, the properties included in this paper are a small proportion of those that apply to the Fibonacci numbers. These numbers arise in so many different capacities that they are beloved by many. As such, the Fibonacci Association was incorporated in 1963. Their primary publication is titled *The Fibonacci Quarterly.*

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