Cancellation Properties of Summands of Direct Products of Groups

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Abstract

This paper investigates the "cancellation properties" of a group C where, for some groups A and B, $A \oplus C \cong B \oplus C$ implies that $A \cong B$. Guided by the results of the Krull-Schmidt Theorem and Walker's Cancellation Theorem, this paper claims and proves by induction on the order of group C that $A \cong B$ if C is of finite order. Then, this paper introduces a discussion regarding the class of other possible cancellable structures.

Contents

1 Introduction

Often times, we as mathematicians can find parallels between an elementary study of mathematics and higher mathematics in an attempt to better understand the concept. When developing mathematicians are introduced to the study of group theory, some ideas are so foreign that it is often necessary to find these parallels to demystify the study. Unfortunately, we sometimes find that the seemingly obvious parallels that we want to draw between elementary and advanced concepts do not apply in way we would hope. For example, we try to understand the idea of a group isomorphism in the same way we might try to understand an equality, but we find that although a parallel can be drawn between these operations, we cannot use them synonymously. This instance can also be seen in the idea of "cancellation": for some integers a, b, c, $a + c = b + c$ implies that $a = b$ because we can cancel the number c from both sides.

What if, for some groups A, B, C, $A \oplus C \cong B \oplus C$ (with $\oplus$ as the External Direct Product operator as discussed below)? When, if ever, can we say that $A \cong B$? In fact, should we be able to do this in every instance?

The answer to this question is no. There are many instances in which the C group cannot be cancelled. There are two instances proven in this investigation, first in Section 3.2 during a discussion of Walker's Cancellation Theorem, and another in Section 4.1.

Knowing that cancellation does not always occur, our natural inclination is to wonder: what properties are necessary to ensure that cancellation can occur? Before we approach this question, we will provide some background information to aid the reader in understanding the proofs provided.

2 Background Information

2.1 External Direct Products of Groups

Suppose we have two groups, A and B. We represent the external direct product of A and B as $A \oplus B$. This follows the notation of [2].

The external direct product of groups is represented by a component-wise association between elements in the two groups. More formally, we say that $A \oplus B = \{(a, b)|a \in A, b \in B\}$. In other words, the external direct product of A and B gives the set of paired elements of the two groups.

We can also define a group operation on the external direct product of groups. Suppose that group A is the set A under the operation $*$, and suppose that group B is the set B under the operation $\star$. Then, for $(a, b), (a', b') \in A \oplus B$, $(a, b)(a', b') = (a * a', b \star b')$ where $(a * a') \in A$ and $(b \star b') \in B$ since groups are closed under their respective operations.

Finally, the external direct product can apply to more than two groups as well. For example, for groups $A_1, A_2, A_3, \cdots, A_n$, the external direct product $A_1 \oplus A_2 \oplus \cdots \oplus A_n = \{(a_1, a_2, \cdots, a_n)|a_1 \in A_1, a_2 \in A_2, \cdots, a_n \in A_n\}$. Similarly, the operations upon the group function in the same way as above.

2.2 Internal Direct Products of Groups

The notion of an internal direct product is different than that of an external direct product in the sense that it requires "splitting" a group rather than joining multiple groups. We say that G is the internal direct product of A and B if $G = A \times B$ also following the notation of [2]. For the $\times$ operation to make sense, the following conditions must be true of groups A and B:

1. Groups A and B are normal subgroups of G,

2. $A \cap B = \{e\}$ where e is the necessary identity element in G, and

3. $G = AB = \{a * b | a \in A, b \in B\}$ where $*$ is the operation of G.

The internal direct product of groups $H_1, H_2, \cdots, H_n$ is defined as the product

$$G = \prod H_i = H_1 \times H_2 \times \cdots \times H_n$$

where multiplication in $\prod H_i$ is multiplication in G.

Since it quickly becomes clear that there are many more restrictions on the properties of internal direct products, we question how the two products may be related. The following proof shows the relationship between the two products.

2.3 Relationship Between the External and Internal Direct Products of Groups

Let $G = H \oplus K$. By definition, $G = \{(h_i, k_j)|, h_i \in H, k_j \in K\}$. Define group $H' = \{(h_i, k_e)|h_i \in H\}$ where k_e is the identity element of K, and define group $K' = \{(h_e, k_j)|k_j \in K\}$ where h_e is the identity element of H. Notice that H' and K' are both normal subgroups of G, $H' \cap K' = (h_e, k_e)$ and $G = H'K'$. It follows that $G = H' \times K'$. Further, we see that $H' \cong H$; define $\phi_1 : H' \to H$ as $\phi_1(h_i, k_e) = h_i$. To see that ϕ_1 is an isomorphism, we will first show that it is a homomorphism, and then that it is both injective and surjective. Since it is easy to see that ϕ_1 is well defined, we will not show this proof here.

First, look at elements $(h_i, k_e), (h_j, k_e) \in H'$. Then, $\phi_1((h_i, k_e)(h_j, k_e)) = \phi_1(h_i h_j, k_e) = h_i h_j = \phi_1(h_i, k_e)\phi_1(h_j, k_e)$. Thus, ϕ_1 is a homomorphism. Now, suppose $\phi_1(h_i, k_e) = \phi_1(h_j, k_e)$ but $h_i \neq h_j$. This is impossible since $h_i = \phi_1(h_i, k_e) = \phi_1(h_j, k_e) = h_j$, which is a contradiction. Thus, ϕ_1 is injective. Finally, it follows that for any $h_i \in H$, there exists an element $(h_i, k_e) \in H'$ such that $\phi_1(h_i, k_e) = h_i$. Thus, ϕ_1 is surjective. Thus, since ϕ_1 is a bijective homomorphism, ϕ_1 is an isomorphism.

Similarly, $K' \cong K$; define $\phi_2 : K' \to K$ as $\phi_2(h_e, k_j) = k_j$, and notice that ϕ_2 is also an isomorphism.

From this result, we see that there exist normal, distinct subgroups of G, namely H' and K', such that $H' \times K' = G = H \oplus K$ with $H \cong H'$ and $K \cong K'$. Thus, given groups $H \oplus K$, we can find an easy isomorphism to convert our external direct product into internal direct product $H' \times K'$.

2.4 Ascending and Descending Chain Conditions

A group satisfies the Ascending Chain Condition (ACC) if there is no infinite ascending chain

$$G_1 \subset G_2 \subset G_3 \cdots$$

with G_i a normal subgroup of G. Any finite group of order n satisfies the ACC since any chain of normal subgroups must terminate after including all elements in G.

Similarly, G satisfies the Descending Chain Condition (DCC) if there is no infinite descending chain

$$G_1 \supset G_2 \supset G_3 \cdots$$

with G_i a normal subgroup of G. For example, Z, the group of integers under addition, does not satisfy DCC since

$$\langle 2 \rangle \supset \langle 2^2 \rangle \supset \langle 2^3 \rangle \cdots$$

is an infinite decreasing sequence of normal subgroups.

We say that a group G is of *finite chief length* if G satisfies both the ACC and DCC.

2.5 First and Second Isomorphism Theorems for Groups

The following are the statements of the first two isomorphism theorems for groups. These will be used in later proofs in this investigation.

The First Isomorphism Theorem for Groups: For $\phi : G \to H$ where ϕ is a homomorphism, $G/\ker\phi \cong \phi(G)$ is an isomorphism.

The Second Isomorphism Theorem for Groups: if K is a subgroup of group G, and N is a normal subgroup of G, then $K/(K \cap N) \cong KN/N$.

3 Overview and Historical Investigations

The primary topic of this paper is to investigate the cancellation properties of direct products of groups. By cancellation, we mean the following statement:

For groups A, B, and C where $A \oplus C \cong B \oplus C$, we say C is *cancellable* if $A \cong B$.

As shown earlier, the internal and external direct products are very closely related. Thus, we also investigate the following question:

If G is a group, where $G = A_1 \times B_1 = A_2 \times B_2$, and $B_1 \cong B_2$, when is it true that $A_1 \cong A_2$?

First, we will take a look at some historical investigations before proposing a possible solution to this problem.

3.1 Krull-Schmidt Theorem

The following discussion and proof are as in [3] and [4]

With relation to direct products of groups a direct product of groups $\prod H_i = H_1 \times H_2 \times \cdots \times H_n$, there is always an "easy" homomorphism that we can induce with direct products where, for any $\phi_i : H_i \to K_i$, the induced homomorphism is also done component-wise:

$$\phi_1 \times \phi_2 \cdots \times \phi_n : H_1 \times H_2 \times \cdots \times H_n \to K_1 \times K_2 \times \cdots \times K_n.$$

The homomorphism maps $(x_1 x_2 \cdots x_n)$ to $(\phi_1(x_1)\phi_2(x_2) \cdots \phi_n(x_n))$.

If we have some $\prod H_i = G$, then each H_i is normal in G, and we can say that $H_i \cong 1 \times \cdots 1 \times H_i \times 1 \times \cdots \times 1$.

The proof of the Krull-Schmidt Theorem uses the idea of endomorphisms of group G and expressing the decomposition $G = \prod H_i$ as the equation

$$1 = \sum \varepsilon_i$$

where ε_i is the projection to H_i as an endomorphism of G. Remember that an endomorphism ε is a homomorphism from a group onto itself. Further, we will look at endomorphisms as additive where the endomorphism $\alpha + \beta$ is the endomorphism

$$(\alpha + \beta)(x) = \alpha(x)\beta(x)$$

while $\alpha\beta$ denotes composition as

$$(\alpha\beta)(x) = \alpha(\beta(x)).$$

For sake of notation, the identity endomorphism of G will be expressed as 1 such that $1\alpha = \alpha 1 = \alpha$, while the trivial endomorphism of G will be expressed as 0 such that $\alpha + 0 = 0 + \alpha = \alpha$.

Note that $\alpha+\beta$ is not generally an endomorphism of G. In fact, $\alpha+\beta$ is an endomorphism of G if and only if $\alpha(x)$ commutes with $\beta(y)$ for all $x, y \in G$. We can see this with the following proof:

$\alpha + \beta : G \to G$ is an endomorphism if and only if $(\alpha+\beta)(xy) = (\alpha+\beta)(x)(\alpha+\beta)(y)$ for all $x, y \in G$. In other words,

$$\alpha(x)\alpha(y)\beta(x)\beta(y) = \alpha(xy)\beta(xy) = \alpha(x)\beta(x)\alpha(y)\beta(y).$$

By cancelling $\alpha(x)$ and $\beta(y)$, we get the equivalence that $\alpha(y)\beta(x) = \beta(x)\alpha(y)$. This completes the proof.

Now, we will investigate the idea that that a group that of finite chief length can be expressed as a unique direct product of finitely many indecomposable normal subgroups.

First, we notice that, if group G is of finite chief length, then any normal endomorphism α cannot be surjective while not injective (otherwise we get an infinite increasing subsequence $\ker \alpha^k$). Similarly, α cannot be injective while not surjective (else we get an infinite decreasing subsequence imα^k). With this fact, we claim that for some group G of finite chief length, G is indecomposable if and only if every normal endomorphism is either nilpotent or an automorphism.

We see that, if G decomposes, then the projection onto one factor (the H_i normal subgroups) is a normal endomorphism which is neither nilpotent nor an automorphism. Now suppose that every normal endomorphism β of indecomposable group G of finite chief length is not an automorphism. Thus, as we showed earlier, β is neither injective nor surjective. If β is also not nilpotent, then $\ker \beta^k$ and imβ^k are proper subgroups of G. Then, since G satsifies the chain conditions, there exists a k such that $\ker \beta^k = \ker \beta^l$ and im$\beta^k =$ imβ^l for all l ¿ k. Now, for proper normal subgroups K, H of G, then $G = K \times H$ by Fitting's Lemma (which shows that if im$\beta^k =$ im$(\beta^k)^2)$ = H, and $\ker \beta^k = \ker(\beta^k)^2$ = K, then $G = H \times K$), which is a contradiction.

This leads us to the statement of the Krull-Schmidt Theorem:

The Krull-Schmidt Theorem: Let G be a nontrivial group of finite chief length. Then group G can be decomposed as $G = G_1 \times G_2 \times \cdots \times G_n$ where each group G_i is indecomposable. This decomposition is unique up to permutation; if group G can be written as any decomposition $G = H_1 \times H_2 \times \cdots \times H_m$ where each H_i are indecomposable, then $n = m$, and, after reorganization, $G_i \cong H_i$ for each i. Further, for all $k < n$, $G = G_1 \times \cdots \times$

$G_k \times H_{k+1} \times \cdots \times H_n$.

The existence of the decomposition was proved earlier, so now we will look at the uniqueness claim. Suppose we have two decompositions of G. Then, we notice that the two resulting decompositions of the identity endomorphism of G are $1 = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n$ and $1 = \eta_1 + \eta_2 + \cdots + \eta_m$. Then

$$\varepsilon_1 = \varepsilon_1\varepsilon_1 = \varepsilon_1(1)\varepsilon_1 = \sum \varepsilon_1\eta_i\varepsilon_1$$

is not nilpotent, so at least one of the summands $(\varepsilon_1\eta_i\varepsilon_1)$ is not nilpotent. By reorganization, assume that i = 1. But $\varepsilon_1\eta_1\varepsilon_1$ is essentially a normal endomorphism of H_1 which, as we showed earlier, must be an automorphism of H_1. But $\varepsilon_1\eta_1\varepsilon_1 = \varepsilon_1\eta_1\eta_i\varepsilon_1$ is a composition of two homomorphisms

$$\psi = \eta_1\varepsilon_1 : H_1 \to K_1 \text{ and } \phi = \varepsilon_1\eta_1 : K_1 \to H_1.$$

Since K_1 is indecomposable and $K_1 = \ker\phi \times \text{ im}\psi$, either $\ker\phi$ or $\text{im}\psi$ is trivial. Since ψ is injective, $\ker\phi = 1$, so both $\psi = \eta_1\varepsilon_1 : H_1 \to K_1$ and $\phi = \varepsilon_1\eta_1 : K_1 \to H_1$ are isomorphisms. Now, we claim that

$$G = K_1 \times H_2 \times \cdots \times H_n.$$

First, $H_2 \times \cdots \times H_n = \ker\varepsilon_1$ and $K_1 = \text{im}\eta_1$ connot meet nontrivially since $\phi = \varepsilon_1\eta_1 : K_1 \to H_n$ is an injective homomorphism. This means that K_1 and $H_2 \times \cdots \times H_n$ centralize each other (i.e., each element of K commutes with each element of N) so $\beta = \eta_1\varepsilon_1 + (\varepsilon_2 + \cdots + \varepsilon_n)$ is a normal endomorphism of G. But β induces an isomorphism from G onto the subgroup $G^* = K_1 \times H_2 \times \cdots \times H_n$ so β is an injective homomorphism and thus onto. Thus, $G = G^*$.

Now, since we know that

$$G/K_1 = H_2 \times \cdots \times H_n = K_2 \times \cdots \times K_m,$$

the rest of the theorem follows by induction.

This theorem tells us that cancellation occurs in an internal direct product when the groups in question are of finite chief length. Since the internal and external product are related as mentioned in Section 2, the Krull-Schmidt Theorem tells us that cancellation in an external direct product is possible when the groups in the product are of finite chief length.

The next historical investigation that we will look at nudges our investigation into a similar direction.

3.2 Walker's Cancellation Theorem

Unlike the Krull-Schmidt Theorem, the proof of Walker's Cancellation Theorem will not be given here. Rather, the statement of the theorem and its consequences will be discussed.

Walker's Cancellation Theorem: For groups that are abelian, if $F \oplus H \cong F' \oplus G$ and $F \cong F'$ and F is finitely generated, then $H \cong G$. [1]

Remember than an abelian group is a group where elements are commutative by the operation of the group. Also, remember that an abelian group F is finitely generated if there exist finitely many elements $x_i \in F$ where, for any element $x \in F$, $x = x_1 n_1 + x_2 n_2 + \cdots + x_m n_m$ with each n_i an integer. In other words, F is finitely generated if any element in F can be created by the combination of the elements of a finite subset of F. For example, Z, the group of integers under addition, can be finitely generated since any element can be generated by the element $\langle 1 \rangle$. Further, any finite group G can be easily finitely generated by $\langle G \rangle$.

Walker also presented a case in [1] in which it is easy to see that H not $\cong G$. For this example, let each C_i be cyclic of order 2. Walker shows that for groups $F = C_2 \oplus C_3 \oplus C_4 \oplus \cdots$, $G = C_1$, $F' = C_1 \oplus C_3 \oplus C_5 \cdots$, and $H = C_2 \oplus C_4 \oplus C_6 \oplus \cdots$, then clearly $F \oplus G = F' \oplus H$ and $F \cong F'$, but $G \ncong H$.

Clearly, since the groups F and F' are not finitely generated since both groups are infinite in order.

Interestingly, both of these theorems point towards an idea of finitude. If group G is finite, then it will also be of finite chief length. In the abelian case, as discussed earlier, if group G is finite, it will obviously be generated by $\langle G \rangle$. This leads us to wonder if cancellation can occur if we know that the known isomorphic groups in an external direct product are finite. This is the primary investigation of this paper.

4 Introduction of Investigation

In this paper, we will investigate the hypothesis that cancellation is possible when the two known isomorphic groups in the external direct product are finite. To do this, we will look at two theorems.

THEOREM 1. If G is a group, $G = A_1 \times B_1 = A_2 \times B_2, B_1 \cong B_2$, and $|B_i| < \infty$, then $A_1 \cong A_2$

THEOREM 2 Suppose that Theorem 1 is true for B_i for order k. If $H_1 \oplus K_1 \cong H_2 \oplus K_2$, $K_1 \cong K_2$, and $|K_i| = k$, then $H_1 \cong H_2$.

In other words, we hypothesize that if the two internal direct products are equal and B_1 and B_2 are isomorphic, then A_1 and A_2 are isomorphic if the B_i groups are not infinite in size.

We will prove the validity of THEOREM 2, and then conduct a modified proof by induction to prove THEOREM 1. In the end, we will attempt to show that, as in THEOREM 2, $H_1 \cong H_2$ when K_i is finite.

Though the internal and external direct products of groups are very closely related, the notation of the groups as $A_i, B_i, H_i,$ and K_i are noted as such in order to maintain the separation of THEOREM 1 and THEOREM 2. Of course, the groups could be noted with more parallelism and structure, but it will become apparent as the proof continues that this notation makes this organization most simple.

4.1 Infinite Groups are Not Necessarily Cancellable

Before we conduct our investigation, we will provide another example to note when cancellation does not occur. This time, we will look at the effect of infinite non-abelian groups. The following is an example of such.

Here, we will use an existence proof.

Let group $G = R[x]$, the group of polynomials with real coefficients. We will prove that

$$G \oplus G \cong G \oplus \{0\}.$$

Define a mapping $\phi : G \oplus G \to G$ as such:

$$\phi(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, b_0 + b_1x + b_2x^2 + \cdots + b_nx^n) =$$
$$a_0 + b_0x + a_1x^2 + b_1x^3 + a_2x^4 + b_2x^5 + \cdots + a_nx^{2n} + b_nx^{2n+1}$$

and notice that ϕ is an isomorphism. Thus, $G \oplus G \cong G$. Since it is obvious that $G \oplus \{0\} \cong G$, it follows directly that $G \oplus G \cong G \oplus \{0\}$. It is also obvious that $G \cong G$, but clearly $G \ncong \{0\}$. Thus, Theorem 1 in general is false.

5 Proof

Here, we begin our proof of Theorem 1 and Theorem 2. Each point listed is to be considered a Lemma of the Theorems, each helping lead to the desired conclusion.

5.1 Proof of Theorem 2

Suppose Theorem 1 is true for B_i of order k, and that $H_1 \oplus K_1 \cong H_2 \oplus K_2$, $K_1 \cong K_2$, and $|K_i| = k$. There exist normal subgroups of $G = H_1 \oplus K_1$, namely $H_1' = \{(H_1, k_e)\}$ and $K_1' = \{(h_e, K_1)\}$, where h_e, k_e are identity elements of H_1 and K_1, respectively, such that $H_1' \times K_1' = H_1 \oplus K_1$. Similarly, there are groups H_2' and K_2' such that $H_2' \times K_2' = H_2 \oplus K_2$. Thus, we have $H_1' \times K_1' \cong H_2' \times K_2'$. As defined, we can show that $H_1 \cong H_1'$, $K_1 \cong K_1'$, $H_2 \cong H_2'$, and $K_2 \cong K_2'$. To see this, define $\phi_1 : H_1' \to H_1$ as $\phi_1(h_i, k_e) = h_i$ and notice that ϕ_1 is an isomorphism. Similarly, define $\phi_2 : K_1' \to K_1$ as $\phi_2(h_e, k_i) = k_i$ and notice that ϕ_2 is also an isomorphism. Thus, we get the desired isomorphisms. Finally, since $K_1' \cong K_1 \cong K_2 \cong K_2'$, $K_1' \cong K_2'$ with $|K_i'| = k$. By Theorem 1, $H_1 \cong H_1' \cong H_2' \cong H_2$, so $H_1 \cong H_2$. This completes the proof.

5.2 Inductive Base Case

Suppose that $|B_i| = 1$, and suppose that $A_1 \times B_1 \cong A_2 \times B_2$. Then, $A_1 \times B_1 = A_1$ since B_i is a trivial group. Similarly, $A_2 \times B_2 = A_2$. Thus, $A_1 \cong A_2$.

From here we consider what the induction step would provide for us. We suppose that our hypothesis (Theorem 1) is now true for B_i of order $1, \cdots, k$, and we will progress for $|B_i| = k + 1$. If we can now prove for arbitrary groups A and B such that $A \oplus C \cong B \oplus C$ for some group C such that $|C| \leq k$, we will be able to show that $A \cong B$ by Theorem 2. We move forward to find a way to do this.

5.3 If $G = A \times B$ for some A, B, then $G/B \cong A$

This result follows directly from the Second Isomorphism Theorem for Groups.

By definition of $A \times B$, groups A and B are both normal subgroups of G, $A \cap B = \{e\}$, and $AB = \{ab | a \in A, b \in B\} = G$. Thus, $A/(A \cap B) = A/\{e\} = A$, and $AB/B = G/B$. Thus, $G/B \cong A$.

Using this result, we will look at two different scenarios:

1. $A_1 \cap B_2 = \{e\}$, and

2. $A_1 \cap B_2 \neq \{e\}$

5.4 If $A_1 \cap B_2 = \{e\}$, then $G = A_1 \times B_2$

In order to prove this, we will look at cosets. First, since $G = A_1 \times B_1$, then by Section 5.3 it follows that $G/A_1 \cong B_1$, and so the number of cosets of A_1 is $|B_1| = k$. To prove that $G = A_1 \times B_2$, we want to look at cosets of A_1 with elements of B_2. Look at the set $\{A_1 b_i | b_i \in B_2\}$. If indeed $G = A_1 \times B_2$, then it must be true that for any two distinct elements $b_i, b_j \in B_2$, $A_1 b_i \neq A_1 b_j$.

Suppose that $b_i \neq b_j$ and $A_1 b_i = A_1 b_j$. It follows that $b_i b_j^{-1} \in A_1$. By definition of b_i and b_j, it also follows that $b_i b_j^{-1} \in B_2$, so $b_i b_j^{-1} \in A_1 \cap B_2$. But since we assumed that $A_1 \cap B_2 = \{e\}$, it must be true that $b_i b_j^{-1} = e$, and so $b_i = b_j$ and we have reached a contradiction.

Thus, since $A_1 b_i \neq A_1 b_j$ for all $b_i, b_j \in B_2$, then $A_1 b_1 \cdots A_1 b_k$ with $b_i \in B_2$ are all the cosets. Now, we know that

$$\bigcup_{b \in B_2} A_1 b = G$$

and $|G : A_1| = |B_1| = |B_2|$, and so $A_1 \times B_2 = G$.

Thus, if $A_1 \cap B_2 = \{e\}$, $A_1 \cong G/B_2 \cong A_2$ by the Second Isomorphism Theorem of Groups. Similarly, we get the same result if $A_2 \cap B_1 = \{e\}$.

Now we will assume that $H_1 = A_1 \cap B_2 \neq \{e\}$ and $H_2 = A_2 \cap B_1 \neq \{e\}$ and observe the properties.

5.5 H_1, H_2, and $H_1 \times H_2$ are normal subgroups of G

Since we know that $G = A_1 \times B_1 = A_2 \times B_2$, each subgroup A_1, A_2, B_1, B_2 is normal in G. Then, for all $g \in G$, $a \in A_1$, and $b \in B_2$, it follows that $gag^{-1} \in A_1$ and $gbg^{-1} \in B_2$, and so for all $h \in H_1$, $ghg^{-1} \in H_1$. Since now we know that $gH_1g^{-1} \subseteq H_1$ for all $g \in G$, H_1 is a normal subgroup of G.

In order to prove that H_2 is a normal subgroup of G, the proof is similar. This time, we specify the proof for $h \in H_2$, and so it now follows, as in the above proof, that for all $g \in G$, $gH_2g^{-1} \subseteq H_2$ and so H_2 is normal in G.

Now, H_1 and H_2 are normal in G and $H_1 \cap H_2 = \{e\}$ since $A_1 \cap B_1 = \{e\}$ and $A_2 \cap B_2 = \{e\}$, so now we may see if $H_1 \times H_2$ is normal in G. To do this, we must show that for all $g \in G$, $g(H_1H_2)g^{-1} \subseteq H_1 \times H_2$. Remember that $gH_1g^{-1} \subseteq H_1$ and $gH_2g^{-1} \subseteq H_2$, so $(gH_1g^{-1})(gH_2g^{-1}) = gH_1H_2g^{-1}$, so $gH_1H_2g^{-1} \subseteq H_1 \times H_2$. Thus, $H_1 \times H_2$ is a normal subgroup of G.

The understanding of H_1 and H_2 are normal subgroups of G allows us to look at H_1 and H_2 by means of factor groups more easily and even allows us to simplify what we know by means of the Isomorphism Theorems of Groups.

From here, we will combine what we know to show that we can find something of the form $A_1 \oplus C \cong A_2 \oplus C$ where $|C| \leq k$.

5.6 $G/(H_1 \times H_2) \cong A_1/H_1 \oplus B_1/H_2$

Remember that $G = A_1 \times B_1$, so we will show that $(A_1 \times B_1)/(H_1 \times H_2) \cong A_1/H_1 \oplus B_1/H_2$.

Define a homomorphism $\phi : (A_1 \times B_1) \to A_1/H_1 \oplus B_1/H_2$ as

$$\phi(a_1 b_1) = (a_1 H_1, b_1 H_2).$$

To see that ϕ is in fact a homomorphism, notice that for all elements $a_1, a_1' \in A_1$ and $b_1, b_1' \in B_1$,

$\phi((a_1 b_1)(a_1' b_1')) = \phi((a_1 a_1')(b_1 b_1')) = (a_1 a_1' H_1, b_1 b_1' H_2) = (a_1 H_1, b_1 H_2)(a_1' H_1, b_1' H_2) = \phi(a_1 b_1)\phi(a_1' b_1')$.

This fact about commutativity of elements in an internal direct product is theorized and proven in Gallian.

Remember that $H_1 \subseteq A_1$ and $H_2 \subseteq B_1$, so ϕ is defined on $H_1 \times H_2$. But since $H_1 \cap B_1 = \{e\}$ and $H_2 \cap A_1 = \{e\}$, $\ker \phi \supseteq H_1 \times H_2$. Further, if an element exists in $\ker \phi$, the element itself must not exist in A_1 or B_1, meaning that the element must live in H_1 or H_2, or $H_1 \times H_2$. Thus, $\ker \phi \subseteq H_1 \times H_2$. Thus, $\ker \phi = H_1 \times H_2$. Thus, by the First Isomorphism Theorem of Groups, $(A_1 \times B_1)/(H_1 \times H_2) \cong A_1/H_1 \oplus B_1/H_2$.

Similarly, of course, $G/(H_1 \times H_2) = (A_2 \times B_2)/(H_1 \times H_2) \cong A_2/H_2 \oplus B_2/H_1$. Further, $(A_1 \times B_1)/(H_1 \times H_2) = (A_2 \times B_2)/(H_1 \times H_2)$, so

$$A_1/H_1 \oplus B_1/H_2 \cong A_2/H_2 \oplus B_2/H_1.$$

Now, since $B_1 \cong B_2$, we can also say that $B_1 \oplus A_1/H_1 \oplus B_1/H_2 \cong B_2 \oplus A_2/H_2 \oplus B_2/H_1$. We will call this FORMULA 1. FORMULA 1 provides us with further insight into what we may want; particularly, we want to find something of the form $A_1 \oplus C \cong A_2 \oplus C$ where the C group is a factor group of some form of the B_i groups which will definitely have order less than or equal to k. From here, we will work to derive additional formulas to reach this desired conclusion.

5.7 $B_1 \oplus A_1/H_1 \oplus B_1/H_2 \cong (B_1 \times A_1)/H_1 \oplus B_1/H_2$

Define a homomorphism $\phi : B_1 \times A_1 \to B_1 \oplus A_1/H_1$ as $\phi(b_i a_i) = (b_i, a_i H_1)$. To see that this is in fact a homomorphism, notice that $\phi((b_i a_i)(b'_i a'_i)) = \phi((b_i b'_i)(a_i a'_i)) = (b_i b'_i, a_i a'_i H_1) = (b_i, a_i H_1)(b'_i, a'_i H_1) = \phi(b_i a_i)\phi(b'_i a'_i)$.

Since $H_1 \cap B_1 = \{e\}$, for all $h \in H_1$, it follows that $\phi(h) = (e, hH_1) = (e, eH_1)$, so $\ker \phi = H_1$. Thus, by the First Isomorphism Theorem of Groups, $(B_1 \times A_1)/H_1 \cong B_1 \oplus A_1/H_1$. Thus,

$$B_1 \oplus A_1/H_1 \oplus B_1/H_2 \cong (B_1 \times A_1)/H_1 \oplus B_1/H_2.$$

Now, by the same reasoning, we see that $(B_2 \times A_2)/H_1 \oplus B_1/H_2 \cong A_2 \oplus B_2/H_1 \oplus B_1/H_2$. To see this, we modify the above proof to show that, with the adjusted homomorphism, $\ker \phi = H_1$, since $H_1 \cap A_2 = \{e\}$.

Now, combining the two equations we have identified, we will get the following equation:

$$B_1 \oplus A_1/H_1 \oplus B_1/H_2 \cong A_2 \oplus B_2/H_1 \oplus B_1/H_2.$$

We will call this FORMULA 2.

5.8 $B_2 \oplus A_2/H_2 \oplus B_2/H_1 \cong A_1 \oplus B_1/H_2 \oplus B_2/H_1$

By the proof in section 5.7, we know that $(A_1 \times B_1)/H_2 \cong A_1 \oplus B_1/H_2$ and $(B_2 \times A_2)/H_2 \cong B_2 \oplus A_2/H_2$ since $(A_1 \times B_1)/H_2 = (B_2 \times A_2)/H_2$. It follows directly that $A_1 \oplus B_1/H_2 \cong B_2 \oplus A_2/H_2$. Thus, it follows that $B_2 \oplus A_2/H_2 \oplus B_2/H_1 \cong A_1 \oplus B_1/H_2 \oplus B_2/H_1$. We will call this last equation FORMULA 3.

Now, we have our three formulas:

FORMULA 1 $= B_1 \oplus A_1/H_1 \oplus B_1/H_2 \cong B_2 \oplus A_2/H_2 \oplus B_2/H_1$

FORMULA 2 $= B_1 \oplus A_1/H_1 \oplus B_1/H_2 \cong A_2 \oplus B_2/H_1 \oplus B_1/H_2$

FORMULA 3 $= B_2 \oplus A_2/H_2 \oplus B_2/H_1 \cong A_1 \oplus B_1/H_2 \oplus B_2/H_1$

By combining FORMULA 2 to FORMULA 1 and then FORMULA 1 to FORMULA 3, we get the following equation:

$$A_2 \oplus B_2/H_1 \oplus B_1/H_2 \cong B_2 \oplus A_2/H_2 \oplus B_2/H_1 \cong A_1 \oplus B_1/H_2 \oplus B_2/H_1,$$

or,

$$A_1 \oplus B_1/H_2 \oplus B_2/H_1 \cong A_2 \oplus B_1/H_2 \oplus B_2/H_1.$$

With this final equation, we find what we were looking for. Since the group B_2/H_1 has order less than or equal to our inducted k value. Thus, by the induction hypothesis,

$$A_1 \oplus B_1/H_2 \cong A_2 \oplus B_1/H_2.$$

Similarly, it follows that

$$A_1 \cong A_2.$$

Thus, we have proven that if $A_1 \oplus C \cong A_2 \oplus C$ for some finite group C, then $A_1 \cong A_2$.

6 Consequences and Generalizations

This result supports the claim that the finite quality that was included in both the Krull-Schmidt Theorem and Walker's Cancellation Theorem was a necessary component of cancellation. And in fact, this proof acts as a companion theorem to Walker's Cancellation Theorem: this proof allows us to look at all groups regardless of the abelian structure of the group. In fact, it allows us to ignore Walker's Cancellation Theorem except for the case of infinite groups. Now, the notion of cancelling infinite groups may be surprising, but we note that the three theorems in question (Walker's, Krull-Schmidt, and that of this investigation) are not complete answers to the cancellation question, but each address different cases. Walker's Cancellation Theorem provides us with an interesting infinite case as well. Suppose, for groups A and B, $Z \oplus A \cong Z \oplus B$. Walker's Cancellation Theorem would tell us directly that $A \cong B$ if A and B are abelian groups. Thus, though finitude is one category of cancellation, there is a large class of groups that can be cancelled.

This raises the question: what other structures have the characteristic of cancellation? Some structures come up immediately, and we will begin to investigate these.

6.1 $C_1 \oplus C_2$ has the characteristic if and only if C_1 and C_2 are cancellable.

The first direction of this proof is simple, and was proved earlier. If C_1 and C_2 are cancellable, and we know that $A \oplus C_1 \oplus C_2 \cong B \oplus C_1 \oplus C_2$, then we know that, by cancelling the C_2 group, $A \oplus C_1 \cong B \oplus C_1$, and then it immediately follows that $A \oplus B$ since the C_1 group is cancellable.

To prove the other direction, we remember that if $C_1 \oplus C_2$ is cancellable, and $A \oplus C_1 \oplus C_2 \cong B \oplus C_1 \oplus C_2$, then $A \cong B$. Suppose $C_1 \oplus C_2$ is cancellable and $A \oplus C_1 \cong B \oplus C_1$. It follows that we can always "build up" groups even if we do not know if they are cancellable or not. Then, if $A \oplus C_1 \cong B \oplus C_1$, $A \oplus C_1 \oplus C_2 \cong B \oplus C_1 \oplus C_2$, and thus $A \oplus B$, and the C_1 group has effectively been cancelled. Similarly, we show that $A \oplus C_2 \cong B \oplus C_2$ implies that $A \oplus C_2 \oplus C_1 \cong B \oplus C_2 \oplus C_1$ which implies that $A \oplus B$, and so the C_1 group has effectively been cancelled. Thus, we have proven the if and only if statement.

What types of things also have the properties that we have been investigating? To what other types of structures can we generalize our proof? In the proof that we conducted, it seems that the only two major structural identities necessary were 1) the idea of the external direct product and 2) the idea of finite size. Thus, it seems that we can generalize this proof for vector spaces. We give these proofs below.

6.2 Generalization to Vector Spaces

Suppose we look at vector spaces and allow the "finite" criterion be apply to the dimension of the vector space. Then, we notice that, if we have vector spaces V and W, it follows that $V \cong W$ if and only if V and W have the same dimension. We also have the concept of an external direct product of two vector spaces. In fact, we also know the following fact:

If the dimension of V is n and the dimension of W is m, then the dimension of $V \oplus W$ is n + m.

Now, we ask the following question:

Suppose, for some vector spaces V, W, and Z, $V \oplus Z \cong W \oplus Z$ where the dimension of Z is some finite number z. Is it true that $V \cong W$?

To answer this question, we do a basic counting method. Suppose V has dimension v and W has dimension w. Then, $V \oplus Z$ has dimension v+z while $W \oplus Z$ has dimension w+z. Then, since the vector spaces are isomorphic, v+z = w+z. Since z is a finite number, we can subtract z from both sides of the equation to see that v = w. Thus, $V \cong W$.

After this investigation, we present two questions for further investigation.

6.3 If group C is cancellable, and $C' \subseteq C$, is C' is cancellable?

This question is interesting because we already know a partial answer. If C can be decomposed into normal subgroups $C = C' \times D$ for some group D, we can easily prove that C' is cancellable. But what if this is nott the case? What if we do nott know that C decomposes as such? Thus, the author presents this question for those interested.

6.4 Can we generalize our proof for Rings?

As mentioned earlier, it seems as though the only two things necessary for our proof is a notion of an external direct product paired with the idea of finite structures. Surely, we can look at finite rings and finite ideals, but can our proof show cancellation properties for rings? Especially with the binary operations upon a ring, there seem to be complicated results with respect to this generalization. The scope of this investigation doesn't reach this question, but the author hopes that this question is a source of inspiration.

7 Conclusion

The proof provided in Section 5 provides a very clear road map of where the question of cancellation can be taken in the future. As mentioned earlier, it seems that we can generalize our proof for many more structures in order to provide more knowledge about the study of group theory as a whole. Though finite groups were already included in the proof of the Krull-Schmidt Theorem, it becomes apparent that the proof provided in Section 5 is easier to understand and requires fewer necessary components.

What other properties allow cancellation to occur? It has been a very long time since the Krull-Schmidt and Walker's Cancellation Theorems have been formalized and proven, are there more properties to be proven? This remains a mystery. The author hopes that, in the proof of the question of generalization to rings draws towards a better understanding of cancellation in the future.

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