Trigonometry in the Hyperbolic Plane

Tiffani Traver

May 16, 2014

Abstract

The primary objective of this paper is to discuss trigonometry in the context of hyperbolic geometry. This paper will be using the Poincaré model. In order to accomplish this, the paper is going to explore the hyperbolic trigonometric functions and how they relate to the traditional circular trigonometric functions. In particular, the angle of parallelism in hyperbolic geometry will be introduced, which provides a direct link between the circular and hyperbolic functions. Using this connection, triangles, circles, and quadrilaterals in the hyperbolic plane will be explored. The paper is also going to look at the ways in which familiar formulas in Euclidean geometry correspond to analogous formulas in hyperbolic geometry. While hyperbolic geometry is the main focus, the paper will briefly discuss spherical geometry and will show how many of the formulas we consider from hyperbolic and Euclidean geometry also correspond to analogous formulas in the spherical plane.

1 Introduction

The *axiomatic method* is a method of proof that starts with definitions, axioms, and postulates and uses them to logically deduce consequences, which are called theorems, propositions or corollaries [4]. This method of organization and logical structure is still used in all of modern mathematics. One of the earliest, and perhaps most important, examples of the axiomatic method was contained in Euclid's well-known book, *The Elements*, which starts with his famous set of five postulates. From these postulates, he goes on to deduce most of the mathematics known at that time.

Euclid's original postulates are shown in Table 1 [9]. As one could imagine, Euclid's original wording can be difficult to understand. Since Euclid's initial introduction, the axioms have been reworded. An equivalent, more modern, wording of the postulates is also shown in Table 1 [4]. For our discussion, the most important of these axioms is the fifth one, also known as the *Euclidean Parallel Postulate.*

For over a thousand years, Euclid's *Elements* was the most important and most frequently studied mathematical text and so his postulates continued to be the foundation of nearly all mathematical knowledge. But from the very beginning his fifth postulate was controversial. The first four postulates appeared self-evidently true; however, the fifth postulate was thought to be redundant and it was thought that it could be proven from the first four. In other words, there was question as to whether it was necessary to assume the fifth postulate or could it be derived as a consequence of the other four. Why was the fifth postulate so controversial?

In Marvin Jay Greenberg's book *Euclidean and Non-Euclidean Geometries*, a detailed history of the Euclidean Parallel Postulate is given, including a large portion dedicated to the many attempts to prove the fifth postulate. According to Greenberg, it was difficult to accept the postulate because "we cannot verify empirically whether two drawn lines meet since we can draw only segments, not complete lines," whereas the other four postulates seemed reasonable from experience working with compasses and straightedges [5]. Many famous mathematicians, such as Proclus and Adrien-Marie Legendre, thought they had discovered correct proofs. However, all such attempts proved unsuccessful and ended in failure as holes and gaps were found in their reasoning.

Although the mathematicians were unable to prove the fifth postulate, the efforts of mathematicians such as Bolyai, Lobachevsky, and Gauss with respect to the fifth postulate were rewarded. Their "consideration of alternatives to Euclid's parallel postulate resulted in the development of non-Euclidean geometries" [5]. One of these non-Euclidean geometries is now called *hyperbolic* and is the main subject of this paper. We will

	Original Postulate	Reworded Postulate
I.	To draw a straight line from any point to any point.	For every point P and for every point Q not equal to P, there exists a unique line ℓ that passes through P and Q.
II.	To produce a finite straight line continuously in a straight line.	For every segment AB and for every segment CD, there exists a unique point E such that B is between A and E and segment CD is congruent to segment BE.
III.	To describe a circle with any center and distance.	For every point O and every point A not equal to O, there exists a circle with center O and radius OA.
IV.	That all right angles are equal to one another.	All right angles are congruent to each other.
V.	That if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines, if produced indefinitely, will meet on that side on which the angles are less than two right angles.	For every line ℓ and for every point P that does not lie on ℓ, there exists a unique line m through P that is parallel to ℓ.

Table 1: Euclid's five postulates.

not have the time or space for a detailed development of hyperbolic geometry, so we will start by describing two models of it: the Poincaré model and the Klein model. The Klein model, though it is easier to describe, is ultimately harder to use; so our main focus will be on the Poincaré model.

Our primary objective is to discuss trigonometry in the context of hyperbolic geometry. So we will need to consider the hyperbolic trigonometric functions and how they relate to the traditional circular trigonometric functions. In particular, we will introduce the *angle of parallelism* in hyperbolic geometry, which provides a direct link between the circular and hyperbolic functions. Then, we will use this connection to explore triangles, circles, and quadrilaterals in hyperbolic geometry and how familiar formulas in Euclidean geometry correspond to analogous formulas in hyperbolic geometry.

In fact, besides hyperbolic geometry, there is a second non-Euclidean geometry that can be characterized by the behavior of parallel lines: *elliptic* geometry. The three types of plane geometry can be described as those having constant curvature; either negative (hyperbolic), positive (spherical), or zero (Euclidean). Spherical geometry is intimately related to elliptic geometry and we will show how many of the formulas we consider from Euclidean and hyperbolic geometry also correspond to analogous formulas in spherical geometry. For example, there are three different versions of the Pythagorean Theorem; one each for hyperbolic, Euclidean, and spherical right triangles.

2 Models of the Hyperbolic Plane

Hyperbolic geometry is a non-Euclidean geometry in which the parallel postulate from Euclidean geometry (refer to Chapter 4) is replaced. As a result, in hyperbolic geometry, there is more than one line through a certain point that does not intersect another given line. Since the hyperbolic plane is a plane with constant negative curvature, the fact that two parallel lines exist to a given line visually makes sense. This curvature results in shapes in the hyperbolic plane differing from what we are used to seeing in the Euclidean plane. For example, in Figure 6 we can see different forms of a triangle in the hyperbolic plane. These triangles are different than typical triangles we are used to seeing in the Euclidean plane. Luckily, there are different models that help us visualize the hyperbolic plane.

In this section, we will cover two of the main models of hyperbolic geometry, namely the Poincaré and Klein models. For each model, we will define points, lines, and betweenness, along with distance and angle measures. It is important to note that, unless otherwise specified, we will be working in the Poincaré model.

2.1 Poincaré Model

The Poincaré Model is a disc model used in hyperbolic geometry. In other words, the Poincaré Model is a way to visualize a hyperbolic plane by using a unit disc (a disc of radius 1). While some Euclidean concepts, such as angle congruences, transfer over to the hyperbolic plane, we will see that things such as lines are defined differently.

2.1.1 Points, Lines, and Betweenness

Consider a unit circle γ in the Euclidean plane.

Definition 2.1. In the hyperbolic plane, *points* are defined as the points interior to γ.

In other words, all hyperbolic points are in the set $\{(x, y)|x^2 + y^2 < 1\}$. In Figure 1, P and Q are examples of hyperbolic points whereas R is not a point in the hyperbolic plane since it lies outside the unit disc γ.

Definition 2.2. *Lines* of the hyperbolic plane are the diameters of γ and arcs of circles that are perpendicular to γ.

Note that in Figure 1, ℓ is a diameter of γ, hence ℓ is a line in the hyperbolic plane. Similarly, circle δ is perpendicular to γ and therefore, m is considered a line in the hyperbolic plane. In this case, we were given the fact that circle δ is orthogonal to circle γ but this will not always be known. In order to determine when a circle is perpendicular to γ in the Poincaré model, we need to define the *inverse* of a point.

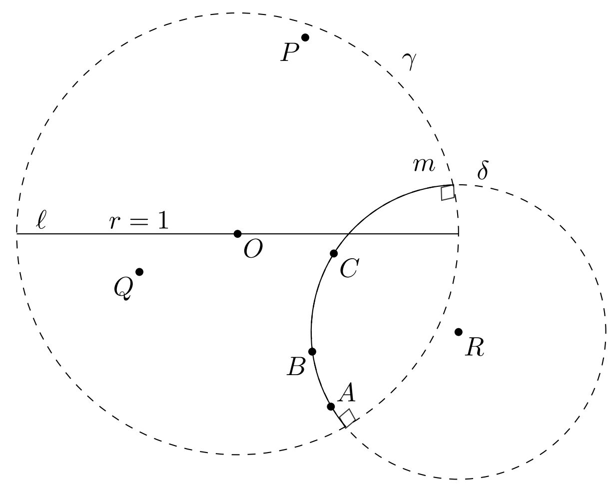


Figure 1: Poincaré Model for hyperbolic geometry where P and Q are hyperbolic points and ℓ and m are hyperbolic lines.

Definition 2.3. Let γ be a circle of radius r with center O. For any point $P \neq O$ the *inverse* P' of P with respect to γ is the unique point P' on ray $\overrightarrow{OP}$ such that $(OP)(OP') = r^2$.

Since we are working the the Poincaré model, the radius r is always equal to 1. This modifies the definition above; two points P and P' are inverses if $(OP)(OP') = 1$. Refering to Figure 2, point P has inverse P'. The following proposition allows us to determine if a circle is orthogonal to γ or not.

Proposition 2.4. *Let P be any point that does not lie on circle γ and that does not coincide with the center O of γ, and let δ be a circle through P. Then δ cuts γ orthogonally if and only if δ passes through the inverse point P' of P with respect to γ.*

This is to say that if P, a point in the circle γ not equal to the origin O, and it's inverse P', a point outside the circle γ, lie on the same circle δ, then δ is orthogonal to γ and hence the sector of δ that lies inside of γ is a line in the Poincaré model [1]. This is shown in Figure 2. We conclude this collection of definitions with the notion of betweenness.

Definition 2.5. Let A, B, and C be on an open arc m coming from an orthogonal circle δ with center R. We define B to be *between* A and C if $\overrightarrow{RB}$ is between $\overrightarrow{RA}$ and $\overrightarrow{RC}$.

Betweenness is one of the several things that is defined the same in both the Euclidean and hyperbolic planes.

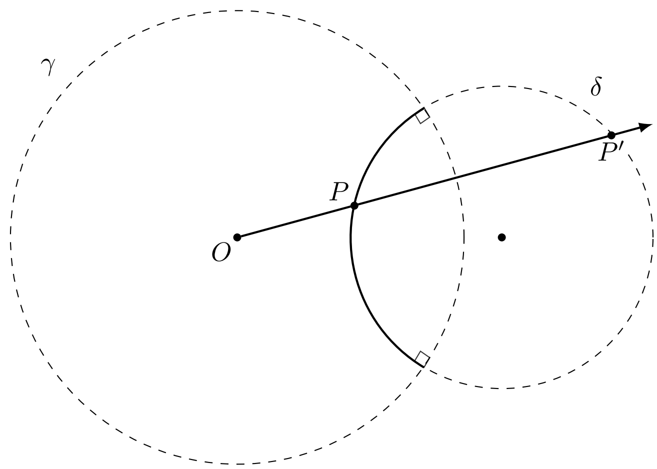
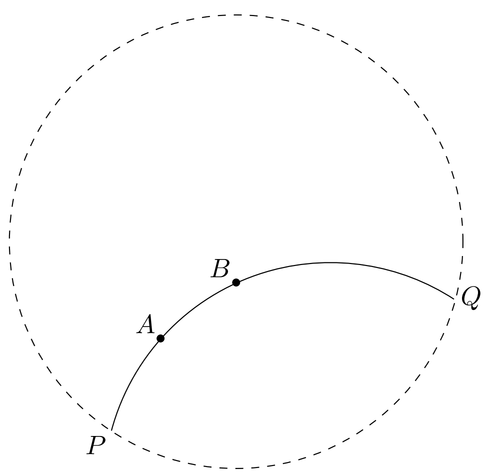
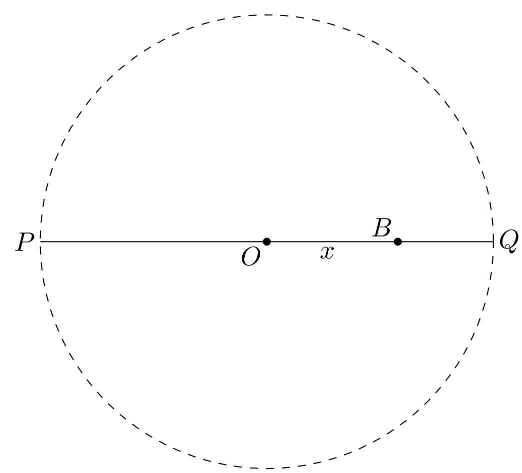


Figure 2: The inverse of a point. In particular, point P' is the inverse of point P. As a result, we conclude that circle δ is perpendicular to circle γ.



(a) Cross-ratio of four distinct points.



(b) Poincaré distance from the origin.

Figure 3: Poincaré distance.

2.1.2 Poincaré Distance

In the hyperbolic plane, distance from one point to another is different than what we call distance in the Euclidean plane. In order to determine the distance, we must first define *cross-ratio.*

Definition 2.6. If P, Q, A, and B are distinct points in $\mathbb{R}^2$, then their *cross-ratio* is

$$[P, Q, A, B] = \frac{PB \cdot QA}{PA \cdot QB}$$

where PB, QA, PA, and QB are the Euclidean lengths of those segments.

Example 2.7. Referring to Figure 3a, suppose $PB = 1$, $QA = 3/2$, $PA = 1/2$, and $QB = 1$. We then have

$$\begin{aligned}[P, Q, A, B] &= \frac{PB \cdot QA}{PA \cdot QB} \\ &= \frac{3/2}{1/2} \\ &= 3.\end{aligned}$$

The cross-ratio of four distinct points is important when we want to find the Poincaré length of a line segment.

Definition 2.8. If P, Q, A, and B are distinct points in $\mathbb{R}^2$, then in hyperbolic geometry, the *Poincaré length* $d(A, B)$ is defined as

$$d(A, B) = |\ln([P, Q, A, B])|.$$

We can interpret this formula for the Poincaré distance in an interesting way by applying it to the diameter of a circle. Consider Figure 3b. If we let that be a circle of radius 1 with center O, then $d(B, O) = |\ln([P, Q, O, B])|$. We then have

$$d(B, O) = \left|\ln\left(\frac{PB \cdot QO}{PO \cdot QB}\right)\right|.$$

Note that we are denoting the Euclidean distance from the origin O to the point B as x. Similarly, since this is a circle of radius 1, we know that the Euclidean distance from P to Q is 2. We can see that the Euclidean length of PB is $(1 + x)$ and that of QO is 1. Hence, $PB \cdot QO = (1 + x)$. Similarly, we find that $PO \cdot QB = (1 - x)$. Therefore,

$$d(B, O) = \left|\ln\left(\frac{1 + x}{1 - x}\right)\right|.$$

This yields the following theorem:

Theorem 2.9. *If a point B inside the unit disc is at a Euclidean distance x from the origin O, then the Poincaré length from B to O is given by*

$$d(B, O) = \left|\ln\left(\frac{1 + x}{1 - x}\right)\right|.$$

Note, the Euclidean distance $x = OB$ can never be equal to 1 because B is a point in the Poincaré model and thus is inside the unit circle. However, as x approaches 1, the Poincaré length from B to O is going off to infinity.

2.1.3 Angles

Another similarity between the Euclidean and hyperbolic planes is angle congruence. This has the same meaning in both planes. For the Poincaré model, since lines can be circular arcs, we need to define how to find the measure of an angle.

In the hyperbolic plane, the way we find the degrees in an angle is conformal to the Euclidean plane. In the Poincaré model, we have three cases to consider:

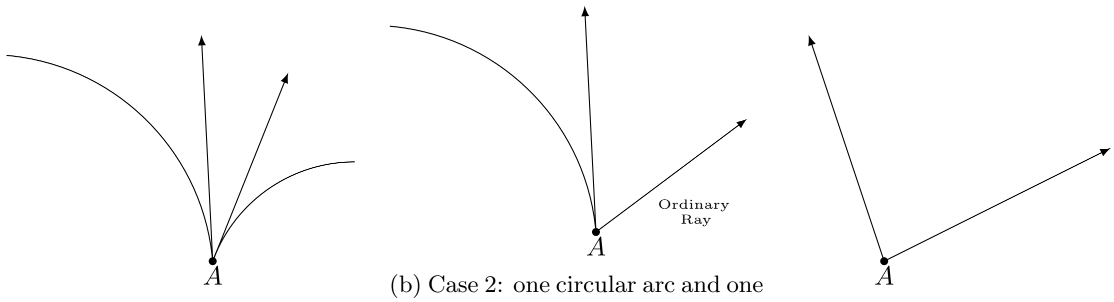
- Case 1: where two circular arcs intersect
- Case 2: where one circular arc intersects an ordinary ray
- Case 3: where two ordinary rays intersect

Note that an ordinary ray, is a ray as we think of it in the Euclidean plane. More formally, an ordinary ray is a line that starts at a point and goes off in a certain direction to infinity.

Consider case 1. If two circular arcs intersect at a point A, the number of degrees in the angle they make is the number of degrees in the angle between their tangent rays at A. Refer to Figure 4a.

For case 2, suppose one circular arc intersects an ordinary ray at a point A, the number of degrees in the angle they make is the number of degrees in the angle between the tangent ray of the circular arc at A and the ordinary ray at A. Refer to Figure 4b.

Lastly, for case 3, the angle between two ordinary rays that intersect at a point A is interpreted the same as the degrees of an angle in the Euclidean plane. Refer to Figure 4c.



(a) Case 1: two circular arcs (b) Case 2: one circular arc and one ordinary ray (c) Case 3: two ordinary rays

Figure 4: Degrees of an angle in the Poincaré model.

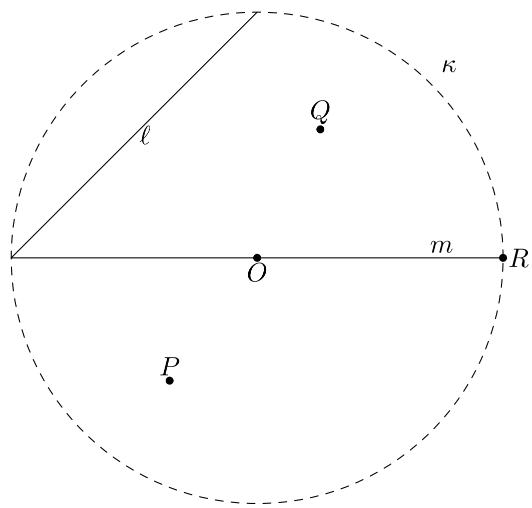


Figure 5: Klein model for hyperbolic geometry where P and Q are points and ℓ and m are lines.

2.2 Klein Model

The Klein Model (also known as the Beltrami-Klein Model) is a disc model of hyperbolic geometry via projective geometry. This model is a projective model because it is derived using stereographic projection [8]. Within this model, there is a fixed circle κ in a Euclidean plane. This model is very similar to the Poincaré model defined in Section 2.1. Recall, in the Poincaré model, circle γ has radius 1 because it is a unit disc model. Here, circle κ in the Klein model does not have a fixed radius. We will see in Section 2.2.1 that definitions of points and lines also differ between the two models.

2.2.1 Points and Lines

Let κ be a circle in the Euclidean plane with center O and radius OR, see Figure 5.

Definition 2.10. In the hyperbolic plane, *points* are defined as all points X such that $OX < OR$.

In Figure 5, P and Q are examples of points in the Klein model. In contrast, point R is not considered to be a point because $OR = OR$ thus $OR \not< OR$. This idea of points in the Klein model is very similar to points in the Poincaré model. The difference in definitions is because the radius OR in the Klein model is not fixed to 1. The main distinction between the two models is how lines are defined.

Definition 2.11. *Lines* of the hyperbolic plane are chords inside circle κ excluding their endpoints.

Comparing Definitions 2.2 and 2.11, we can see that in the Klein model, rather than lines being arcs of circles orthogonal to γ, lines are the chords within the circle κ. Note that the set of chords of κ also includes diameters of κ. Thus, in Figure 5, ℓ and m are lines in the Klein model.

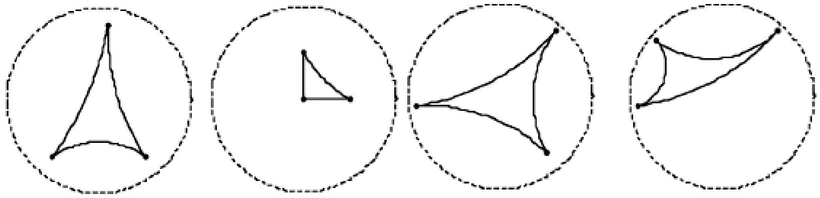


Figure 6: Examples of hyperbolic triangles in the Poincaré model.

(taken from http://euler.slu.edu/escher/index.php/Hyperbolic_Geometry)

As mentioned in Section 2.1.1, betweenness in the hyperbolic plane is the same as in the Euclidean plane and thus betweenness in the Klein model is defined the same as betweenness in the Poincaré model (refer to Definition 2.5).

2.2.2 Distance in the Klein Model

Finding a distance between two points in the Klein model differs from that in the Poincaré model but in both cases, we use the cross-ratio (refer to Definition 2.6).

Definition 2.12. Let A and B be two points in circle κ and P and Q be the endpoints of the chord AB. Then, the *Klein distance* $d_k(A, B)$ between points A and B is

$$d_k(A, B) = \frac{|\ln([P, Q, B, A])|}{2}.$$

We see that the Klein distance is the Poincaré distance divided by 2.

2.2.3 Angle Measurements in the Klein Model

As mentioned in Section 2.1.3, finding the measurement of an angle in the hyperbolic plane is conformal to the Euclidean plane. In the Poincaré model, we had to consider three cases, however, for the Klein model it is much more complicated. The Klein model is only conformal at the origin [8]. As a result, finding the measurement of angles at the origin is the same as finding them in the Euclidean plane. The difficulty begins when an angle is not at the origin. In Section 5, we introduce an isomorphism between the Klein and Poincaré models. This isomorphism allows us to map lines and points from the Klein model into the Poincaré model. Hence, to ease the process of measuring an angle in the Klein model, we will map that angle into the Poincaré model and then measure it there.

3 Hyperbolic Trigonometry

Trigonometry is the study of the relationships between the angles and the sides of a triangle [6]. Before digging deeper, we will cover the general notation that will be used. For $\triangle ABC$ with sides a, b, and c, we will use the notation $a = BC$, b=AC, and $c = AB$ for the lengths of the sides. That is to say that $\angle A$ is opposite side a, $\angle B$ is opposite side b, and $\angle C$ is opposite side c.

In the Euclidean plane, the idea of similar triangles was used to help define the sine, cosine, and tangent of an acute angle in a right triangle. From these definitions, we were able to extend the same ideas to find the cosecant, secant, and cotangent of such an angle. For example, in the Euclidean plane, given a right triangle ABC where $\angle C$ is the right angle, we define $\cos(A)$ as the ratio of the adjacent side to the hypotenuse. In other words, $\cos(A) = b/c$.

In the hyperbolic plane, triangles come in all different forms. Some examples are shown in Figure 6. As a result, the Euclidean ratios no longer hold true in all cases and hence, we define trigonometric functions differently in the hyperbolic plane. For the circular functions, their definitions are in terms of their Taylor series expansions:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

(and $\tan x = \sin x / \cos x$, etc.). Trigonometry in the hyperbolic plane not only involves the circular functions but also the *hyperbolic functions* defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \tag{1}$$

(and $\tanh x = \sinh x / \cosh x$, etc.). Similar to the circular functions, these hyperbolic functions can also be defined using the Taylor series:

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \qquad \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}. \tag{2}$$

By comparing the Taylor series expansions for the circular functions and the hyperbolic functions, we can see that the hyperbolic functions are the circular functions without the coefficients $(-1)^n$. The name "hyperbolic functions" comes from the hyperbolic

$$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1, \tag{3}$$

from which the parametric equations $x = \cosh\theta$ and $y = \sinh\theta$ give one part of the hyperbola $x^2 - y^2 = 1$ in the Cartesian plane [6]. It is important not to confuse θ in this sense with θ in the Euclidean plane. Here (refer to Figure 8), θ geometrically represents twice the area bounded by the hyperbola, x-axis, and the line joining the origin to the point $(\cosh\theta, \sinh\theta)$. By examining the unit hyperbola in the next section, we will have a better idea of what θ is in the hyperbolic sense. In contrast to the hyperbolic θ, in the Euclidean plane, θ can be used to represent the angle.

3.1 Unit Circle and Unit Hyperbola

As we have seen, the circular and hyperbolic trigonometric functions differ. In this section, we will see that there is an interesting similarity between the two when considering the area of a sector of a unit circle and the area between the x-axis and a hyperbolic sector drawn to the parabola $y = \sqrt{x^2 - 1}$, both in the Euclidean plane.

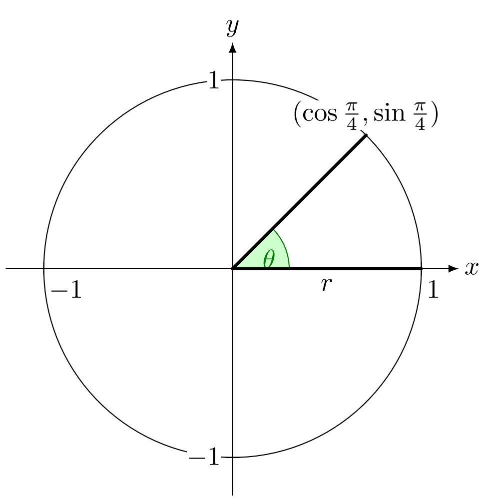


Figure 7: Unit circle in the Euclidean plane.

Consider the unit circle. Note that a point on the unit circle has coordinates $(\cos\theta, \sin\theta)$. The following example illustrates the method of finding the area of a sector of the unit circle.

Example 3.1. For the example, refer to Figure 7. We want to find the area of the sector formed when $\theta = \pi/4$ of the unit circle. The arc length of a circle is $\theta \cdot r$. So, for this example,

$$\text{arc length} = \pi/4 \cdot 1 = \pi/4.$$

The area of a sector of a circle is (arc length $\cdot\, r/2$). We then have,

$$\text{Sector Area} = \pi/4 \cdot 1/2 = \pi/8.$$

Hence, when $\theta = \pi/4$ the area of the sector of the unit circle is $\pi/8$.

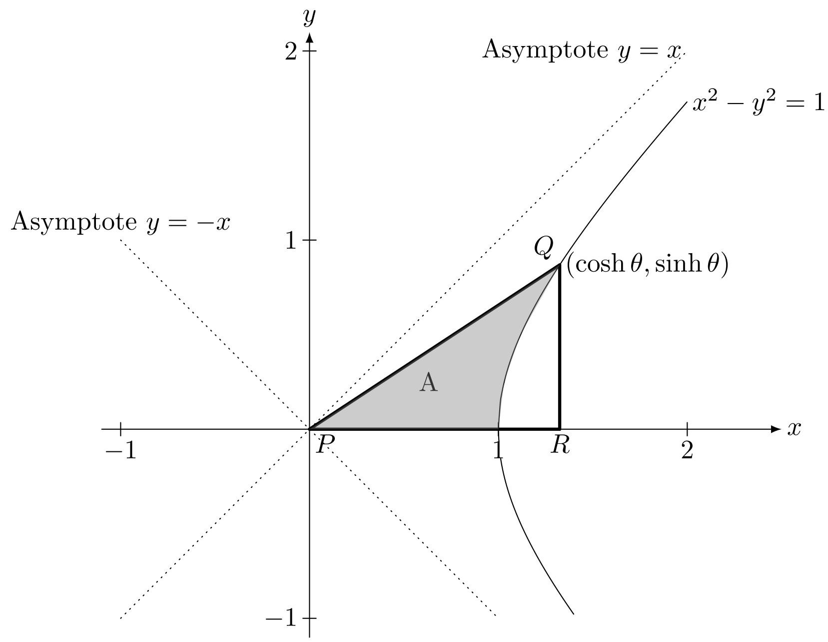


Figure 8: Unit hyperbola in the Euclidean plane. The area of the shaded region A is $\theta/2$.

Since the unit circle has a radius of 1, the formula for the area of a sector of the unit circle is $\theta/2$, as shown in Example 3.1.

Now, rather than using the unit circle, we are going to use the curve $x^2 - y^2 = 1$, which is called the unit hyperbola. We are interested in the area bounded by this curve, a hyperbolic sector to the curve, and the x-axis.

This area of interest is shown in Figure 8 and is the shaded region labeled A. To find this area, note that we will take the area of $\triangle PQR$ and subtract the area under the curve $x^2 - y^2 = 1$ from $x = \cosh 0$ to $x = \cosh\theta$. We then have

$$\text{Area} = \frac{\sinh\theta \cdot \cosh\theta}{2} - \int_{\cosh 0}^{\cosh\theta} \sqrt{x^2 - 1}\, dx.$$

We then substitute $\theta = \cosh^{-1}(x)$ which implies $x = \cosh\theta$ and hence $dx = \sinh\theta\, d\theta$. Applying equation (1), we have

$$\begin{aligned}
\text{Area} &= \frac{\sinh\theta \cdot \cosh\theta}{2} - \int_0^\theta \sinh^2\theta\, d\theta \\
&= \frac{2 \cdot \sinh\theta \cdot \cosh\theta}{4} - \int_0^\theta \frac{e^{2\theta} - 2 + e^{-2\theta}}{4}\, d\theta \\
&= \frac{\sinh(2\theta)}{4} - \left(\frac{\sinh(2\theta) - 2\theta}{4}\right)\Bigg|_0^\theta \\
&= \frac{\theta}{2}.
\end{aligned}$$

The similarity is that the area we found with respect to the unit circle is the same as the area we found with respect to the unit hyperbola. Both areas are equal to $\theta/2$.

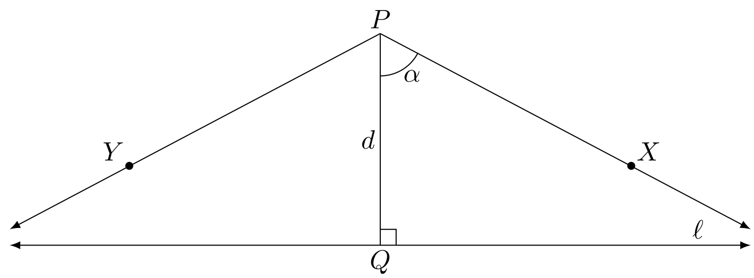


Figure 9: The angle of parallelism is α.

4 The Angle of Parallelism

While the result in Section 3.1 provides an interesting similarity between the unit circle and the unit hyperbola, the *angle of parallelism* provides a direct link between the circular and hyperbolic functions. In this section, we will present a theorem that allows us to directly solve for the angle of parallelism.

Recall that in the hyperbolic plane, the parallel postulate from the Euclidean plane is replaced with the Hyperbolic Parallel Postulate.

The Euclidean Parallel Postulate. *For every line ℓ and for every point P that does not lie on ℓ, there exists a unique line m through P that is parallel to ℓ.*

This Euclidean Postulate establishes the uniqueness of parallel lines. This part of the postulate differs in the hyperbolic plane [2].

The Hyperbolic Parallel Postulate. *Given any line ℓ and any point P not on ℓ, there exist more than one line M such that P is on M and M is parallel to ℓ.*

As a result of the existence of more than one parallel line in the hyperbolic plane, the following theorem holds.

Theorem 4.1. *Given any line ℓ and any point P not on ℓ, there exist limiting parallel rays $\overrightarrow{PX}$ and $\overrightarrow{PY}$.*

This is shown in Figure 9. The important thing to remember about these two limiting parallel rays is that they are situated such that they are symmetric about the perpendicular line PQ to ℓ, where Q lies on line ℓ. It can be shown that $\angle XPQ \cong \angle YPQ$. From this congruence relation, we conclude that either of these angles can be called the *angle of parallelism* for P with respect to ℓ [6].

Theorem 4.2. *Let α be the angle of parallelism for P with respect to ℓ and d be the Euclidean distance from P to Q, where PQ is perpendicular to ℓ. We then have, the formula of Bolyai-Lobachevsky:*

$$\tan\left(\frac{\alpha}{2}\right) = e^{-d}. \tag{4}$$

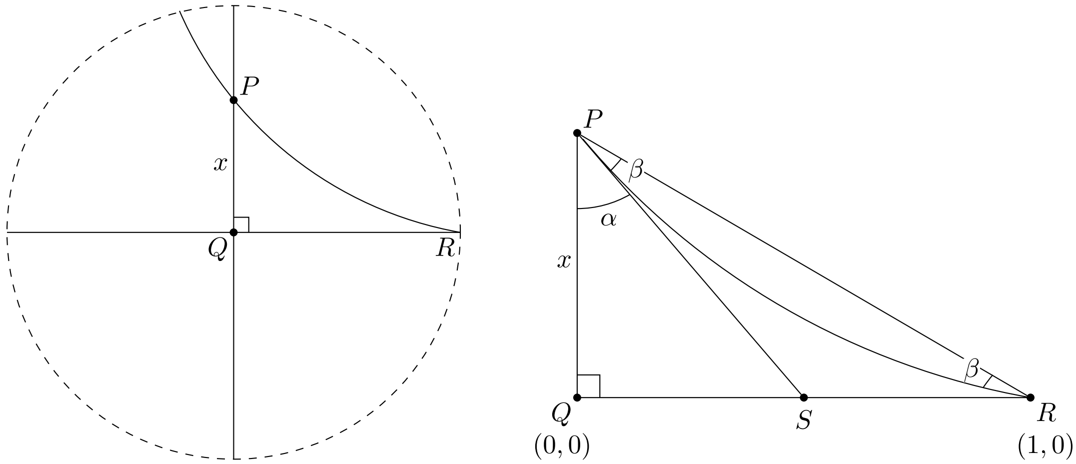
Proof. Consider Figure 10a. We have a circular arc that contains points P and R. With Q being the origin of the unit disc, we can draw a triangle in the Euclidean sense, namely $\triangle$QPR. This triangle is shown in Figure 10b. Note that S is the point of intersection between QR and the tangent line to the circular arc at point P.

Recall that $d = \left|\ln((1+x)/(1-x))\right|$. Note, $-d = \left|\ln((1-x)/(1+x))\right|$. Hence,

$$e^{-d} = \left(\frac{1-x}{1+x}\right).$$

We then have, $\angle SPR = (1/2)\widehat{PR}$ and $\angle SRP = (1/2)\widehat{PR}$ thus, $\angle SPR \cong \angle SRP \cong \beta$. Then, using $\triangle$QPR,

$$\begin{aligned} \pi &= \pi/2 + \alpha + 2\beta \\ \pi/4 - \beta &= \alpha/2. \end{aligned}$$



(a) (b)

Figure 10: Proof of the Bolyai-Lobachevsky formula. Note that the triangle in (b) is the same triangle $\triangle PQR$ shown in (a).

Recall that

$$\tan(\theta - \gamma) = \frac{\tan(\theta) - \tan(\gamma)}{1 + \tan(\theta)\tan(\gamma)}.$$

Applying this formula, we have

$$\begin{aligned} \tan(\alpha/2) &= \tan(\pi/4 - \beta) \\ &= \frac{1 - \tan(\beta)}{1 + \tan(\beta)} \\ &= \frac{1 - x}{1 + x}. \end{aligned}$$

Therefore,

$$e^{-d} = \frac{1 - x}{1 + x} = \tan\left(\frac{\alpha}{2}\right).$$

$\square$

To see how to apply Formula (4), we will do an example.

Example 4.3. Suppose that we want to find the distance d when $\alpha = \pi/3$ in Figure 9. We then have,

$$\tan\left(\frac{\pi/3}{2}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3} = e^{-d}.$$

Therefore,

$$\begin{aligned} -d &= \ln\left(\frac{\sqrt{3}}{3}\right) \\ d &= -(1/2 \cdot \ln(3) - \ln(3)) \\ &\approx .55. \end{aligned}$$

This means that if $\alpha = \pi/3$ then the distance d from point P to Q is approximately .55.

It is important to note that in formula (4), the angle of parallelism α is dependent on the Euclidean distance d. If we look closer at formula (4), in particular as the Euclidean distance d goes to 0, then we have

$$\begin{aligned}\lim_{d\to 0} e^{-d} &= \lim_{d\to 0} \tan\left(\frac{\alpha_d}{2}\right)\\ 1 &= \lim_{d\to 0} \tan\left(\frac{\alpha_d}{2}\right).\end{aligned}$$

This implies that as $d \to 0$, $\alpha_d \to \pi/2$. In other words, as the distance d between points P and Q in Figure 9 goes to 0, the angle of parallelism α is getting closer to $\pi/2 = 90°$. Therefore, parallel lines in the hyperbolic plane are looking like parallel lines in the Euclidean plane. We can also transfer this idea to hyperbolic triangles and say that if the sides of the triangle are sufficiently small, then the triangle looks like a regular Euclidean triangle. We will see this in more detail in Section 5.

In contrast, if we look at formula (4) as d goes to ∞, then we see that $\alpha_d \to 0$. So, as the Euclidean distance d between points P and Q in Figure 9 gets infinitely large, the limiting parallel ray $\overrightarrow{PX}$ essentially aligns with line PQ.

4.1 Alternative Forms of the Bolyai-Lobachevsky Formula

The Bolyai-Loabachevsky formula is "certainly one of the most remarkable formulas in all of mathematics." [6] This formula relates the angle of parallelism to distance. By simply rewriting the formula in a different way, we are able to also provide a link between hyperbolic and circular functions. In this section, we are going to establish that this relationship exists by considering the sine, cosine, and tangent of the angle of parallelism.

Note that Lobachevsky denoted α as $\Pi(d)$. From now on, we will use this notation because it makes it clear that the angle of parallelism relies on the hyperbolic distance d. By manipulating equation (4), we find that the radian measure of the angle of parallelism becomes

$$\Pi(d) = 2 \cdot \arctan\left(e^{-d}\right).$$

Theorem 4.4. *Let $\Pi(x)$ be the angle of parallelism and x be the hyperbolic distance. Then,*

$$\sin(\Pi(x)) = \operatorname{sech}(x) = 1/\cosh(x), \tag{5}$$

$$\cos(\Pi(x)) = \tanh(x), \tag{6}$$

$$\tan(\Pi(x)) = \operatorname{csch}(x) = 1/\sinh(x). \tag{7}$$

Proof. Note, we will use double angle formulas and substitution. If we let $y = \arctan\left(e^{-x}\right)$, then $\tan(y) = e^{-x}$. So, $\sec^2(y) = \tan^2(y) + 1$ becomes $\sec^2(y) = e^{-2x} + 1$. We then have

$$\begin{aligned}\frac{1}{\sec^2(y)} &= \frac{1}{e^{-2x}+1}\\ \cos(y) &= \frac{1}{(e^{-2x}+1)^{1/2}}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sin(y) &= \tan(y)\cos(y)\\ \sin(y) &= \frac{e^{-x}}{(e^{-2x}+1)^{1/2}}.\end{aligned}$$

Now, note that the double angle formula for sine is $\sin(2y) = 2 \cdot \sin(y) \cdot \cos(y)$. Therefore, since $\Pi(x) = 2 \cdot \arctan\left(e^{-x}\right) = 2 \cdot y$, we have

$$\begin{aligned}
\sin(\Pi(x)) &= \sin(2 \cdot y) \\
&= 2\sin(y)\cos(y) \\
&= 2 \cdot \frac{e^{-x}}{(e^{-2x}+1)^{1/2}} \cdot \frac{1}{(e^{-2x}+1)^{1/2}} \\
&= \frac{2}{e^{x}+e^{-x}} \\
&= \operatorname{sech}(x).
\end{aligned}$$

Since $1/\cosh(x) = \operatorname{sech}(x)$, this proves equation (5). Recall that the double angle formula for cosine is $\cos(2y) = \cos^2(y) - \sin^2(y)$. Then,

$$\begin{aligned}
\cos(\Pi(x)) &= \cos(2 \cdot y) \\
&= \cos^2(y) - \sin^2(y) \\
&= \frac{1}{(e^{-2x}+1)} - \frac{e^{-2x}}{(e^{-2x}+1)} \\
&= \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} \\
&= \frac{\sinh(x)}{\cosh(x)} \\
&= \tanh(x).
\end{aligned}$$

This proves equation (6). Lastly,

$$\begin{aligned}
\tan(\Pi(x)) &= \frac{\sin(\Pi(x))}{\cos(\Pi(x))} \\
&= \frac{\operatorname{sech}(x)}{\tanh(x)} \\
&= \operatorname{csch}(x).
\end{aligned}$$

Since $1/\sinh(x) = \operatorname{csch}(x)$, this proves equation (7). □

We conclude that the function Π provides a link between the hyperbolic and the circular functions.

5 Hyperbolic Identities

In the Euclidean plane, there are many trigonometric identities. These identities are equations that hold for all angles. In the hyperbolic plane, there are corresponding trigonometric identities that involve both circular and hyperbolic functions. In this section, we will establish an isomorphism between the Klein model and the Poincaré model. This isomorphism will help with our proof of a few hyperbolic identities.

While preference for the Klein or Poincaré model varies, there is a helpful isomorphism between the two that preserves the incidence, betweenness, and congruence axioms. A one-to-one correspondence can be set up between the "points" and "lines" in one model to the "points" and "lines" in the other [6].To establish the isomorphism between the Klein and Poincaré models, we will start with the Klein model. That means we have a circle κ with center O and radius r. In the Euclidean three dimensional space, consider a sphere, also with radius r, sitting on the Klein model such that it is tangent to the origin O. For a visual aid, refer to Figure 11. We then project the entire Klein model upward onto the lower hemisphere of the sphere. This will cause all of the chords in the Klein model to become arcs of circles that are orthogonal to the equator of the sphere. In Figure 11, we can see an example of this projection. The chord PR is projected upward and becomes the arc $P'R'$. We now connect the north pole of the sphere to each point on the arcs of circles that were created by the chords of the Klein model and project them onto the original plane. In the figure, this creates the line $\hat{P}\hat{R}$. The projection of the equator, using the same process, will create a circle larger than the original circle κ. The projection of the lower hemisphere will land inside this new circle. The new, larger circle creates the Poincaré model. By doing this transformation successively, the original chords and points of the Klein model will be mapped one-to-one onto the 'lines' and points in the Poincaré model.

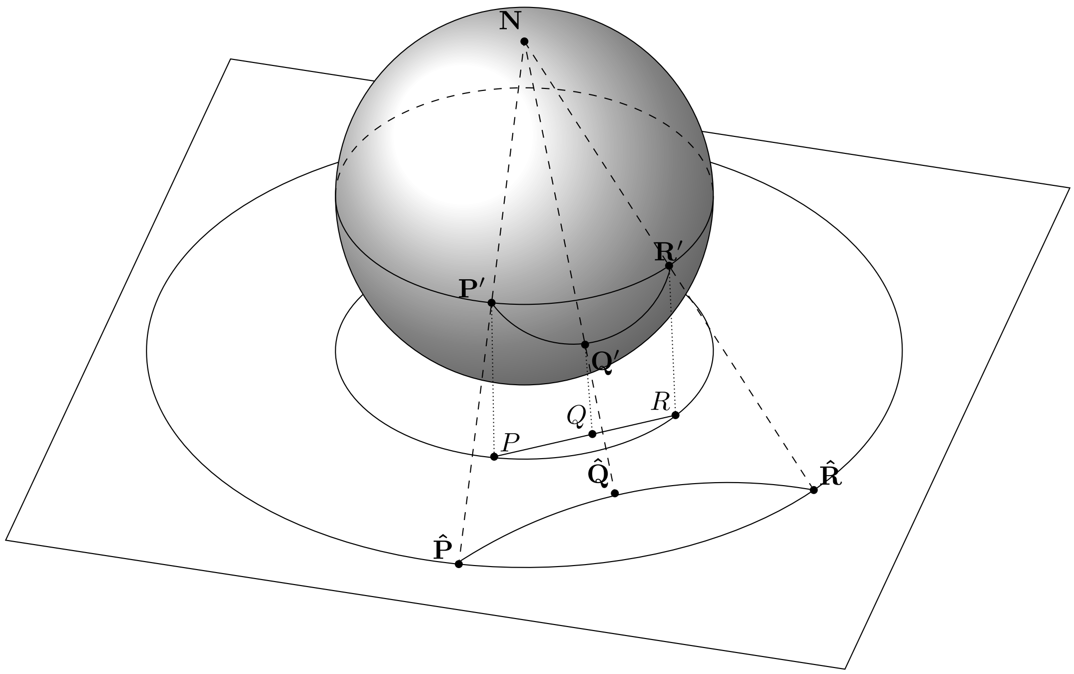


Figure 11: Isomorphism between the Klein and Poincaré models

In general, this shows that an isomorphism between the planes exists. Using equations (8) and (9), we will be more specific and define an isomorphism F. Let κ with center O be a circle of radius 1 and B be a point within the circle. Recall,

$$e^{d(O,B)} = \frac{1+OB}{1-OB}.$$

Note, for brevity, let $x = d(O,B)$ and $t = OB$ in this formula above. Then,

$$\sinh x = \frac{2t}{1-t^2} \qquad \text{and} \qquad \cosh x = \frac{1+t^2}{1-t^2}, \tag{8}$$

so that

$$\tanh x = \frac{2t}{1+t^2}. \tag{9}$$

Let $F(t) = (2t)/(1+t^2)$. We claim that F is the above isomorphism. Recall the definition of the inverse of a point in the hyperbolic plane, Definition 2.3. By Proposition 2.4, if a point in the Poincaré model lies on an orthogonal arc in the Poincaré model, then we can conclude that the corresponding inverse point lies also on the circle containing the arc but outside of the Poincaré model. Now, consider Figure 12.

We will start by showing that F is this isomorphism when considering a point that bisects the chord, which is point A. We will then show that this isomorphism holds for any point along the chord. First, let the distance from point O to point B be equal to t. Since Q is the inverse of B, we know that the distance from O to Q is $1/t$. Similarly, the distance from O to P is the average of the distances from O to B and O to Q. Therefore, $OP = (1/t + t)/2 = (1 + t^2)/2t$. Now, we want to show that the distance from point O to point A is equal to $2t/(1 + t^2)$. To show this, note that we have similar triangles $\triangle OSP$ and $\triangle OAS$. We then have

$$\frac{OA}{OQ} = \frac{OQ}{OP}.$$

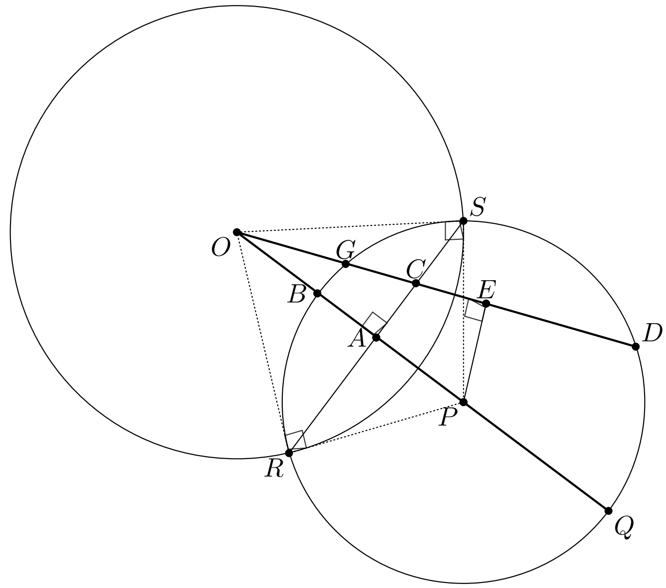


Figure 12: Isomorphism F from the Poincaré model to the Klein model

And hence, $OA \cdot OP = 1$. We can conclude that A and P are inverses. Therefore, $OA = 2t/(1+t^2)$, which is what we wanted to show.

Now, we want to show that this relationship holds for any point on the chord RS. Let $OG = s$. Similar to the argument above, since D is the inverse of G, we know that $OD = 1/s$. The distance OE is the average of the distances OG and OD. Therefore, $OE = (1/s + s)/2 = (1+s^2)/2s$. Again, we have similar triangles $\triangle OAC$ and $\triangle OEP$. We then have

$$\frac{OC}{OA} = \frac{OP}{OE}.$$

Since we have established that points A and P are inverses, we know $OA \cdot OP = 1$. Hence, $OC \cdot OE = OA \cdot OP = 1$. We can conclude that C and E are inverses. Therefore, $OC = 2s/(1+s^2)$, which is what we wanted.

Thus, F is the isomorphism from the Poincaré model to the Klein model. This isomorphism will be helpful when trying to prove Theorem 5.1 in Section 5.1.

5.1 Right Triangle Trigonometric Identities

In the Euclidean plane, there are certain identities that can only be applied to a triangle containing a right angle. Similarly, in the hyperbolic plane, some identities only hold for right triangles. In this section, we present a theorem that contains three identities in which the triangle must contain a right angle in order for the identities to be applied.

Theorem 5.1. *Given any right triangle $\triangle ABC$, with $\angle C$ being the right angle, in the hyperbolic plane. Let a, b, and c denote the hyperbolic lengths of the corresponding sides. Then*

$$\sin A = \frac{\sinh a}{\sinh c} \qquad \textit{and} \qquad \cos A = \frac{\tanh b}{\tanh c}, \tag{10}$$

$$\cosh c = \cosh a \cdot \cosh b = \cot A \cdot \cot B, \tag{11}$$

$$\cosh a = \frac{\cos A}{\sin B}. \tag{12}$$

Proof. We will show that formulas (11) and (12) follow from formula (10). Then we will prove formula (10). Recall the identities $\sin^2 A + \cos^2 A = 1$ and $\cosh^2 a - \sinh^2 a = 1$. We then have

$$\begin{aligned}
1 &= \sin^2 A + \cos^2 A \\
1 &= \frac{\sinh^2 a}{\sinh^2 c} + \frac{\tanh^2 b}{\tanh^2 c} \\
\sinh^2 c &= \sinh^2 a + \cosh^2 c \cdot \tanh^2 b \\
1 + \sinh^2 c &= 1 + \sinh^2 a + \cosh^2 c \cdot \frac{\sinh^2 b}{\cosh^2 b} \\
\cosh^2 c &= \cosh^2 a + \cosh^2 c \cdot \frac{\sinh^2 b}{\cosh^2 b} \\
\cosh^2 c \cdot (\cosh^2 b - \sinh^2 b) &= \cosh^2 a \cdot \cosh^2 b \\
\cosh c &= \cosh a \cdot \cosh b.
\end{aligned}$$

This gives the first equality in formula (11). Now, applying formula (10) to B instead of A, we have

$$\sin B = \frac{\sinh b}{\sinh c}.$$

Therefore,

$$\begin{aligned}
\frac{\cos A}{\sin B} &= \frac{\tanh b}{\tanh c} \cdot \frac{\sinh c}{\sinh b} \\
&= \frac{\cosh c}{\cosh b} \\
&= \cosh a.
\end{aligned}$$

This gives us formula (12). We will use this formula to get the second equality in (11). Note, $\cosh b = \cos B / \sin A$. We then have

$$\begin{aligned}
\cosh c &= \cosh a \cdot \cosh b. \\
&= \frac{\cos A}{\sin B} \cdot \frac{\cos B}{\sin A} \\
&= \cot A \cdot \cot B.
\end{aligned}$$

We conclude that formulas (11) and (12) follow from (10). Now, we need to prove formula (10). We will proceed under the assumption that vertex A of the right triangle coincides with the center O of the circle κ in the Poincaré model. Refer to Figure 13. The points B' and C' are the images of B and C under the isomorphism F. Let B'' be the point of intersection between $\overrightarrow{OB}$ and the orthogonal circle κ_1 that contains the Poincaré line $\overleftrightarrow{BC}$. Note, we will use the same notation as earlier by letting $x = d(O, B)$ and $t = OB$. From the Euclidean triangle $\triangle AB'C'$, we have

$$\cos A = \frac{OC'}{OB'}.$$

Recall that formula (9) says that the hyperbolic tangent of the Poincaré length OB is equal to the Euclidean length OB'. Hence,

$$\cos A = \frac{OC'}{OB'} = \frac{\tanh b}{\tanh c}$$

which is the second formula in (10). Now, we need to prove the first formula. By Proposition 2.4, B'' is the inverse of B in κ, so that

$$BB'' = OB'' - OB = \frac{1}{t} - t = \frac{1 - t^2}{t}.$$

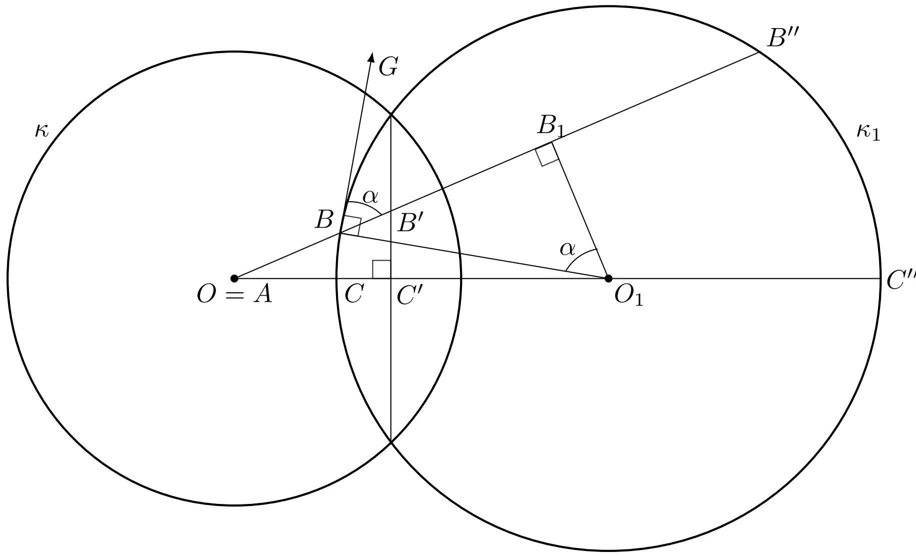


Figure 13: This figure is used in the proof of Theorem 5.1, equation (10). Let κ be a Poincaré circle with center $O = A$ and circle κ_1 be a circle perpendicular to κ with origin O_1. Note that, B and B'' are inverses as well as C and C''.

Using equation (8), we then have

$$BB'' = \frac{2}{\sinh c} \qquad \text{and} \qquad CC'' = \frac{2}{\sinh b}.$$

Now, let B_1 be the midpoint of $\overline{BB''}$. Note B_1 is also the foot of the perpendicular from the center O_1 of κ_1 to $\overline{BB''}$. Let $\overrightarrow{BG}$ be the tangent ray to κ_1 at point B. Therefore, $\angle O_1BG$ is a right angle and $\angle O_1B_1B \cong \angle GBO_1$. Then, $\angle BO_1B_1 \cong \angle GBB_1 = \alpha$, because both of these angles are compliments of $\angle B_1BO_1$. Hence,

$$\begin{aligned} \sin B &= \frac{BB_1}{O_1B} \\ &= \frac{BB''}{2} \cdot \frac{1}{O_1C} \\ &= \frac{BB''}{CC''} \\ &= \frac{2}{\sinh c} \cdot \frac{\sinh b}{2} \\ &= \frac{\sinh b}{\sinh c}. \end{aligned}$$

Since $\angle B$ is an arbitrary acute angle in a right triangle, we can interchange A and B to get the first formula in (10). $\square$

Formula (12) and the second equality in formula (11) do not have Euclidean counterparts but formulas (10) and the first equality in (11) do. First, we will look at the first equality in formula (11) and we will show the correspondence to the Pythagorean theorem in the Euclidean plane. Note that if we use the Taylor series expansions from equation (2), the formula becomes

$$\begin{aligned}
\cosh c &= \cosh a \cdot \cosh b \\
\sum_{n=0}^{\infty} \frac{c^{2n}}{(2n)!} &= \sum_{n=0}^{\infty} \frac{a^{2n} + b^{2n}}{(2n)!} \\
1 + \frac{1}{2}c^2 + \cdots &= 1 + \frac{1}{2}(a^2 + b^2) + \cdots .
\end{aligned}$$

If we assume triangle $\triangle ABC$ is sufficiently small, we can ignore the higher-order terms and thus,

$$c^2 \approx a^2 + b^2.$$

Similarly, under the same assumption, formula (10) becomes

$$\sin A \approx \frac{a}{c} \qquad \text{and} \qquad \cos A \approx \frac{b}{c}.$$

Hence, formula (10) corresponds to the Euclidean ratios of a triangle's sides.

In both of these cases, we have are working under the assumption that a triangle is "sufficiently small." We will explore through example when a triangle fits in this category.

Example 5.2. For this example, we are going to be using equation (11) and the Euclidean Pythagorean Theorem. Let triangle $\triangle ABC$ be a right triangle with $a = 1$ and $b = 2$. We want to find the length of side c. In the hyperbolic plane,

$$\begin{aligned}
\cosh c &= \cosh a \cdot \cosh b \\
&= \cosh 1 \cdot \cosh 2 \\
&= 5.8.
\end{aligned}$$

Therefore, $c \approx 2.45$. Whereas, in the Euclidean plane, $a^2 + b^2 = c^2$ and hence $c \approx 2.24$. We conclude from this example that a triangle of this size is sufficiently small enough for the two formulas to be good approximations for each other.

As a counterexample, we are going to look at a slightly larger triangle in which the two formulas are not good estimates of each other.

Example 5.3. Let triangle $\triangle ABC$ be a right triangle with sides $a = 4$ and $b = 8$. In the hyperbolic plane, we have

$$\begin{aligned}
\cosh c &= \cosh a \cdot \cosh b \\
&= \cosh 4 \cdot \cosh 8 \\
&= 40702.35.
\end{aligned}$$

Therefore, $c \approx 11.31$. Whereas, in the Euclidean plane we find that $c \approx 8.94$. Comparing these two values for c, we can see that the formulas are beginning to separate themselves and that this triangle is not sufficiently small.

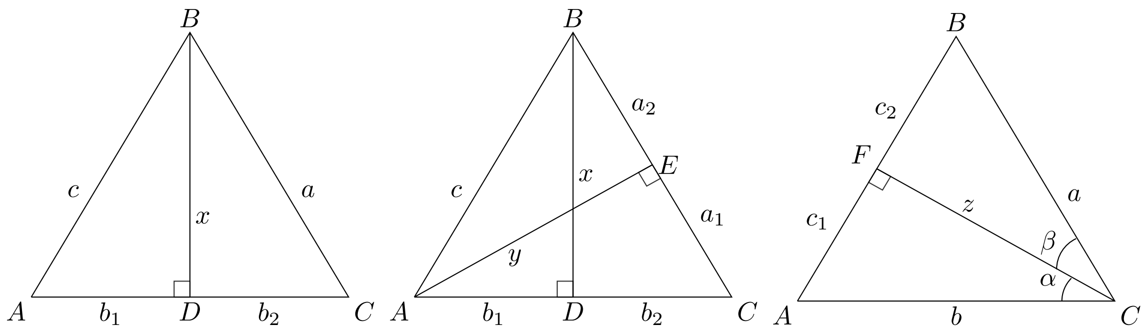
Table 2 shows a few more examples of the difference between the hyperbolic length c and the Euclidean length c for a given right triangle with sides of length a and b. Looking at the last column which shows the difference between the two values for c, we can see that as the triangle is getting bigger, the difference is getting larger. Therefore, in order for the hyperbolic identities to break down into the Euclidean identities, we need the triangle to be sufficiently small.

5.2 Trigonometric Identities for any Triangle

While Theorem 5.1 presents identities for a hyperbolic right triangle, we also have identities that can be applied to any given triangle in the hyperbolic plane.

	Hyperbolic	Euclidean	Difference
$a = 4$, $b = 1$	$c = 4.45$	$c = 4.12$	.33
$a = 5$, $b = 2$	$c = 6.33$	$c = 5.36$	.97
$a = 6$, $b = 3$	$c = 8.31$	$c = 6.71$	1.6
$a = 13$, $b = 10$	$c = 22.3$	$c = 16.4$	5.9
$a = 25$, $b = 30$	$c = 54$	$c = 39.05$	14.95
$a = 40$, $b = 50$	$c = 89.3$	$c = 64$	25.3
$a = 55$, $b = 70$	$c = 124.3$	$c = 89$	35.3

Table 2: A table comparing the hyperbolic length c to the Euclidean length c for a given right triangle with sides a and b.



(a) Equation (13) (b) Equation (14) (c) Equation (15)

Figure 14: This figure is used in the proof of Theorem 5.4. In each case, we have a triangle $\triangle ABC$ and we drop at least one perpendicular from a vertex to the opposite side in order to prove equations (13), (14), and (15).

Theorem 5.4. *For any triangle $\triangle ABC$ in the hyperbolic plane,*

$$\cosh c = \cosh a \cdot \cosh b - \sinh a \cdot \sinh b \cdot \cos C \tag{13}$$

$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c} \tag{14}$$

$$\cosh c = \frac{\cos A \cdot \cos B + \cos C}{\sin A \cdot \sin B}. \tag{15}$$

Proof. Refer to Figure 14. Before we begin, recall that

$$\cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y \tag{16}$$

and

$$\cosh(x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y. \tag{17}$$

Now, we will start with the proof of equation (13). Given a hyperbolic triangle $\triangle ABC$, we will drop a perpendicular from B to $\overline{AC}$, namely BD in Figure 14a. Let the length of $BD = x$ and $b = b_1 + b_2$. Using equations (10), (11), and (17), we then have

$$\begin{aligned}
\cosh c &= \cosh b_1 \cdot \cosh x \\
&= \cosh(b - b_2) \cdot \cosh x \\
&= (\cosh b \cdot \cosh b_2 - \sinh b \cdot \sinh b_2) \cdot \cosh x \\
&= \cosh b \cdot \cosh a - \sinh b \cdot \sinh a \cdot \frac{\cosh a \cdot \sinh b_2}{\cosh b_2 \cdot \sinh a} \\
&= \cosh b \cdot \cosh a - \sinh b \cdot \sinh a \cdot \frac{\tanh b_2}{\tanh a} \\
&= \cosh a \cdot \cosh b - \sinh a \cdot \sinh b \cdot \cos C.
\end{aligned}$$

This proves equation (13). We use a similar triangle for the proof of equation (14), except we are also going to drop a perpendicular from A to $\overline{BC}$, AE in Figure 14b, of length y. Using equation (10), we have

$$\frac{\sin A}{\sinh a} = \frac{1}{\sinh a} \cdot \frac{\sinh x}{\sinh c} = \frac{\sinh x}{\sinh a} \cdot \frac{1}{\sinh c} = \frac{\sin C}{\sinh c}.$$

Similarly,

$$\frac{\sin C}{\sinh c} = \frac{1}{\sinh c} \cdot \frac{\sinh y}{\sinh b} = \frac{\sinh y}{\sinh c} \cdot \frac{1}{\sinh b} = \frac{\sin B}{\sinh b}.$$

Therefore, we have equation (14). Lastly, we will use Figure 14c to prove equation (15). In this case, we are going to drop a perpendicular from C to AB and call the length z. This is going to divide $\angle C$ into two parts, namely $\angle\alpha$ and $\angle\beta$. Using equations (10), (12), and (16), we have

$$\begin{aligned}
\cosh c \ = \ \cosh(c_1 + c_2) &= \cosh c_1 \cdot \cosh c_2 + \sinh c_1 \cdot \sinh c_2 \\
&= \frac{\cos\alpha}{\sin A} \cdot \frac{\cos\beta}{\sin B} + \sinh b \cdot \sin\alpha \cdot \sinh a \cdot \sin\beta \\
&= \frac{\cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta \cdot \sinh^2 z}{\sin A \cdot \sin B} \\
&= \frac{(\cos(\alpha+\beta) + \sin\alpha \cdot \sin\beta) + (\sin\alpha \cdot \sin\beta \cdot \sinh^2 z)}{\sin A \cdot \sin B} \\
&= \frac{\cos C + \sin\alpha \cdot \sin\beta\,(1 + \sinh^2 z)}{\sin A \cdot \sin B} \\
&= \frac{\cos C + (\sin\alpha \cdot \cosh z) \cdot (\sin\beta \cdot \cosh z)}{\sin A \cdot \sin B} \\
&= \frac{\cos C + \cos A \cdot \cos B}{\sin A \cdot \sin B}.
\end{aligned}$$

Therefore, we have equation (15). It is important that we note for this proof, we are working under the assumption that the dropped perpendiculars fall within the hyperbolic triangle $\triangle ABC$. Without this assumption, we could show in a proof that is generally the same as above that when the dropped perpendicular falls outside of the $\triangle ABC$ the equations (13), (14), and (15) still hold. □

In the same way that equation (10) and the first part of formula (11) in Theorem 5.1 corresponded to known Euclidean identities, equations (13) and (14) also have corresponding identities in the Euclidean plane. Formula (13) is the *hyperbolic law of cosines* and thus relates to the Euclidean law of cosines. Likewise, formula (14) is the *hyperbolic law of sines* and is analogous to the Euclidean law of sines. Similar to the discussion at the end of Section 5.1, we can see the relationships between these hyperbolic and Euclidean identities by applying them to a sufficiently small triangle. By doing this, the hyperbolic identities essentially reduce to their Euclidean counterparts.

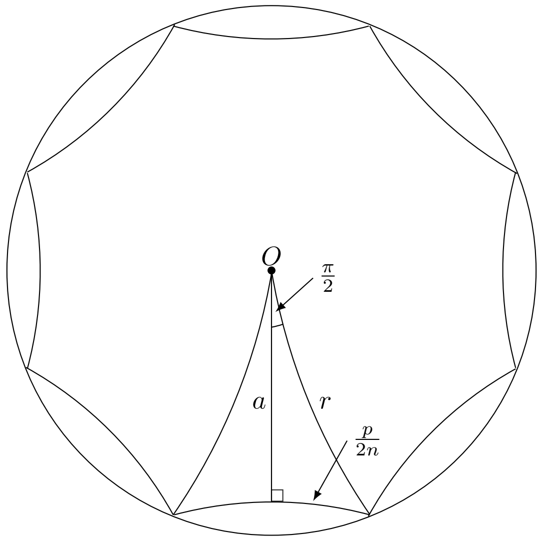


Figure 15: Derivation for the formula of the circumference of a circle. Note, a is the length of the perpendicular from the origin O to a side of the n-gon, p is the sum of the lengths of the sides of the n-gon, and r is the radius of the circle and the n-gon.

6 Circumference and Area of a Circle

In Sections 5.1 and 5.2, we presented a few hyperbolic identities for triangles that, when the triangle is sufficiently small, reduce to their corresponding Euclidean identities. In this section, we are going to consider another common geometric shape in the hyperbolic plane, a circle. Similar to triangles, we will see that when the radius of the circle is small enough, the formulas for the circumference and area of a circle in the hyperbolic plane reduce to the Euclidean formulas [6].

6.1 Circumference of a Hyperbolic Circle

Before we begin working in the hyperbolic plane, recall how the formula for the circumference of a Euclidean circle, $C = 2\pi r$, is derived in the Euclidean plane. Let p_n be the perimeter of a regular n-gon drawn inside of a circle. Figure 15 shows this scenario for the hyperbolic plane but if we replace the Poincaré lines of the n-gon with Euclidean lines, then the figure can be applied to the Euclidean scenario. Note, as $n \to \infty$, the n-gon increases to fill the circle. Therefore, we define the circumference C as $C = \lim_{n\to\infty} p_n$. Using something like Figure 15 in the Euclidean plane and Euclidean trigonometry, we have

$$\begin{aligned} p_n &= r \cdot 2n \cdot \sin\left(\frac{\pi}{n}\right) \\ &= r \cdot 2n \cdot \left[\frac{\pi}{n} - \frac{1}{3!}\left(\frac{\pi}{n}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{n}\right)^5 - \cdots\right] \\ &= 2\pi r - \frac{2r\pi^2}{n^2}\left[\frac{\pi}{3!} - \frac{1}{5!}\left(\frac{\pi^3}{n^2}\right) + \cdots\right]. \end{aligned}$$

Hence,

$$C = \lim_{n\to\infty} p_n = 2\pi r.$$

We will use this same idea to prove formula (18) in the following theorem.

Theorem 6.1. (Gauss) *In the hyperbolic plane, the circumference C of a circle of radius r is given by*

$$C = 2\pi \sinh r. \tag{18}$$

Proof. Since we are now working in the hyperbolic plane, we use formula (10) and Figure 15 to find that

$$\sinh\left(\frac{p_n}{2n}\right) = \sinh r \cdot \sinh\left(\frac{\pi}{n}\right),$$

which by series expansion becomes

$$\frac{p_n}{2n}\left[1 + \frac{1}{3!}\left(\frac{p_n}{2n}\right)^2 + \frac{1}{5!}\left(\frac{p_n}{2n}\right)^4 + \cdots\right] = \frac{\pi}{n} \cdot \sinh r\left[1 - \frac{1}{3!}\left(\frac{\pi}{n}\right)^2 + \frac{1}{5!}\left(\frac{\pi}{n}\right)^4 - \cdots\right].$$

Multiplying both sides by $2n$, we then have

$$p_n\left[1 + \frac{1}{3!}\left(\frac{p_n}{2n}\right)^2 + \frac{1}{5!}\left(\frac{p_n}{2n}\right)^4 + \cdots\right] = 2\pi \cdot \sinh r\left[1 - \frac{1}{3!}\left(\frac{\pi}{n}\right)^2 + \frac{1}{5!}\left(\frac{\pi}{n}\right)^4 - \cdots\right].$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} p_n\left[1 + \frac{1}{3!}\left(\frac{p_n}{2n}\right)^2 + \frac{1}{5!}\left(\frac{p_n}{2n}\right)^4 + \cdots\right] &= \lim_{n\to\infty} 2\pi \cdot \sinh r\left[1 - \frac{1}{3!}\left(\frac{\pi}{n}\right)^2 + \frac{1}{5!}\left(\frac{\pi}{n}\right)^4 - \cdots\right] \\ C = \lim_{n\to\infty} p_n &= 2\pi \sinh r. \end{aligned}$$

□

Formula (18) is similar to the Euclidean formula for the circumference of a circle. If we let r approach 0 in formula (18), then it resembles the Euclidean formula $C = 2\pi r$.

Example 6.2. In this example, we are going to look at how small the radius r must be in order for the Euclidean circumference to be a relatively good approximation for the hyperbolic circumference.

First, suppose we are given a hyperbolic circle κ such that the radius $r = 2$. We then have

$$\begin{aligned} C_h &= 2\pi \sinh(2) \\ &\approx 2\pi \cdot (3.63) \\ &\approx 22.79. \end{aligned}$$

Note, the Euclidean circumference of κ is

$$\begin{aligned} C_e &= 2\pi \cdot (2) \\ &\approx 12.57. \end{aligned}$$

This shows that if the radius of the hyperbolic circle is 2, the Euclidean circumference is nearly half the hyperbolic circumference and thus is not a good approximation. In contrast, suppose we are given a hyperbolic circle γ such that the radius $r = 1$. We then have

$$\begin{aligned} C_h &= 2\pi \sinh(1) \\ &\approx 2\pi \cdot (1.18) \\ &\approx 7.38. \end{aligned}$$

Note, the Euclidean circumference of γ is

$$\begin{aligned} C_e &= 2\pi \cdot (1) \\ &\approx 6.28. \end{aligned}$$

Comparing these two values for the circumference of circle γ, we can see that if the radius is 1, the hyperbolic circumference is approximately the Euclidean circumference. We can conclude that a circle of radius 2 is too large for the formula of the hyperbolic circumference to reduce to the Euclidean version. Whereas, a circle of radius 1 has a hyperbolic circumference that is approximately equal to the Euclidean circumference. To emphasize this concept, if we looked at a circle of radius < 1, we would see that C_h and C_e would be even closer in value.

Theorem 6.1 provides the link used to write the *law of sines* in a form valid in neutral geometry. This form of the law of sines is described in Corollary 6.3.

Corollary 6.3. (J. Bolyai) *The sines of the angles of a triangle are to one another as the circumference of the circles whose radii are equal to the opposite sides.*

Bolyai denoted the circumference of a circle of radius r by $\circ r$ [6]. Hence, this result can be written as

$$\circ a : \circ b : \circ c = \sin A : \sin B : \sin C. \tag{19}$$

Proof. To show how this holds in the hyperbolic plane, looking at formula (14), if we divide everything by 2π, we then have

$$\frac{\sin A}{2\pi \cdot \sinh a} = \frac{\sin B}{2\pi \cdot \sinh b} = \frac{\sin C}{2\pi \cdot \sinh c}.$$

By applying Theorem 6.1, this implies that

$$\frac{\sin A}{\circ a} = \frac{\sin B}{\circ b} = \frac{\sin C}{\circ c}.$$

Therefore, we have equation (19). Hence, Corollary 6.3 holds in hyperbolic geometry. A similar argument concludes that it also holds in the Euclidean and spherical planes. We can conclude that the version of the law of sines described in Corollary 6.3 is valid in neutral geometry. □

6.2 Area of a Hyperbolic Circle

Next, we will introduce the definition of the *defect* of a triangle and two theorems which will lead to the formula for the area of a hyperbolic circle.

Definition 6.4. Given a triangle $\triangle ABC$, the *defect*, denoted $\delta(ABC)$, is defined as the difference between $180°$ and the angle sum of $\triangle ABC$:

$$\delta(ABC) = 180° - (\angle A)° - (\angle B)° - (\angle C)°.$$

(It is important to note that the defect of a triangle can also be measured in radians.)

Theorem 6.5. *In hyperbolic geometry, the area of triangle $\triangle ABC$ is*

$$\text{Area}(\triangle ABC) = \delta(ABC).$$

Let $K = \text{Area}(\triangle ABC)$ in radians. If $\triangle ABC$ is a right triangle where $\angle C = \pi/2$, then $K = \pi/2-(A+B)$. Using this formula for K where $\triangle ABC$ is a right triangle, we have a formula that relates the area to the side lengths a and b, namely formula (20).

Theorem 6.6. *Given a right triangle $\triangle ABC$ with area K, we have*

$$\tan\frac{K}{2} = \tanh\frac{a}{2} \cdot \tanh\frac{b}{2}. \tag{20}$$

Like most of the other theorems we have discussed, this theorem has a corresponding formula in the Euclidean plane. That is, for Euclidean geometry, formula (20) becomes $K/2 = a/2 \cdot b/2$ [6].

Recall, the formula for the area of a circle in the Euclidean plane is $A = \pi r^2$. Similar to the derivation of the circumference in the Euclidean plane, we will use Figure 15. Note, the triangle area is $1/2 \cdot p/n \cdot a$ and hence, the n-gon area is $1/2 \cdot pa = a/2 \cdot p$. Recall that p is the perimeter of the n-gon, therefore as $n \to \infty$ we have $p \to 2\pi r$. Similarly, as $n \to \infty$ we also have that $a \to r$. Substituting these into the area of the polygon, we have as $n \to \infty$, the area goes to $r/2 \cdot 2\pi r = \pi r^2$. Keeping this derivation in mind, we now present the formula for the area of a hyperbolic circle.

Theorem 6.7. *The area of a circle of radius r is* $4\pi \sinh^2(r/2) = 2\pi(\cosh r - 1)$.

Proof. Let A be the area of a circle and let K_n be the area of the inscribed n-gon. We then have

$$A = \lim_{n\to\infty} K_n.$$

Applying formula (20) and using Figure 15, we have

$$\begin{aligned}
\tan \frac{K/2n}{2} &= \tanh \frac{p/2n}{2} \cdot \tanh \frac{a}{2} \\
4n \tan \frac{K}{4n} &= 4n \tanh \frac{p}{4n} \cdot \tanh \frac{a}{2}.
\end{aligned}$$

Note, due to the continuity of tangent and hyperbolic tangent and since

$$4n \tan \frac{K}{4n} = K + \frac{K}{3} \cdot \frac{K}{4n}^2 + \cdots ,$$

$$4n \tanh \frac{p}{4n} = p - \frac{p}{3} \cdot \frac{p}{4n}^2 + \cdots ,$$

we have

$$\begin{aligned}
\lim_{n\to\infty} \left(4n \tan \frac{K_n}{4n} \right) &= \lim_{n\to\infty} \left(4n \tanh \frac{p_n}{4n} \cdot \tanh \frac{a_n}{2} \right) \\
\lim_{n\to\infty} K_n &= \lim_{n\to\infty} p_n \cdot \lim_{n\to\infty} \tanh \frac{a_n}{2}.
\end{aligned}$$

Using similar logic to the Euclidean case presented above, as $n \to \infty$, we have $p_n \to C$ and $a_n \to r$. From equation (18) in Theorem 6.1, we have

$$A = 2\pi \sinh r \cdot \tanh \frac{r}{2}.$$

Applying the identities

$$\tanh \frac{r}{2} = \frac{\sinh r}{\cosh r + 1},$$

$$\sinh^2 r = \cosh^2 r - 1,$$

$$2 \sinh^2 \frac{r}{2} = \cosh r - 1,$$

we have

$$A = \frac{2\pi \sinh^2 r}{\cosh r + 1} = \frac{2\pi(\cosh^2 r - 1)}{\cosh r + 1} = \frac{2\pi(\cosh r - 1)(\cosh r + 1)}{\cosh r + 1} = 2\pi(\cosh r - 1).$$

This gives one version of the formula presented in Theorem 6.7. Applying the last identity above, we obtain the other formula:

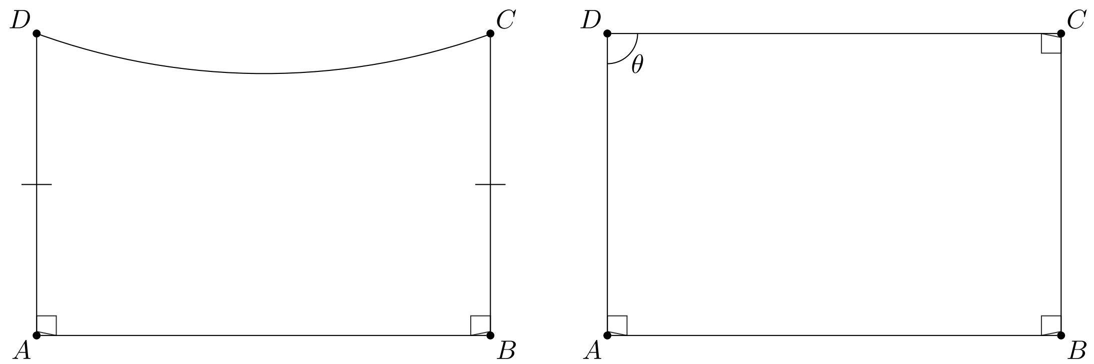
$$A = 2\pi(\cosh r - 1) = 2\pi \left(2 \sinh^2 \frac{r}{2} \right) = 4\pi \sinh^2 \frac{r}{2}.$$

□

7 Saccheri and Lambert Quadrilaterals

In section 6, we discussed different formulas in regards to hyperbolic circles. In this section, we will explore Saccheri and Lambert quadrilaterals. Recall, the definitions of such quadrilaterals.

Definition 7.1. A *Saccheri quadrilateral* is a quadrilateral $\diamond ABCD$ such that $\angle A$ and $\angle B$ are right angles and $\overline{AD} \cong \overline{BC}$.



(a) A Saccheri quadrilateral. (b) A Lambert quadrilateral.

Figure 16: Quadrilaterals in the hyperbolic plane.

It is important to note that we are not assuming anything about the angles $\angle C$ and $\angle D$. Referring to Figure 16a, side AB will be called the *base*, sides AD and BC will be called the *legs*, and side CD will be called the *summit*. Similarly, angles $\angle C$ and $\angle D$ will be called *summit angles*.

Definition 7.2. A *Lambert quadrilateral* is a quadrilateral that has at least three right angles.

Similar to the Saccheri quadrilateral, we are not assuming anything about angle $\angle D$ in the Lambert quadrilateral shown in Figure 16b. However, it should be noted that the measure of angle $\angle D$ varies depending on the geometry in which the quadrilateral is contained. If the Lambert quadrilateral is in the Euclidean plane, $\angle D = 90°$, if it is in the hyperbolic plane, $\angle D < 180°$, and if it is in the spherical plane, $\angle D > 180°$.

Besides the definitions above, we also need to recall a few properties about geometry in the hyperbolic plane. In the hyperbolic plane, the following properties hold: the angle sum of every triangle is $< 180°$, the summit angles of all Saccheri quadrilaterals are acute, the fourth angle of every Lambert quadrilateral is acute, and rectangles do not exist [6].

7.1 Saccheri Quadrilaterals

In this section, we are going to consider the Saccheri quadrilateral shown in Figure 17 with base of length b, legs of length a, and summit of length c.

Theorem 7.3. *For a Saccheri quadrilateral,*

$$\sinh\frac{c}{2} = \cosh a \cdot \sinh\frac{b}{2}.$$

Furthermore, since $\cosh^2 a = 1 + \sinh^2 a > 1$, *we conclude that* $\sinh(c/2) > \sinh(b/2)$ *and hence,* $c > b$.

Proof. Refer to Figure 17. Let $\theta = \angle DAC$ and $d = AC$. Applying equation (15), we have

$$\cosh c = \cosh a \cosh d - \sinh a \sinh d \cos\theta.$$

Since d does not show up in our desired equation, we want to ultimately eliminate it. By using equations (10) and (11), we are able to do so. Note,

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) = \frac{\sinh a}{\sinh d}.$$

This implies that

$$\sinh d = \frac{\sinh a}{\cos\theta}.$$

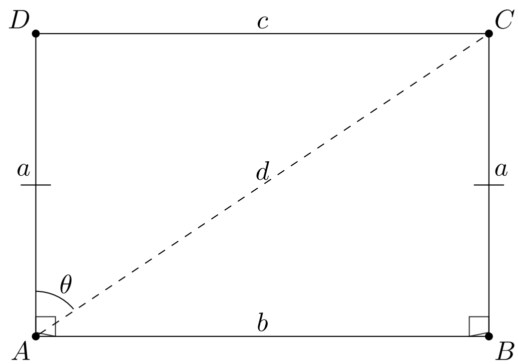


Figure 17: Saccheri quadrilateral used in the proof of Theorem 7.3.

Similarly, we have

$$\cosh d = \cosh a \cosh b.$$

Therefore,

$$\begin{aligned}
\cosh c &= \cosh a(\cosh a \cosh b) - \sinh a\left(\frac{\sinh a}{\cos\theta}\right)\cos\theta \\
&= \cosh^2 a \cosh b - \sinh^2 a \\
&= \cosh^2 a \cosh b - (\cosh^2 a - 1) \\
&= \cosh^2 a(\cosh b - 1) + 1 \\
\cosh c - 1 &= \cosh^2 a(\cosh b - 1)
\end{aligned}$$

Recall the identity from the proof of Theorem 6.7, $2\sinh^2(x/2) = \cosh x - 1$. We then have

$$2\sinh^2\frac{c}{2} = \cosh^2 a\left(2\sinh^2\frac{b}{2}\right).$$

Dividing both sides by two and taking the square root yields the desired result. $\square$

7.2 Lambert Quadrilaterals

Lambert quadrilaterals are closely related to Saccheri quadrilaterals. More specifically, a Saccheri quadrilateral is two Lambert quadrilaterals [7].

Theorem 7.4. *A Lambert quadrilateral is one-half of a Saccheri quadrilateral.*

As a result of this relationship between these two quadrilaterals, we have the corresponding theorem to Theorem 7.3 for Lambert quadrilaterals.

Theorem 7.5. *Given a Lambert quadrilateral $\diamond ABCD$ where $\angle D$ is the acute angle, if c is the length of a side adjacent to $\angle D$, b is the length of a side opposite $\angle D$, and a is the length of the other adjacent side, then*

$$\sinh c = \cosh a \sinh b.$$

Proof. This directly follows from Theorem 7.4 and Theorem 7.3. $\square$

8 Notes on Spherical Trigonometry

Many of the theorems presented above for the hyperbolic plane were analogous to a formula in the Euclidean plane such as the law of cosines, law of sines, and the Pythagorean theorem. In this section, we will see that these formulas also have a counterpart in the spherical plane.

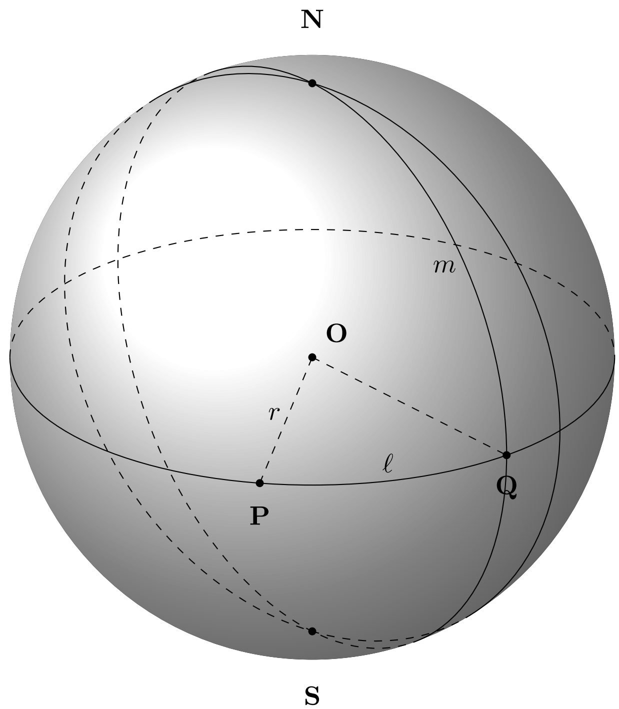


Figure 18: Spherical plane: the surface of a sphere with radius r. Note, P and Q are spherical points and $\ell = PQ$ and $m = NQ$ are spherical lines.

Spherical geometry is geometry that takes place on the surface of a sphere, as shown in Figure 18. In the spherical plane, points are defined as what we know to be a point in the Euclidean plane. For example, in Figure 18, P and Q are spherical points. Lines, however, are arcs of *great circles* [10]. A great circle on a sphere is any circle whose origin coincides with the origin of the sphere. Another way to interpret lines is the shortest distance between two points along the sphere. For example, in Figure 18, ℓ and m are spherical lines. Another important concept is the length of a line. In the spherical plane, the length of a line $a = AB$ is equal to the size of angle $\angle AOB$ in radians, where O is the origin of the sphere. For example, in Figure 18, the length of ℓ is equal to the radian measure of angle $\angle POQ$. Lastly, similar to hyperbolic geometry, the Euclidean parallel postulate doesn't hold in spherical geometry. In the spherical plane, there are no parallel lines at all [3].

8.1 Spherical Triangles

Before taking a closer look at triangles in spherical geometry and the theorems that relate, we need to define an angle measure in the spherical plane. In the spherical plane, the angle measure is determined by the measure of the angle created at the origin of the sphere by the two great circles containing the arcs that make the angle of interest. For example, in Figure 18 the measure of angle $\angle PQN$ is determined by the measure of the angle created by the great circle that creates the equator of the sphere and the great circle containing line m.

An example of a spherical triangle is shown in Figure 19. In this figure, we have a sphere with radius $r = 1$ and origin O and we have a spherical right triangle $\triangle ABC$ where angle $\angle BCA$ is a right angle. Let angles $\angle BOA = \theta$ and $\angle COA = \delta$ and $\angle BOC = \alpha$. Thus, the lengths of the side $a = \alpha$, $b = \delta$, and $c = \theta$. We then construct the point C' by dropping a perpendicular from B to OC. Note that B and C' orthogonally project onto the same point, namely A'. We can also conclude that angles $\angle BAC$ and $\angle BA'C'$ are congruent. Now, we can look at the four Euclidean triangles that we have created inside this sphere.

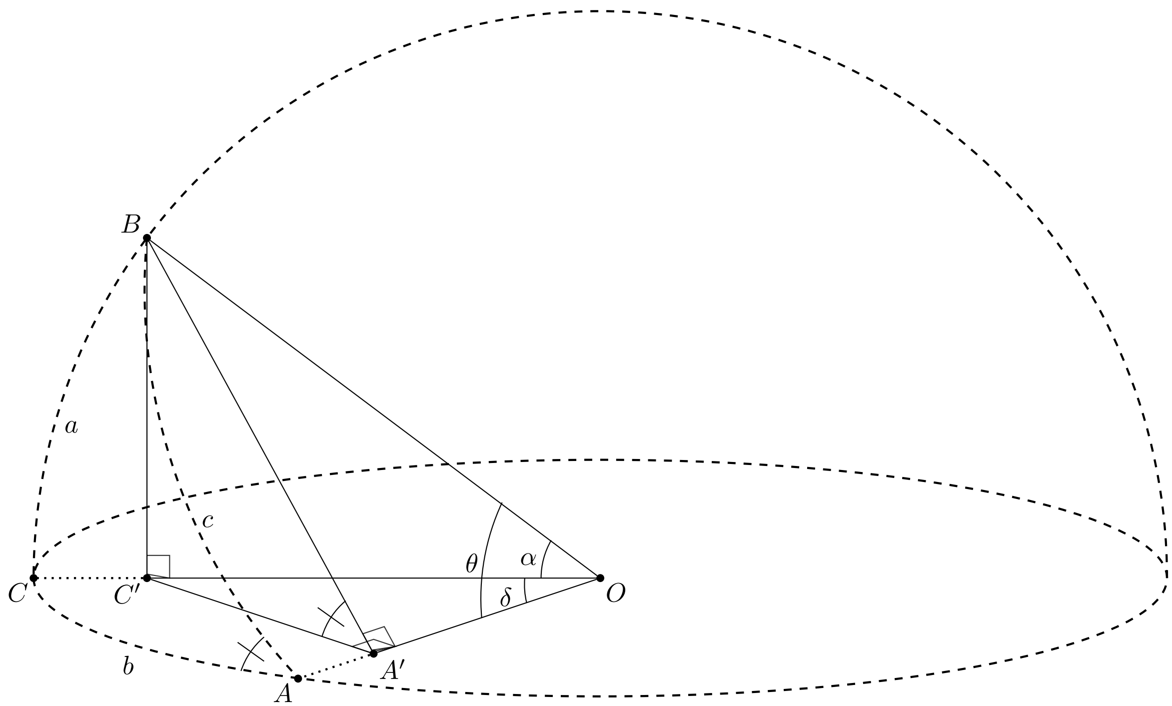


Figure 19: A sphere with radius $r = 1$ and origin O. A spherical right triangle $\triangle ABC$ on the surface of the sphere where angle $\angle BCA$ is a right angle. Construct the point C' by dropping a perpendicular from B to OC. Note that B and C' orthogonally project onto the same point, namely A', and angles $\angle BAC$ and $\angle BA'C'$ are congruent.

To start, consider $\triangle BOC'$. We have $\sin\alpha = BC'/OB$ and thus

$$\sin\alpha = BC'. \tag{21}$$

Similarly, we have

$$\cos\alpha = OC'. \tag{22}$$

From $\triangle BOA'$. we have

$$\sin\theta = BA' \tag{23}$$

and

$$\cos\theta = OA'. \tag{24}$$

These above equations will become important when we prove Theorems 8.1 and 8.2.

8.1.1 Trigonometry and the Ratio of Sides

As we noted above, for a right triangle in the Euclidean plane, the sine of an angle can be interpreted as the ratio of the opposite side to the hypotenuse. There are also relationships between the other trigonometric functions and the sides of a triangle in the Euclidean plane. For the hyperbolic plane, formula (10) is the same idea. In the spherical plane, we also have the spherical analogue of formula (10).

Theorem 8.1. *Let triangle $\triangle ABC$ be a spherical right triangle with the right angle at $\angle C$. Then,*

$$\sin A = \frac{\sin a}{\sin c} \qquad \cos A = \frac{\tan b}{\tan c}. \tag{25}$$

Proof. Consider triangle $\triangle BA'C'$ in Figure 19. Using equations (21) and (22), we obtain the first part of equation (25). Note,

$$\begin{aligned}
\sin A &\cong \sin A' \\
&= \frac{BC'}{BA'} \\
&= \frac{\sin\alpha}{\sin\theta} \\
&= \frac{\sin a}{\sin c}.
\end{aligned}$$

Using $\triangle OA'C'$, we have

$$\sin\delta = \frac{C'A'}{\cos\alpha}$$

and thus $C'A' = \sin b \cos a$. We will use this result and equations (23) and (24) to obtain the second equation in (25). We have

$$\begin{aligned}
\cos A &\cong \cos A' \\
&= \frac{C'A'}{BA'} \\
&= \frac{\sin b \cos a}{\sin c} \\
&= \frac{\tan b \cos b \cos c}{\tan c \cos c} \\
&= \frac{\tan b}{\tan c}.
\end{aligned}$$

□

8.1.2 Pythagorean Theorem

Theorem 8.2. *Let triangle $\triangle ABC$ be a spherical right triangle with the right angle at $\angle C$ and let the sphere have radius r. Then,*

$$\cos\frac{c}{r} = \cos\frac{a}{r}\cos\frac{b}{r}.$$

It is important to note that we will only be working with spheres of radius $r = 1$ and thus, Theorem 8.2 becomes

$$\cos c = \cos a \cos b, \tag{26}$$

which is the equation we will prove.

Proof. Note, we will be using Figure 19 and equations (22) and (24). Consider triangle $\triangle OC'A'$. We have

$$\begin{aligned}
\cos\delta &= \frac{OA'}{OC'} = \frac{\cos\theta}{\cos\alpha} \\
\cos b &= \frac{\cos c}{\cos a}
\end{aligned}$$

Hence, $\cos c = \cos a \cos b$. □

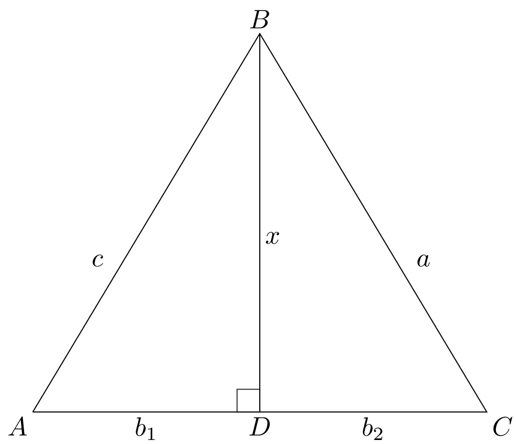


Figure 20: Spherical triangle $\triangle ABC$ used in the proof of Theorems 8.3 and 8.4.

8.1.3 Law of Cosines

Theorem 8.3. *Let triangle $\triangle ABC$ be a spherical triangle. Then,*

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

Proof. Before we begin, we need to recall the identity $\cos(x+y) = \cos s \cos y + \sin x \sin y$. Now, we will be using this identity, Figure 20, and equations (25) and (26). Let $BD = x$ be the perpendicular dropped from B to AC. We then have

$$\begin{aligned}
\cos c &= \cos b_1 \cos x \\
&= \cos(b - b_2) \cos x \\
&= (\cos b \cos b_2 + \sin b \sin b_2) \cdot \cos x \\
&= \cos a \cos b + \sin b \sinh b_2 \frac{\cos a}{\cos b_2} \\
&= \cos a \cos b + \sin b \sin a \cdot \frac{\tan b_2}{\tan a} \\
&= \cos a \cos b + \sin a \sin b \cos C.
\end{aligned}$$

□

8.1.4 Law of Sines

Theorem 8.4. *Let triangle $\triangle ABC$ be a spherical triangle. Then,*

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Proof. Note, we will be using Figure 20. Let $BD = x$ be the perpendicular dropped from B to AC. Applying Theorem 8.1 to triangles $\triangle ABD$ and $\triangle CBD$ we have

$$\sin A = \frac{\sin x}{\sin c}$$

and

$$\sin C = \frac{\sin x}{\sin a}.$$

Therefore, solving both equations for $\sin x$ we obtain $\sin A \sin c = \sin a \sin C$ and hence

$$\frac{\sin A}{\sin a} = \frac{\sin C}{\sin c}.$$

If we drop another perpendicular from A to BC and go through similar steps, we find that

$$\frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

and thus we have the desired equation. $\square$

8.2 Circumference and Area of a Circle

The formulas for circumference and area of a circle of radius r are very similar to the formulas used in the hyperbolic plane.

Theorem 8.5. *A circle in the spherical plane with radius r has circumference $C = 2\pi \sin r$.*

Theorem 8.6. *A circle in the spherical plane with radius r has area $A = 4\pi \sin^2(r/2)$.*

Rather than using the hyperbolic sine function as we did when working in the hyperbolic plane, here we are using the circular sine function. Other than this difference, the formulas for circumference and area in the hyperbolic and spherical planes are the same.

9 Conclusion

Euclid's book, *The Elements*, made large waves in all of mathematics. In particular, it changed the field of geometry. At the time, Euclidean geometry was the only geometry known. The discovery of hyperbolic geometry has impacted a variety of fields. It is not only used in mathematics but also in physics and even astrophysics. In all of these fields, models of the hyperbolic plane prove helpful. In this paper, we introduced the Poincaré and Klein models but there are many others out there. Through the use of these models, we can see how Euclidean geometry and hyperbolic geometry relate to each other.

Through the use of the Poincaré model, we were able to explore the similarities and differences between hyperbolic and Euclidean shapes. In particular, for triangles we explored the relationship between the angles and sides and saw how well-known Euclidean formulas had analogous formulas in the hyperbolic plane. Similarly, we looked at the formulas for the area and circumference of a circle in both the hyperbolic and Euclidean plane. We also noted that the hyperbolic equations began to resemble the Euclidean equations as the shapes got smaller and smaller.

While spherical geometry was briefly discussed, this area could be pursued further. Interesting results could be found by examining the spherical formulas in comparison to the Euclidean and hyperbolic formulas. We showed that there is a form of the Law of Sines for neutral geometry. By studying the three planes together, it is possible that other interesting formulas also hold in neutral geometry. In conclusion, it is intriguing to study non-Euclidean geometry and see how closely hyperbolic geometry is related to Euclidean geometry.

References

[1] Carslaw, H. S. (1916). The Elements of Non-Euclidean Plane Geometry and Trigonometry. Longmans, Green and co. 160-162.

[2] Dobbs, D. E. (2001). Verification of the Hyperbolic Parallel Postulate in a Half-Plane Model. Mathematics and Computer Education, 35(2), 140-146.

[3] Elliptic Geometry. (2014, February 14). In Wikipedia, The Free Encyclopedia. Retrieved 04:04, April 3, 2014, from `http://en.wikipedia.org/wiki/Elliptic_geometry`

[4] Greenberg, Marvin Jay. Euclidean and Non-Euclidean Geometries. 4th ed. New York: W.H. Freeman, 2008. 14-20. Print.

[5] Greenberg, Marvin Jay. Euclidean and Non-Euclidean Geometries. 4th ed. New York: W.H. Freeman, 2008. 20-25. Print.

[6] Greenberg, Marvin Jay. Euclidean and Non-Euclidean Geometries. 4th ed. New York: W.H. Freeman, 2008. 302-333, 480, 487-506. Print.

[7] Kay, D. C. (2011). College Geometry: A Unified Development. CRC Press. 166-173.

[8] Otten, S., Zin, C. (2012). In a Class with Klein: Generating a Model of the Hyperbolic Plane. Primus, 22(2), 85-96.

[9] Morrow, G. R. (Ed.). (1992). A commentary on the First Book of Euclid's Elements. Princeton University Press. 185-192.

[10] Spherical geometry. (2014, March 26). In Wikipedia, The Free Encyclopedia. Retrieved 04:09, April 3, 2014, from `http://en.wikipedia.org/wiki/Spherical_geometry`