3

Rules for Finding Derivatives

It is tedious to compute a limit every time we need to know the derivative of a function. Fortunately, we can develop a small collection of examples and rules that allow us to compute the derivative of almost any function we are likely to encounter. Many functions involve quantities raised to a constant power, such as polynomials and more complicated combinations like $y = (\sin x)^4$. So we start by examining powers of a single variable; this gives us a building block for more complicated examples.

3.1 THE POWER RULE

We start with the derivative of a power function, $f(x) = x^n$. Here n is a number of any kind: integer, rational, positive, negative, even irrational, as in x^π. We have already computed some simple examples, so the formula should not be a complete surprise:

$$\frac{d}{dx}x^n = nx^{n-1}.$$

It is not easy to show this is true for any n. We will do some of the easier cases now, and discuss the rest later.

The easiest, and most common, is the case that n is a positive integer. To compute the derivative we need to compute the following limit:

$$\frac{d}{dx}x^n = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}.$$

For a specific, fairly small value of n, we could do this by straightforward algebra.

EXAMPLE 3.1   Find the derivative of $f(x) = x^3$.

$$\begin{aligned}
\frac{d}{dx}x^3 &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x}. \\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - x^3}{\Delta x}. \\
&= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3}{\Delta x}. \\
&= \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + \Delta x^2 = 3x^2.
\end{aligned}$$

$\square$

The general case is really not much harder as long as we don't try to do too much. The key is understanding what happens when $(x+\Delta x)^n$ is multiplied out:

$$(x+\Delta x)^n = x^n + nx^{n-1}\Delta x + a_2 x^{n-2}\Delta x^2 + \cdots + a_{n-1}x\Delta x^{n-1} + \Delta x^n.$$

We know that multiplying out will give a large number of terms all of the form $x^i\Delta x^j$, and in fact that $i+j = n$ in every term. One way to see this is to understand that one method for multiplying out $(x+\Delta x)^n$ is the following: In every $(x+\Delta x)$ factor, pick either the x or the Δx, then multiply the n choices together; do this in all possible ways. For example, for $(x+\Delta x)^3$, there are eight possible ways to do this:

$$\begin{aligned}
(x+\Delta x)(x+\Delta x)(x+\Delta x) &= xxx + xx\Delta x + x\Delta xx + x\Delta x\Delta x \\
&\quad + \Delta xxx + \Delta xx\Delta x + \Delta x\Delta xx + \Delta x\Delta x\Delta x \\
&= x^3 + x^2\Delta x + x^2\Delta x + x\Delta x^2 \\
&\quad + x^2\Delta x + x\Delta x^2 + x\Delta x^2 + \Delta x^3 \\
&= x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3
\end{aligned}$$

No matter what n is, there are n ways to pick Δx in one factor and x in the remaining $n-1$ factors; this means one term is $nx^{n-1}\Delta x$. The other coefficients are somewhat harder to understand, but we don't really need them, so in the formula above they have simply been called a_2, a_3, and so on. We know that every one of these terms contains Δx to at least the power 2. Now let's look at the limit:

$$\begin{aligned}
\frac{d}{dx}x^n &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\Delta x + a_2 x^{n-2}\Delta x^2 + \cdots + a_{n-1}x\Delta x^{n-1} + \Delta x^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + a_2 x^{n-2}\Delta x^2 + \cdots + a_{n-1}x\Delta x^{n-1} + \Delta x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} nx^{n-1} + a_2 x^{n-2}\Delta x + \cdots + a_{n-1}x\Delta x^{n-2} + \Delta x^{n-1} = nx^{n-1}.
\end{aligned}$$

Now without much trouble we can verify the formula for negative integers. First let's look at an example:

EXAMPLE 3.2   Find the derivative of $y = x^{-3}$. Using the formula, $y' = -3x^{-3-1} = -3x^{-4}$.

$\square$

Here is the general computation. Suppose n is a negative integer; the algebra is easier to follow if we use $n = -m$ in the computation, where m is a positive integer.

$$\begin{aligned}
\frac{d}{dx}x^n &= \frac{d}{dx}x^{-m} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^{-m} - x^{-m}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{1}{(x+\Delta x)^m} - \frac{1}{x^m}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^m - (x+\Delta x)^m}{(x+\Delta x)^m x^m \Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^m - (x^m + mx^{m-1}\Delta x + a_2 x^{m-2}\Delta x^2 + \cdots + a_{m-1}x\Delta x^{m-1} + \Delta x^m)}{(x+\Delta x)^m x^m \Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-mx^{m-1} - a_2 x^{m-2}\Delta x - \cdots - a_{m-1}x\Delta x^{m-2} - \Delta x^{m-1})}{(x+\Delta x)^m x^m} \\
&= \frac{-mx^{m-1}}{x^m x^m} = \frac{-mx^{m-1}}{x^{2m}} = -mx^{m-1-2m} = nx^{-m-1} = nx^{n-1}.
\end{aligned}$$

We will later see why the other cases of the power rule work, but from now on we will use the power rule whenever n is any real number. Let's note here a simple case in which the power rule applies, or almost applies, but is not really needed. Suppose that $f(x) = 1$; remember that this "1" is a function, not "merely" a number, and that $f(x) = 1$ has a graph that is a horizontal line, with slope zero everywhere. So we know that $f'(x) = 0$. We might also write $f(x) = x^0$, though there is some question about just what this means at $x = 0$. If we apply the power rule, we get $f'(x) = 0x^{-1} = 0/x = 0$, again noting that there is a problem at $x = 0$. So the power rule "works" in this case, but it's really best to just remember that the derivative of any constant function is zero.

Exercises 3.1.

Find the derivatives of the given functions.

1. $x^{100} \Rightarrow$
2. $x^{-100} \Rightarrow$
3. $\frac{1}{x^5} \Rightarrow$
4. $x^\pi \Rightarrow$
5. $x^{3/4} \Rightarrow$
6. $x^{-9/7} \Rightarrow$

3.2 LINEARITY OF THE DERIVATIVE

An operation is linear if it behaves "nicely" with respect to multiplication by a constant and addition. The name comes from the equation of a line through the origin, $f(x) = mx$, and the following two properties of this equation. First, $f(cx) = m(cx) = c(mx) = cf(x)$, so the constant c can be "moved outside" or "moved through" the function f. Second, $f(x+y) = m(x+y) = mx + my = f(x) + f(y)$, so the addition symbol likewise can be moved through the function.

The corresponding properties for the derivative are:

$$(cf(x))' = \frac{d}{dx}cf(x) = c\frac{d}{dx}f(x) = cf'(x),$$

and

$$(f(x) + g(x))' = \frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x).$$

It is easy to see, or at least to believe, that these are true by thinking of the distance/speed interpretation of derivatives. If one object is at position $f(t)$ at time t, we know its speed is given by $f'(t)$. Suppose another object is at position $5f(t)$ at time t, namely, that it is always 5 times as far along the route as the first object. Then it "must" be going 5 times as fast at all times.

The second rule is somewhat more complicated, but here is one way to picture it. Suppose a flat bed railroad car is at position $f(t)$ at time t, so the car is traveling at a speed of $f'(t)$ (to be specific, let's say that $f(t)$ gives the position on the track of the rear end of the car). Suppose that an ant is crawling from the back of the car to the front so that its position *on the car* is $g(t)$ and its speed *relative to the car* is $g'(t)$. Then in reality, at time t, the ant is at position $f(t) + g(t)$ along the track, and its speed is "obviously" $f'(t) + g'(t)$.

We don't want to rely on some more-or-less obvious physical interpretation to determine what is true mathematically, so let's see how to verify these rules by computation.

We'll do one and leave the other for the exercises.

$$\frac{d}{dx}(f(x) + g(x)) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - (f(x) + g(x))}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x) + g(x + \Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

$$= f'(x) + g'(x)$$

This is sometimes called the **sum rule** for derivatives.

EXAMPLE 3.3   Find the derivative of $f(x) = x^5 + 5x^2$. We have to invoke linearity twice here:

$$f'(x) = \frac{d}{dx}(x^5 + 5x^2) = \frac{d}{dx}x^5 + \frac{d}{dx}(5x^2) = 5x^4 + 5\frac{d}{dx}(x^2) = 5x^4 + 5 \cdot 2x^1 = 5x^4 + 10x.$$

□

Because it is so easy with a little practice, we can usually combine all uses of linearity into a single step. The following example shows an acceptably detailed computation.

EXAMPLE 3.4   Find the derivative of $f(x) = 3/x^4 - 2x^2 + 6x - 7$.

$$f'(x) = \frac{d}{dx}\left(\frac{3}{x^4} - 2x^2 + 6x - 7 \right) = \frac{d}{dx}(3x^{-4} - 2x^2 + 6x - 7) = -12x^{-5} - 4x + 6.$$

□

Exercises 3.2.

Find the derivatives of the functions in 1–6.

1. $5x^3 + 12x^2 - 15 \Rightarrow$
2. $-4x^5 + 3x^2 - 5/x^2 \Rightarrow$
3. $5(-3x^2 + 5x + 1) \Rightarrow$
4. $f(x) + g(x)$, where $f(x) = x^2 - 3x + 2$ and $g(x) = 2x^3 - 5x \Rightarrow$
5. $(x + 1)(x^2 + 2x - 3) \Rightarrow$
6. $\sqrt{625 - x^2} + 3x^3 + 12$ (See section 2.1.) $\Rightarrow$
7. Find an equation for the tangent line to $f(x) = x^3/4 - 1/x$ at $x = -2$. $\Rightarrow$

8. Find an equation for the tangent line to $f(x) = 3x^2 - \pi^3$ at $x = 4$. $\Rightarrow$
9. Suppose the position of an object at time t is given by $f(t) = -49t^2/10 + 5t + 10$. Find a function giving the speed of the object at time t. The acceleration of an object is the rate at which its speed is changing, which means it is given by the derivative of the speed function. Find the acceleration of the object at time t. $\Rightarrow$
10. Let $f(x) = x^3$ and $c = 3$. Sketch the graphs of f, cf, f', and $(cf)'$ on the same diagram.
11. The general polynomial P of degree n in the variable x has the form $P(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + \ldots + a_n x^n$. What is the derivative (with respect to x) of P? $\Rightarrow$
12. Find a cubic polynomial whose graph has horizontal tangents at $(-2, 5)$ and $(2, 3)$. $\Rightarrow$
13. Prove that $\frac{d}{dx}(cf(x)) = cf'(x)$ using the definition of the derivative.
14. Suppose that f and g are differentiable at x. Show that $f - g$ is differentiable at x using the two linearity properties from this section.

3.3   THE PRODUCT RULE

Consider the product of two simple functions, say $f(x) = (x^2 + 1)(x^3 - 3x)$. An obvious guess for the derivative of f is the product of the derivatives of the constituent functions: $(2x)(3x^2 - 3) = 6x^3 - 6x$. Is this correct? We can easily check, by rewriting f and doing the calculation in a way that is known to work. First, $f(x) = x^5 - 3x^3 + x^3 - 3x = x^5 - 2x^3 - 3x$, and then $f'(x) = 5x^4 - 6x^2 - 3$. Not even close! What went "wrong"? Well, nothing really, except the guess was wrong.

So the derivative of $f(x)g(x)$ is NOT as simple as $f'(x)g'(x)$. Surely there is some rule for such a situation? There is, and it is instructive to "discover" it by trying to do the general calculation even without knowing the answer in advance.

$$\frac{d}{dx}(f(x)g(x)) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x + \Delta x)g(x) + f(x + \Delta x)g(x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x)g(x + \Delta x) - f(x + \Delta x)g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{f(x + \Delta x)g(x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} f(x + \Delta x)\frac{g(x + \Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}g(x)$$

$$= f(x)g'(x) + f'(x)g(x)$$

A couple of items here need discussion. First, we used a standard trick, "add and subtract the same thing", to transform what we had into a more useful form. After some rewriting, we realize that we have two limits that produce $f'(x)$ and $g'(x)$. Of course, $f'(x)$ and

$g'(x)$ must actually exist for this to make sense. We also replaced $\lim_{\Delta x \to 0} f(x + \Delta x)$ with $f(x)$—why is this justified?

What we really need to know here is that $\lim_{\Delta x \to 0} f(x + \Delta x) = f(x)$, or in the language of section 2.5, that f is continuous at x. We already know that $f'(x)$ exists (or the whole approach, writing the derivative of fg in terms of f' and g', doesn't make sense). This turns out to imply that f is continuous as well. Here's why:

$$\lim_{\Delta x \to 0} f(x + \Delta x) = \lim_{\Delta x \to 0} (f(x + \Delta x) - f(x) + f(x))$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\Delta x + \lim_{\Delta x \to 0} f(x)$$

$$= f'(x) \cdot 0 + f(x) = f(x)$$

To summarize: the product rule says that

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x).$$

Returning to the example we started with, let $f(x) = (x^2 + 1)(x^3 - 3x)$. Then $f'(x) = (x^2 + 1)(3x^2 - 3) + (2x)(x^3 - 3x) = 3x^4 - 3x^2 + 3x^2 - 3 + 2x^4 - 6x^2 = 5x^4 - 6x^2 - 3$, as before. In this case it is probably simpler to multiply $f(x)$ out first, then compute the derivative; here's an example for which we really need the product rule.

EXAMPLE 3.5   Compute the derivative of $f(x) = x^2\sqrt{625 - x^2}$. We have already computed $\frac{d}{dx}\sqrt{625 - x^2} = \frac{-x}{\sqrt{625 - x^2}}$. Now

$$f'(x) = x^2\frac{-x}{\sqrt{625 - x^2}} + 2x\sqrt{625 - x^2} = \frac{-x^3 + 2x(625 - x^2)}{\sqrt{625 - x^2}} = \frac{-3x^3 + 1250x}{\sqrt{625 - x^2}}.$$

□

Exercises 3.3.

In 1–4, find the derivatives of the functions using the product rule.

1. $x^3(x^3 - 5x + 10) \Rightarrow$
2. $(x^2 + 5x - 3)(x^5 - 6x^3 + 3x^2 - 7x + 1) \Rightarrow$
3. $\sqrt{x}\sqrt{625 - x^2} \Rightarrow$
4. $\frac{\sqrt{625 - x^2}}{x^{20}} \Rightarrow$
5. Use the product rule to compute the derivative of $f(x) = (2x - 3)^2$. Sketch the function. Find an equation of the tangent line to the curve at $x = 2$. Sketch the tangent line at $x = 2$. $\Rightarrow$

6. Suppose that f, g, and h are differentiable functions. Show that $(fgh)'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.
7. State and prove a rule to compute $(fghi)'(x)$, similar to the rule in the previous problem.

Product notation.   Suppose $f_1, f_2, \ldots f_n$ are functions. The product of all these functions can be written

$$\prod_{k=1}^{n} f_k.$$

This is similar to the use of $\sum$ to denote a sum. For example,

$$\prod_{k=1}^{5} f_k = f_1 f_2 f_3 f_4 f_5$$

and

$$\prod_{k=1}^{n} k = 1 \cdot 2 \cdot \ldots \cdot n = n!.$$

We sometimes use somewhat more complicated conditions; for example

$$\prod_{k=1, k \neq j}^{n} f_k$$

denotes the product of f_1 through f_n except for f_j. For example,

$$\prod_{k=1, k \neq 4}^{5} x^k = x \cdot x^2 \cdot x^3 \cdot x^5 = x^{11}.$$

8. The **generalized product rule** says that if $f_1, f_2, \ldots, f_n$ are differentiable functions at x then

$$\frac{d}{dx}\prod_{k=1}^{n} f_k(x) = \sum_{j=1}^{n} \left(f_j'(x) \prod_{k=1, k \neq j}^{n} f_k(x) \right).$$

Verify that this is the same as your answer to the previous problem when $n = 4$, and write out what this says when $n = 5$.

3.4   THE QUOTIENT RULE

What is the derivative of $(x^2 + 1)/(x^3 - 3x)$? More generally, we'd like to have a formula to compute the derivative of $f(x)/g(x)$ if we already know $f'(x)$ and $g'(x)$. Instead of attacking this problem head-on, let's notice that we've already done part of the problem: $f(x)/g(x) = f(x) \cdot (1/g(x))$, that is, this is "really" a product, and we can compute the derivative if we know $f'(x)$ and $(1/g(x))'$. So really the only new bit of information we need is $(1/g(x))'$ in terms of $g'(x)$. As with the product rule, let's set this up and see how

far we can get:

$$\frac{d}{dx}\frac{1}{g(x)} = \lim_{\Delta x \to 0}\frac{\frac{1}{g(x+\Delta x)} - \frac{1}{g(x)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\frac{\frac{g(x) - g(x+\Delta x)}{g(x+\Delta x)g(x)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\frac{g(x) - g(x+\Delta x)}{g(x+\Delta x)g(x)\Delta x}$$

$$= \lim_{\Delta x \to 0}-\frac{g(x+\Delta x) - g(x)}{\Delta x}\frac{1}{g(x+\Delta x)g(x)}$$

$$= -\frac{g'(x)}{g(x)^2}$$

Now we can put this together with the product rule:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = f(x)\frac{-g'(x)}{g(x)^2} + f'(x)\frac{1}{g(x)} = \frac{-f(x)g'(x) + f'(x)g(x)}{g(x)^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

EXAMPLE 3.6 Compute the derivative of $(x^2+1)/(x^3-3x)$.

$$\frac{d}{dx}\frac{x^2+1}{x^3-3x} = \frac{2x(x^3-3x) - (x^2+1)(3x^2-3)}{(x^3-3x)^2} = \frac{-x^4 - 6x^2 + 3}{(x^3-3x)^2}.$$
□

It is often possible to calculate derivatives in more than one way, as we have already seen. Since every quotient can be written as a product, it is always possible to use the product rule to compute the derivative, though it is not always simpler.

EXAMPLE 3.7 Find the derivative of $\sqrt{625-x^2}/\sqrt{x}$ in two ways: using the quotient rule, and using the product rule.

Quotient rule:

$$\frac{d}{dx}\frac{\sqrt{625-x^2}}{\sqrt{x}} = \frac{\sqrt{x}(-x/\sqrt{625-x^2}) - \sqrt{625-x^2}\cdot 1/(2\sqrt{x})}{x}.$$

Note that we have used $\sqrt{x} = x^{1/2}$ to compute the derivative of $\sqrt{x}$ by the power rule.

Product rule:

$$\frac{d}{dx}\sqrt{625-x^2}x^{-1/2} = \sqrt{625-x^2}\frac{-1}{2}x^{-3/2} + \frac{-x}{\sqrt{625-x^2}}x^{-1/2}.$$

With a bit of algebra, both of these simplify to

$$-\frac{x^2+625}{2\sqrt{625-x^2}x^{3/2}}.$$
□

Occasionally you will need to compute the derivative of a quotient with a constant numerator, like $10/x^2$. Of course you can use the quotient rule, but it is usually not the easiest method. If we do use it here, we get

$$\frac{d}{dx}\frac{10}{x^2} = \frac{x^2\cdot 0 - 10\cdot 2x}{x^4} = \frac{-20}{x^3},$$

since the derivative of 10 is 0. But it is simpler to do this:

$$\frac{d}{dx}\frac{10}{x^2} = \frac{d}{dx}10x^{-2} = -20x^{-3}.$$

Admittedly, x^2 is a particularly simple denominator, but we will see that a similar calculation is usually possible. Another approach is to remember that

$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-g'(x)}{g(x)^2},$$

but this requires extra memorization. Using this formula,

$$\frac{d}{dx}\frac{10}{x^2} = 10\frac{-2x}{x^4}.$$

Note that we first use linearity of the derivative to pull the 10 out in front.

Exercises 3.4.

Find the derivatives of the functions in 1–4 using the quotient rule.

1. $\dfrac{x^3}{x^3 - 5x + 10} \Rightarrow$ 2. $\dfrac{x^2 + 5x - 3}{x^5 - 6x^3 + 3x^2 - 7x + 1} \Rightarrow$

3. $\dfrac{\sqrt{x}}{\sqrt{625-x^2}} \Rightarrow$ 4. $\dfrac{\sqrt{625-x^2}}{x^{20}} \Rightarrow$

5. Find an equation for the tangent line to $f(x) = (x^2-4)/(5-x)$ at $x = 3$. $\Rightarrow$
6. Find an equation for the tangent line to $f(x) = (x-2)/(x^3+4x-1)$ at $x = 1$. $\Rightarrow$
7. Let P be a polynomial of degree n and let Q be a polynomial of degree m (with Q not the zero polynomial). Using sigma notation we can write

$$P = \sum_{k=0}^{n}a_k x^k, \qquad Q = \sum_{k=0}^{m}b_k x^k.$$

Use sigma notation to write the derivative of the **rational function** P/Q.

8. The curve $y = 1/(1+x^2)$ is an example of a class of curves each of which is called a **witch of Agnesi**. Sketch the curve and find the tangent line to the curve at $x = 5$. (The word

---

witch here is a mistranslation of the original Italian, as described at

http://mathworld.wolfram.com/WitchofAgnesi.html

and

http://instructional1.calstatela.edu/sgray/Agnesi/
WitchHistory/Historynamewitch.html.)

$\Rightarrow$

9. If $f'(4) = 5$, $g'(4) = 12$, $(fg)(4) = f(4)g(4) = 2$, and $g(4) = 6$, compute $f(4)$ and $\dfrac{d}{dx}\dfrac{f}{g}$ at 4. $\Rightarrow$

3.5 THE CHAIN RULE

So far we have seen how to compute the derivative of a function built up from other functions by addition, subtraction, multiplication and division. There is another very important way that we combine simple functions to make more complicated functions: function composition, as discussed in section 2.3. For example, consider $\sqrt{625-x^2}$. This function has many simpler components, like 625 and x^2, and then there is that square root symbol, so the square root function $\sqrt{x} = x^{1/2}$ is involved. The obvious question is: can we compute the derivative using the derivatives of the constituents $625 - x^2$ and $\sqrt{x}$? We can indeed. In general, if $f(x)$ and $g(x)$ are functions, we can compute the derivatives of $f(g(x))$ and $g(f(x))$ in terms of $f'(x)$ and $g'(x)$.

EXAMPLE 3.8 Form the two possible compositions of $f(x) = \sqrt{x}$ and $g(x) = 625 - x^2$ and compute the derivatives. First, $f(g(x)) = \sqrt{625-x^2}$, and the derivative is $-x/\sqrt{625-x^2}$ as we have seen. Second, $g(f(x)) = 625 - (\sqrt{x})^2 = 625 - x$ with derivative -1. Of course, these calculations do not use anything new, and in particular the derivative of $f(g(x))$ was somewhat tedious to compute from the definition.
□

Suppose we want the derivative of $f(g(x))$. Again, let's set up the derivative and play some algebraic tricks:

$$\frac{d}{dx}f(g(x)) = \lim_{\Delta x \to 0}\frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\frac{f(g(x+\Delta x)) - f(g(x))}{g(x+\Delta x)) - g(x)}\frac{g(x+\Delta x) - g(x)}{\Delta x}$$

Now we see immediately that the second fraction turns into $g'(x)$ when we take the limit. The first fraction is more complicated, but it too looks something like a derivative. The denominator, $g(x+\Delta x) - g(x)$, is a change in the value of g, so let's abbreviate it as

---

$\Delta g = g(x+\Delta x) - g(x)$, which also means $g(x+\Delta x) = g(x) + \Delta g$. This gives us

$$\lim_{\Delta x \to 0}\frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g}.$$

As Δx goes to 0, it is also true that Δg goes to 0, because $g(x+\Delta x)$ goes to $g(x)$. So we can rewrite this limit as

$$\lim_{\Delta g \to 0}\frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g}.$$

Now this looks exactly like a derivative, namely $f'(g(x))$, that is, the function $f'(x)$ with x replaced by $g(x)$. If this all withstands scrutiny, we then get

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

Unfortunately, there is a small flaw in the argument. Recall that what we mean by $\lim_{\Delta x \to 0}$ involves what happens when Δx is close to 0 *but not equal to 0*. The qualification is very important, since we must be able to divide by Δx. But when Δx is close to 0 but not equal to 0, $\Delta g = g(x+\Delta x)) - g(x)$ is close to 0 *and possibly equal to 0*. This means it doesn't really make sense to divide by Δg. Fortunately, it is possible to recast the argument to avoid this difficulty, but it is a bit tricky; we will not include the details, which can be found in many calculus books. Note that many functions g do have the property that $g(x+\Delta x) - g(x) \neq 0$ when Δx is small, and for these functions the argument above is fine.

The chain rule has a particularly simple expression if we use the Leibniz notation for the derivative. The quantity $f'(g(x))$ is the derivative of f with x replaced by g; this can be written df/dg. As usual, $g'(x) = dg/dx$. Then the chain rule becomes

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}.$$

This looks like trivial arithmetic, but it is not: dg/dx is not a fraction, that is, not literal division, but a single symbol that means $g'(x)$. Nevertheless, it turns out that what looks like trivial arithmetic, and is therefore easy to remember, is really true.

It will take a bit of practice to make the use of the chain rule come naturally—it is more complicated than the earlier differentiation rules we have seen.

EXAMPLE 3.9 Compute the derivative of $\sqrt{625-x^2}$. We already know that the answer is $-x/\sqrt{625-x^2}$, computed directly from the limit. In the context of the chain rule, we have $f(x) = \sqrt{x}$, $g(x) = 625 - x^2$. We know that $f'(x) = (1/2)x^{-1/2}$, so

$f'(g(x)) = (1/2)(625 - x^2)^{-1/2}$. Note that this is a two step computation: first compute $f'(x)$, then replace x by $g(x)$. Since $g'(x) = -2x$ we have

$$f'(g(x))g'(x) = \frac{1}{2\sqrt{625 - x^2}}(-2x) = \frac{-x}{\sqrt{625 - x^2}}.$$

□

EXAMPLE 3.10 Compute the derivative of $1/\sqrt{625 - x^2}$. This is a quotient with a constant numerator, so we could use the quotient rule, but it is simpler to use the chain rule. The function is $(625 - x^2)^{-1/2}$, the composition of $f(x) = x^{-1/2}$ and $g(x) = 625 - x^2$. We compute $f'(x) = (-1/2)x^{-3/2}$ using the power rule, and then

$$f'(g(x))g'(x) = \frac{-1}{2(625 - x^2)^{3/2}}(-2x) = \frac{x}{(625 - x^2)^{3/2}}.$$

□

In practice, of course, you will need to use more than one of the rules we have developed to compute the derivative of a complicated function.

EXAMPLE 3.11 Compute the derivative of

$$f(x) = \frac{x^2 - 1}{x\sqrt{x^2 + 1}}.$$

The "last" operation here is division, so to get started we need to use the quotient rule first. This gives

$$f'(x) = \frac{(x^2 - 1)'x\sqrt{x^2 + 1} - (x^2 - 1)(x\sqrt{x^2 + 1})'}{x^2(x^2 + 1)}$$

$$= \frac{2x^2\sqrt{x^2 + 1} - (x^2 - 1)(x\sqrt{x^2 + 1})'}{x^2(x^2 + 1)}.$$

Now we need to compute the derivative of $x\sqrt{x^2 + 1}$. This is a product, so we use the product rule:

$$\frac{d}{dx}x\sqrt{x^2 + 1} = x\frac{d}{dx}\sqrt{x^2 + 1} + \sqrt{x^2 + 1}.$$

Finally, we use the chain rule:

$$\frac{d}{dx}\sqrt{x^2 + 1} = \frac{d}{dx}(x^2 + 1)^{1/2} = \frac{1}{2}(x^2 + 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + 1}}.$$

And putting it all together:

$$f'(x) = \frac{2x^2\sqrt{x^2 + 1} - (x^2 - 1)(x\sqrt{x^2 + 1})'}{x^2(x^2 + 1)}.$$

$$= \frac{2x^2\sqrt{x^2 + 1} - (x^2 - 1)\left(x\frac{x}{\sqrt{x^2 + 1}} + \sqrt{x^2 + 1}\right)}{x^2(x^2 + 1)}.$$

This can be simplified of course, but we have done all the calculus, so that only algebra is left.

□

EXAMPLE 3.12 Compute the derivative of $\sqrt{1 + \sqrt{1 + \sqrt{x}}}$. Here we have a more complicated chain of compositions, so we use the chain rule twice. At the outermost "layer" we have the function $g(x) = 1 + \sqrt{1 + \sqrt{x}}$ plugged into $f(x) = \sqrt{x}$, so applying the chain rule once gives

$$\frac{d}{dx}\sqrt{1 + \sqrt{1 + \sqrt{x}}} = \frac{1}{2}\left(1 + \sqrt{1 + \sqrt{x}}\right)^{-1/2}\frac{d}{dx}\left(1 + \sqrt{1 + \sqrt{x}}\right).$$

Now we need the derivative of $\sqrt{1 + \sqrt{x}}$. Using the chain rule again:

$$\frac{d}{dx}\sqrt{1 + \sqrt{x}} = \frac{1}{2}\left(1 + \sqrt{x}\right)^{-1/2}\frac{1}{2}x^{-1/2}.$$

So the original derivative is

$$\frac{d}{dx}\sqrt{1 + \sqrt{1 + \sqrt{x}}} = \frac{1}{2}\left(1 + \sqrt{1 + \sqrt{x}}\right)^{-1/2}\frac{1}{2}\left(1 + \sqrt{x}\right)^{-1/2}\frac{1}{2}x^{-1/2}.$$

$$= \frac{1}{8\sqrt{x}\sqrt{1 + \sqrt{x}}\sqrt{1 + \sqrt{1 + \sqrt{x}}}}$$

□

Using the chain rule, the power rule, and the product rule, it is possible to avoid using the quotient rule entirely.

EXAMPLE 3.13 Compute the derivative of $f(x) = \frac{x^3}{x^2 + 1}$. Write $f(x) = x^3(x^2 + 1)^{-1}$, then

$$f'(x) = x^3\frac{d}{dx}(x^2 + 1)^{-1} + 3x^2(x^2 + 1)^{-1}$$

$$= x^3(-1)(x^2 + 1)^{-2}(2x) + 3x^2(x^2 + 1)^{-1}$$

$$= -2x^4(x^2 + 1)^{-2} + 3x^2(x^2 + 1)^{-1}$$

$$= \frac{-2x^4}{(x^2 + 1)^2} + \frac{3x^2}{x^2 + 1}$$

$$= \frac{-2x^4}{(x^2 + 1)^2} + \frac{3x^2(x^2 + 1)}{(x^2 + 1)^2}$$

$$= \frac{-2x^4 + 3x^4 + 3x^2}{(x^2 + 1)^2} = \frac{x^4 + 3x^2}{(x^2 + 1)^2}$$

Note that we already had the derivative on the second line; all the rest is simplification. It is easier to get to this answer by using the quotient rule, so there's a trade off: more work for fewer memorized formulas.

□

Exercises 3.5.

Find the derivatives of the functions. For extra practice, and to check your answers, do some of these in more than one way if possible.

1. $x^4 - 3x^3 + (1/2)x^2 + 7x - \pi \Rightarrow$
2. $x^3 - 2x^2 + 4\sqrt{x} \Rightarrow$
3. $(x^2 + 1)^3 \Rightarrow$
4. $x\sqrt{169 - x^2} \Rightarrow$
5. $(x^2 - 4x + 5)\sqrt{25 - x^2} \Rightarrow$
6. $\sqrt{r^2 - x^2}$, r is a constant $\Rightarrow$
7. $\sqrt{1 + x^4} \Rightarrow$
8. $\frac{1}{\sqrt{5 - \sqrt{x}}} \Rightarrow$
9. $(1 + 3x)^2 \Rightarrow$
10. $\frac{(x^2 + x + 1)}{(1 - x)} \Rightarrow$
11. $\frac{\sqrt{25 - x^2}}{x} \Rightarrow$
12. $\sqrt{\frac{169}{x} - x} \Rightarrow$
13. $\sqrt{x^3 - x^2 - (1/x)} \Rightarrow$
14. $100/(100 - x^2)^{3/2} \Rightarrow$
15. $\sqrt[3]{x + x^3} \Rightarrow$
16. $\sqrt{(x^2 + 1)^2 + \sqrt{1 + (x^2 + 1)^2}} \Rightarrow$
17. $(x + 8)^5 \Rightarrow$
18. $(4 - x)^3 \Rightarrow$
19. $(x^2 + 5)^3 \Rightarrow$
20. $(6 - 2x^2)^3 \Rightarrow$
21. $(1 - 4x^3)^{-2} \Rightarrow$
22. $5(x + 1 - 1/x) \Rightarrow$
23. $4(2x^2 - x + 3)^{-2} \Rightarrow$
24. $\frac{1}{1 + 1/x} \Rightarrow$
25. $\frac{-3}{4x^2 - 2x + 1} \Rightarrow$
26. $(x^2 + 1)(5 - 2x)/2 \Rightarrow$

27. $(3x^2 + 1)(2x - 4)^3 \Rightarrow$
28. $\frac{x + 1}{x - 1} \Rightarrow$
29. $\frac{x^2 - 1}{x^2 + 1} \Rightarrow$
30. $\frac{(x - 1)(x - 2)}{x - 3} \Rightarrow$
31. $\frac{2x^{-1} - x^{-2}}{3x^{-1} - 4x^{-2}} \Rightarrow$
32. $3(x^2 + 1)(2x^2 - 1)(2x + 3) \Rightarrow$
33. $\frac{1}{(2x + 1)(x - 3)} \Rightarrow$
34. $((2x + 1)^{-1} + 3)^{-1} \Rightarrow$
35. $(2x + 1)^3(x^2 + 1)^2 \Rightarrow$

36. Find an equation for the tangent line to $f(x) = (x - 2)^{1/3}/(x^3 + 4x - 1)^2$ at $x = 1$. $\Rightarrow$
37. Find an equation for the tangent line to $y = 9x^{-2}$ at $(3, 1)$. $\Rightarrow$
38. Find an equation for the tangent line to $(x^2 - 4x + 5)\sqrt{25 - x^2}$ at $(3, 8)$. $\Rightarrow$
39. Find an equation for the tangent line to $\frac{(x^2 + x + 1)}{(1 - x)}$ at $(2, -7)$. $\Rightarrow$
40. Find an equation for the tangent line to $\sqrt{(x^2 + 1)^2 + \sqrt{1 + (x^2 + 1)^2}}$ at $(1, \sqrt{4 + \sqrt{5}})$. $\Rightarrow$